α-Riesz capacities and Hausdorff measures on a β-set

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ABSTRACT. We consider a compact metric space (X, d) such that X is a β-set ($\beta > 0$). In this space, we define α-Riesz capacities and investigate the relationships between these capacities and Hausdorff measures.

1. Introduction

Let $\mathbf{R}^N$ be the N-dimensional Euclidean space. The Bessel capacity is important as a way of characterizing certain small subsets of $\mathbf{R}^N$. The following estimates for the α-Bessel capacity $B_{\alpha,p}(B_r)$ of a ball B_r of radius r are well-known (cf. [7], [1]).

Theorem A . *Let $1 < p < \infty$ and $\alpha > 0$.*

(i) *If $\alpha p < N$, then there exists a constant $C \geq 1$ such that*

$$C^{-1} r^{N-\alpha p} \leq B_{\alpha,p}(B_r) \leq C r^{N-\alpha p}$$

for all B_r of radius r.

(ii) *If $\alpha p = N$ and $0 < \bar{r} < 1$, then there exists a constant $C \geq 1$ such that*

$$C^{-1}(\log \frac{1}{r})^{1-p} \leq B_{\alpha,p}(B_r) \leq C(\log \frac{1}{r})^{1-p}$$

for $0 < r \leq \bar{r}$.

On the other hand the Hausdorff measure provides another approach to investigate small subsets of $\mathbf{R}^N$. Let $h(r)$ $(0 \leq h(r) \leq +\infty)$ be an increasing function, defined for $r \geq 0$, and satisfying $h(0) = 0$. Let $E \subset \mathbf{R}^N$. For any δ, $0 < \delta \leq \infty$, a set function Λ_h^δ is defined by

$$\Lambda_h^\delta(E) = \inf \sum_{j=1}^{\infty} h(r_j),$$

where the infimum is taken over all coverings by countable unions of balls $B(x_j, r_j)$ satisfying $r_j \leq \delta$. And the value defined by

$$\Lambda_h(E) = \lim_{\delta \to 0} \Lambda_h^\delta(E)$$

is the Hausdorff measure of E with respect to the function h.

The relationships between the Bessel capacity and the Hausdorff measure have been investigated. The upper estimate for the α-Bessel capacity in term of the Hausdorff measure is as follows (cf. [1]).

Theorem B . *Let $1 < p < \infty$, $0 < \alpha p \leq N$ and $E \subset \mathbf{R}^N$. Set $h(r) = r^{N-\alpha p}$ for $\alpha p < N$ and $h(r) = (\log_+ \frac{2}{r})^{1-p}$ for $\alpha p = N$. Then there is a constant $C > 0$ independent of E such that*

$$B_{\alpha,p}(E) \leq C\Lambda_h^1(E),$$

and moreover, $\Lambda_h(E) < \infty$ implies $B_{\alpha,p}(E) = 0$.

The set function Λ_h^∞ is often more useful than the Hausdorff measure itself. It is called the Hausdorff content or the Hausdorff capacity.

The lower estimate for $B_{\alpha,p}$ in term of Hausdorff content is also stated in Adams and Hedberg [1] as follows;

Theorem C . *Let $1 < p < \infty$, $0 < \alpha p \leq N$ and $p' = \frac{p}{p-1}$, and let h be an increasing function on $[0,\infty)$ such that $h(0) = 0$, and*

$$\int_0^1 \left(\frac{h(r)}{r^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty.$$

Let $E \subset \mathbf{R}^N$ be compact satisfying $\Lambda_h^\infty(E) > 0$ and choose δ, $0 \leq \delta \leq 1$, so that

$$h(\delta) \leq \Lambda_h^\infty(E).$$

Set

$$H = \int_0^\delta \left(\frac{h(r)}{r^{N-\alpha p}}\right)^{p'-1} \frac{dr}{r} + \Lambda_h^\infty(E)^{p'-1} \int_\delta^1 \left(\frac{1}{r^{N-\alpha p}}\right)^{p'-1} \frac{dr}{r}.$$

Then there is a constant $C > 0$, independent of h and E, such that

$$\Lambda_h^\infty(E) \leq C H^{p-1} B_{\alpha,p}(E).$$

In particular $B_{\alpha,p}(E) = 0$ implies $\Lambda_h(E) = 0$.

In this paper we consider a compact metric space (X,d). Since the diameter of X is finite, we suppose that the diameter of X is R. Furthermore, suppose that X is a β-set ($\beta > 0$), i.e., there exist a positive Radon measure μ on X and positive real numbers b_1, b_2, R_0 such that

$$b_1 r^\beta \leq \mu(B(x,r)) \leq b_2 r^\beta$$

for all point $x \in X$ and all positive real number $r \leq R_0$, where $B(x,r)$ stands for the open ball of radius r centered at x. Such a measure μ is called a β-measure on X.

Fix a β-measure μ. We may assume that

$$b_1 r^\beta \leq \mu(B(x,r)) \leq \mu(\overline{B(x,r)}) \leq b_2 r^\beta \tag{1.1}$$

for all $r \leq \max\{3,h\}R$ by choosing different constants b_1, b_2 if necessarily. Here h is the constant in Theorem E in §2.

In this spsce X we will find the estimates corresponding to Theorem A. Change of variables is effective in $\mathbf{R}^N$ but we can't directly apply them in a metric space (X,d). We also note that the Bessel capacity with order α is comparable to the Riesz capacity with order α in a fixed ball of $\mathbf{R}^N$.

Let $1 < p < \infty$ and $0 < \alpha < \beta$. Let E be a subset of X. We denote by $\mathcal{M}^+(E)$ the set of all positive Radon measures on E. Then we will define the (α,p,μ)-Riesz capacity $C_{\alpha,p,\mu}$ on a β-set X. We consider an α-Riesz kernel on $X \times X$, i.e.,

$$K_\alpha(x,y) = d(x,y)^{\alpha-\beta} \quad (0 < \alpha < \beta).$$

Let f be a non-negative, μ-measurable function and let $\nu \in \mathcal{M}^+(X)$. We define α-Riesz potentials $K_\alpha f(x)$, and $K_\alpha \nu(x)$ by

$$K_\alpha f(x) = \int_X K_\alpha(x,y) f(y) d\mu(y) \qquad x \in X,$$

$$K_\alpha \nu(x) = \int_X K_\alpha(x,y) d\nu(y) \qquad x \in X.$$

For the kernel K_α on $X \times X$ we also define the (α,p,μ)-Riesz capacity $C_{\alpha,p,\mu}(E)$ of a set E.

Definition 1.1. *Let $1 < p < \infty$ and E be a set. Denote*

$$\mathcal{F}(K_\alpha, p, \mu, E) = \{f \in L^p(\mu) \ : \ f \geq 0,\, K_\alpha f \geq 1 \ on\ E\},$$

and

$$C_{\alpha,p,\mu}(E) = \inf\{\int f^p d\mu \ : \ f \in \mathcal{F}(K_\alpha, p, \mu, E)\}.$$

If $\mathcal{F}(K_\alpha, p, \mu, E)$ is empty, we regard $C_{\alpha,p,\mu}(E)$ as ∞.

By the properties of a β-set and using distribution functions we obtain following two theorems corresponding Theorem A, which will be proved in §4.

Theorem 1. *Let $0 < \alpha < \beta$ and $1 < p < \frac{\beta}{\alpha}$. Then there exists a constant $C > 1$ such that*

$$C^{-1} r^{\beta - \alpha p} \leq C_{\alpha,p,\mu}(B(x_0, r)) \leq C r^{\beta-\alpha p}$$

for all $x_0 \in X$ and $0 < r \leq R$.

Theorem 2. *Let $0 < \alpha < \beta$, $1 < p < \infty$, $\alpha p = \beta$ and b_1, b_2 be numbers in* (1.1). *Furthermore put*

$$\eta_1 = e^{\frac{2b_2}{b_1 \beta}}, \quad \eta_2 = e^{\frac{1}{\beta}} \quad and \quad \zeta = \min\{\frac{1}{2\eta_1}, \frac{1}{2\eta_2}, \frac{1}{2e}\}. \tag{1.2}$$

Then there exists a constant $C > 1$ such that

$$C^{-1}(\log \frac{R}{2r})^{1-p} \leq C_{\alpha,p,\mu}(B(x_0, r)) \leq C(\log \frac{R}{2r})^{1-p},$$

for all $x_0 \in X$ and $0 < r \leq \zeta R$.

Next we define the s-Hausdorff measure of a set E $(s \geq 0)$ by

$$\mathcal{H}^s(E) = \lim_{\delta \to \infty} \mathcal{H}^s_\delta(E) = \sup_{\delta > 0} \mathcal{H}^s_\delta(E),$$

where

$$\mathcal{H}^s_\delta(E) = \Lambda^\delta_h(E) \quad for \quad h(r) = r^s.$$

We next consider the upper estimate for the α-Riesz capacities in term of the s-Hausdorff measure corresponding to Theorem B. The upper estimate is given as follows.

Theorem 3. *Let $0 < \alpha < \beta$, $1 < p \leq \frac{\beta}{\alpha}$ and b_1 be the number in* (1.1). *Further, let $E \subset X$ and set $h(r) = r^{\beta-\alpha p}$ for $\alpha p < \beta$ and $h(r) = (\log \frac{R}{2r})^{1-p}$ for $\alpha p = \beta$. Then there exists a constant $C > 0$, depending only on α, β, p and b_1, such that*

$$C_{\alpha,p,\mu}(E) \leq C \Lambda^\delta_h(E) \tag{1.3}$$

where $\delta = R$ for $\alpha p < \beta$ and $\delta = \zeta R$ with ζ in (1.2) *for $\alpha p = \beta$. Furthermore, if $\alpha p < \beta$ and $\Lambda_h(E) < \infty$, then $C_{\alpha,p,\mu}(E) = 0$.*

We next state the correspondence to Theorem C. Frostman's lemma is usually proved in Euclidean space. But Mattila also has proved, in a compact metric space, the theorem corresponding to Frostman's lemma (cf. [6, 8.17 Theorem]). By using this, we shall prove the following theorem in a metric space.

Theorem 4. *Let $1 < p < \infty$, $0 < \alpha p \leq \beta$ and put $k = \beta - \alpha p + \varepsilon$ for $\varepsilon > 0$. Suppose that $E \subset X$ is compact. If $C_{\alpha,p,\mu}(E) = 0$, then $\mathcal{H}^k(E) = 0$.*

Theorems 3 and 4 will be proved in §5.

2. Maximal functions on a β-set X

Hereafter we assume that a compact metric space X is a β-set such that the diameter of X is R. Two β-measures are equivalent in the following sense (cf. [5, Proposition 1 on p.30]).

Proposition D . *Let μ_1 and μ_2 be β-measures on X. Then there is a constant $C > 1$ such that*

$$C^{-1}\mu_1 \leq \mu_2 \leq C\mu_1.$$

Proposition D is proved in $\mathbf{R}^N$. But it's proof is available by small change under our assumptions.
Fix a β-measure μ on X. And we see by (1.1) that the measure μ satisfies the doubling condition in the following sense;

Proposition 2.1. *There is a constant b such that for any $x \in X$ and $r > 0$*

$$\mu(B(x,2r)) \leq b\mu(B(x,r)). \tag{2.1}$$

We use the following covering theorem of Vitali type in [3, Theorem (1.2) on p.69].

Theorem E . *Suppose that $E \subset B(x,R_1)$ for some $R_1 > 0$. If a family $\{B(x_\tau,r_\tau)\}$ of balls covers E, then there exists a countable system of disjoint balls $\{B(x_j,r_j)\}$ such that $\{B(x_j,hr_j)\}$ covers E for some $h \geq 1$.*

We next define the maximal function on the β-set X and mention some poroperties of it. We define the maximal function Mf of a locally μ-integrable function f by

$$Mf(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|d\mu(y)$$

and, for $0 < \alpha < \beta$, the fractional maximal function $M_\alpha f$ by

$$M_\alpha f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))^{\frac{\beta-\alpha}{\beta}}} \int_{B(x,r)} |f(y)|d\mu(y).$$

We also define a maximal function and a fractional maximal function $M\nu$, $M_\alpha\nu$ of a positive Radon measure ν on X by

$$M\nu(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} d\nu(y),$$

$$M_\alpha f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))^{\frac{\beta-\alpha}{\beta}}} \int_{B(x,r)} d\nu(y).$$

The following propositon can be shown by the usual method.

Proposition 2.2. (i) *If $f \in L^1(\mu)$ and $\lambda > 0$, then*

$$\mu(\{x \in X \ : \ Mf(x) > \lambda\}) \leq \frac{C}{\lambda}||f||_{L^1(\mu)}.$$

(ii) *If $f \in L^p(\mu)$ $(1 < p \leq \infty)$, then $Mf \in L^p(\mu)$ and*

$$||Mf||_{L^p(\mu)} \leq C||f||_{L^p(\mu)}.$$

(iii) *If $\nu \in \mathcal{M}^+(X)$, then*

$$\mu(\{x \in X \ : \ M\nu(x) > \lambda\}) \leq \frac{C}{\lambda}\int d\nu.$$

Here C are constants independent of f, λ and ν.

Proposition 2.3. *If $f \in L^1_{loc}(\mu)$ and $\lambda > 0$, then*

$$\mu(\{x \in X \ : \ M_\alpha f(x) > \lambda\}) \leq C(\frac{1}{\lambda}\int_X |f| d\mu)^{\frac{\beta}{\beta-\alpha}},$$

where C is constant independent of f and λ.

Proof. For any $\lambda > 0$ we put

$$E_\lambda = \{x \in X \ : \ M_\alpha f(x) > \lambda\},$$

Let $x \in E_\lambda$. Then there exists a ball B_x with center x and

$$\lambda < (\frac{1}{\mu(B_x)^{\frac{\beta-\alpha}{\beta}}}\int_{B_x} |f| d\mu,$$

and hence

$$\mu(B_x) < (\frac{1}{\lambda}\int_{B_x} |f| d\mu)^{\frac{\beta}{\beta-\alpha}}.$$

By Theorem E there is a disjoint subfamily $\{B_{x_j}\}$ of $\{B_x\}_{x \in E_\lambda}$, $B_{x_j} = B(x_j, r_j)$ such that

$$E_\lambda \subset \cup_j B(x_j, hr_j).$$

Hence, by (1.1)

$$\begin{aligned} \mu(E_\lambda) &\leq \mu(\cup_j (B(x_j, hr_j)) \leq \sum_j \mu(B(x_j, hr_j)) \leq b_2 h^\beta \sum_j r_j^\beta \\ &\leq \frac{b_2}{b_1} h^\beta \sum_j \mu(B(x_j, r_j)) < C \sum_j (\frac{1}{\lambda}\int_{B_{x_j}} |f| d\mu)^{\frac{\beta}{\beta-\alpha}}, \end{aligned}$$

where $C = \frac{b_2}{b_1} h^\beta$. Since $\{B_{x_j}\}$ is disjoint, we have

$$(\sum_j \frac{1}{\lambda}\int_{B_{x_j}} |f| d\mu)^{\frac{\beta}{\beta-\alpha}} \leq (\frac{1}{\lambda}\int_X |f| d\mu)^{\frac{\beta}{\beta-\alpha}},$$

whence

$$\mu(E_\lambda) \leq C(\frac{1}{\lambda}\int_X |f| d\mu)^{\frac{\beta}{\beta-\alpha}}.$$

□

3. Capacities and potentials

In this section we will define capacities and potentials for a general kernel and state a fundamental properties of these.

By a kernel on $X \times X$ we mean any non-negative symmetric function K on $X \times X$, such that $K(\cdot, y)$ is lower semicontinuous on X for each $y \in X$, and $K(x, \cdot)$ is μ-measurable on X for each $x \in X$. Let f be a non-negative, μ-measurable function and let $\nu \in \mathcal{M}^+(X)$. We define potentials $Kf(x)$, and $K\nu(x)$ by

$$Kf(x) = \int_X K(x, y) f(y) d\mu(y) \qquad x \in X, \tag{3.1}$$

$$K\nu(x) = \int_X K(x, y) d\nu(y) \qquad x \in X. \tag{3.2}$$

For a kernel K on $X \times X$ we also define the L^p-capacity $C_{K,p,\mu}(E)$ of a set E.

Definition 3.1. *Let $1 \leq p < \infty$ and E be a set. Denote*

$$\mathcal{F}(K,p,\mu,E) = \{f \in L^p(\mu) \,:\, f \geq 0,\, Kf \geq 1 \text{ on } E\},$$

and

$$C_{K,p,\mu}(E) = \inf\{\int f^p d\mu \,:\, f \in \mathcal{F}(K,p,\mu,E)\}. \tag{3.3}$$

If $\mathcal{F}(K,p,\mu,E)$ is empty, we regard $C_{K,p,\mu}(E)$ as ∞.

By the usual method we see that the L^p-capacity has the following properties.

Theorem F . *Let $1 \leq p < \infty$. Then*

(i) *$E_1 \subset E_2$ implies $C_{K,p,\mu}(E_1) \leq C_{K,p,\mu}(E_2)$,*

(ii) *for any set E, $C_{K,p,\mu}(E) = \inf\{C_{K,p,\mu}(G) \,:\, E \subset G,\ G \text{ open}\}$,*

(iii) *$E = \cup_{j=1}^{\infty} E_j$ implies $C_{K,p,\mu}(E) \leq \sum_{j=1}^{\infty} C_{K,p,\mu}(E_j)$,*

(iv) *for a decreasing sequence $\{F_j\}$ of compact sets,*

$$C_{K,p,\mu}(\cap_{j=1}^{\infty} F_j) = \lim_{j\to\infty} C_{K,p,\mu}(F_j),$$

(v) *for an increasing sequence $\{E_j\}$,*

$$C_{K,p,\mu}(\cup_{j=1}^{\infty} E_j) = \lim_{j\to\infty} C_{K,p,\mu}(E_j).$$

A capacity satisfying the properties (i), (iv) and (v) in Theorem F is called *Choquet capacity*. Thus, by the capacitability theorem of G. Choquet [2] we see that following theorem holds.

Theorem G . *Let $1 < p < \infty$. Then all Borel sets E are capacitable for $C_{K,p,\mu}$, i.e.,*

$$\begin{aligned} C_{K,p,\mu}(E) &= \sup\{C_{K,p,\mu}(F) \,:\, F \subset E,\ F \text{ compact}\} \\ &= \inf\{C_{K,p,\mu}(G) \,:\, G \supset E,\ G \text{ open}\}. \end{aligned}$$

We say that a property holds (K,p,μ)-q.e., if it holds on X except a set of (K,p,μ)-capacity zero. The following propositions can be proved by the same methods as Theorem 2.3.10, Theorem 2.5.1 and Theorem 2.5.3 in [1].

Proposition 3.1. *Let $1 < p < \infty$ and E be a set. If $C_{K,p,\mu}(E) < \infty$, then there is a unique $f_E \in L_+^p(X,\mu)$ such that $Kf_E \geq 1$ (K,p,μ)-q.e. on E and*

$$\int_X (f_E)^p d\mu = C_{K,p,\mu}(E).$$

We call the function f_E the capacitary function of E, and Kf_E the capacity potencial of E.

Proposition 3.2. *Let $1 < p < \infty$ and F be compact. Then*

$$C_{K,p,\mu}(F)^{\frac{1}{p}} = \sup\{\nu(F) \,:\, \nu \in \mathcal{M}^+(F),\ ||K\nu||_{L^{p'}(\mu)} \leq 1\}.$$

The capacity of a compact set F can be represented by means of a positive measure on F.

Proposition 3.3. *Let $1 < p < \infty$ and F be compact. Assume that $C_{K,p,\mu}(F) < \infty$. Then there is a measure $\nu_F \in \mathcal{M}^+(F)$ such that*

$$\nu_F(F) = \int (K\nu_F)^{p'} d\mu = C_{K,p,\mu}(F).$$

4. (α, p, μ)-Riesz capacities for balls

The main purpose of this section is to prove Theorem 1 and Theorem 2 which estimate the (α, p, μ)-Riesz capacity of balls.

To do it, we begin with elementary formulas.

Lemma 4.1. *Let $\nu \in \mathcal{M}^+(X)$.*

(i) *If $0 < \lambda < \beta$, then*

$$\int_{d(x,y)<r} d(x,y)^{\lambda-\beta} d\nu(y) = (\beta - \lambda) \int_0^r t^{\lambda-\beta-1} \nu(B(x,t)) dt + r^{\lambda-\beta} \nu(B(x,r))$$

for all r satisfying $0 < r \leq R$.

(ii) *If $\lambda < 0$, then*

$$\int_{d(x,y)\geq r} d(x,y)^{\lambda-\beta} d\nu(y) = (\beta - \lambda) \int_r^R t^{\lambda-\beta-1} \nu(B(x,t)) dt - r^{\lambda-\beta} \nu(B(x,r)) + R^{\lambda-\beta} \nu(X)$$

for all r satisfying $0 < r \leq R$.

Proof. Let $0 < \lambda < \beta$. We can write

$$\begin{aligned}
(\beta - \lambda) \int_0^r t^{\lambda-\beta-1} \nu(B(x,t)) dt &= (\beta - \lambda) \int_0^r t^{\lambda-\beta-1} \left(\int_{d(x,y)<t} d\nu(y) \right) dt \\
&= (\beta - \lambda) \int_{d(x,y)<r} \left(\int_{d(x,y)}^r t^{\lambda-\beta-1} dt \right) d\nu(y) \\
&= \int_{d(x,y)<r} d(x,y)^{\lambda-\beta} d\nu(y) - r^{\lambda-\beta} \nu(B(x,r)).
\end{aligned}$$

This leads to (i).

Next let $\lambda < 0$. In a similar way, (ii) follows from

$$\begin{aligned}
&(\beta - \lambda) \int_r^R t^{\lambda-\beta-1} \nu(B(x,t)) dt \\
&= (\beta - \lambda) \int_r^R t^{\lambda-\beta-1} \left(\int_{d(x,y)<t} d\nu(y) \right) dt \\
&= (\beta - \lambda) \{ \int_{d(x,y)\geq r} \left(\int_{d(x,y)}^R t^{\lambda-\beta-1} dt \right) d\nu(y) + \int_{d(x,y)<r} \left(\int_r^R t^{\lambda-\beta-1} dt \right) d\nu(y) \} \\
&= \int_{d(x,y)\geq r} d(x,y)^{\lambda-\beta} d\nu(y) + r^{\lambda-\beta} \nu(B(x,r)) - R^{\lambda-\beta} \nu(X).
\end{aligned}$$

□

The λ-Riesz potentials for balls are as follows;

Proposition 4.2. *There are constants C_1, $C_2 > 0$, depending only on λ, β and b_1, b_2 in the number in (1.1), such that for all $x_0 \in X$ and $0 < r \leq R$*

(i) $C_1^{-1} r^\lambda \leq \int_{d(x_0,x)<r} d(x_0,x)^{\lambda-\beta} d\mu(x) \leq C_1 r^\lambda, \qquad 0 < \lambda < \beta.$

(ii) $\int_{d(x_0,x)\geq r} d(x_0,x)^{\lambda-\beta} d\mu(x) \leq C_2 r^\lambda, \qquad \lambda < 0.$

Proof. The assertion (i) follows from (1.1) and Lemma 4.1, (i). To prove (ii), let $\lambda < 0$. By Lemma 4.1, (ii) we have

$$\begin{aligned}\int_{d(x_0,x)\geq r} d(x,x_0)^{\lambda-\beta}d\mu(x) &\leq (\beta-\lambda)b_2\int_r^R t^{\lambda-1}dt + b_2R^\lambda \\ &= b_2\frac{\beta-\lambda}{\lambda}(R^\lambda - r^\lambda) + b_2R^\lambda \\ &\leq \frac{\lambda-\beta}{\lambda}b_2r^\lambda,\end{aligned}$$

which is the assertion (ii). □

We next consider the estimate corresponding Proposition 4.2 in the case where $\lambda = 0$. We shall apply this proposition to Theorem 2.

Proposition 4.3. *Let $p > 1$, $0 < \alpha < \beta$ and b_1, b_2 be the numbers in* (1.1). *Furthermore, let η_1, η_2 be the numbers in* (1.2).

(i) *If $0 < a < \eta_1 a \leq b \leq R$, then*

$$\int_{a\leq d(x_0,x)\leq b} d(x_0,x)^{-\beta}d\mu(x) \geq C_3\log\frac{b}{a}.$$

(ii) *If $0 < a < \eta_2 a \leq b \leq R$, then*

$$\int_{a\leq d(x_0,x)\leq b} d(x_0,x)^{-\beta}d\mu(x) \leq C_4\log\frac{b}{a}.$$

Here constants C_3, C_4 are independent of a, b.

Proof. We first show (i). By using a distribution function we have

$$\begin{aligned}\int_{a\leq d(x_0,x)\leq b} d(x_0,x)^{-\beta}d\mu(x) &= \int_0^\infty \mu(\{x : a\leq d(x_0,x)\leq b\}\cap\{x : d(x_0,x)^{-\beta} > t\})dt \\ &\geq \int_{b^{-\beta}}^{a^{-\beta}} \mu(\{x : a \leq d(x_0,x) < t^{-\frac{1}{\beta}}\})dt \\ &= \int_{b^{-\beta}}^{a^{-\beta}} \{\mu(B(x_0,t^{-\frac{1}{\beta}})) - \mu(B(x_0,a))\}dt \\ &\geq b_1\int_{b^{-\beta}}^{a^{-\beta}} t^{-1}dt - b_2a^\beta\int_{b^{-\beta}}^{a^{-\beta}} dt \;\geq\; b_1\beta\log\frac{b}{a} - b_2.\end{aligned}$$

The inequality $\eta_1 a \leq b$ implies $b_2 \leq \frac{1}{2}b_1\beta\log\frac{b}{a}$. Hence

$$\int_{a\leq d(x_0,x)\leq b} d(x_0,x)^{-\beta}d\mu(x) \geq \frac{1}{2}b_1\beta\log\frac{b}{a} = C_3\log\frac{b}{a},$$

where $C_3 = \frac{1}{2}b_1\beta$.

We next show (ii). Since

$$\begin{aligned}&\int_{a\leq d(x_0,x)\leq b} d(x_0,x)^{-\beta}d\mu(x) \\ &= \int_0^\infty \mu(\{x : a\leq d(x_0,x)\leq b\}\cap\{x : d(x_0,x)^{-\beta} > t\})dt \\ &\leq \int_0^{b^{-\beta}} \mu(\{x : d(x_0,x)\leq b\})dt + \int_{b^{-\beta}}^{a^{-\beta}} \mu(\{x : d(x_0,x) < t^{-\frac{1}{\beta}}\})dt \\ &\leq b_2b^\beta\int_0^{b^{-\beta}} dt + b_2\int_{b^{-\beta}}^{a^{-\beta}} t^{-1}dt \;=\; b_2(1+\beta\log\frac{b}{a})\end{aligned}$$

and $\eta_2 a \le b$, we have

$$\int_{a\le d(x_0,x)\le b} d(x_0,x)^{-\beta} d\mu(x) \le 2b_2\beta \log\frac{b}{a} = C_4 \log\frac{b}{a},$$

where $C_4 = 2b_2\beta$. $\square$

The capacity $C_{\alpha,p,\mu}(B)$ for a ball B is estimated as follows;

Proposition 4.4. *Let $0<\alpha<\beta$, $1<p\le\frac{\beta}{\alpha}$ and $p'=\frac{p}{p-1}$. Then there is a constant $C_5>0$ such that for all $r>0$*

$$\begin{aligned} 2^{1-p}2^{(\alpha-\beta)p}\min\{\left(\int_{d(x_0,x)\ge 2r} d(x_0,x)^{(\alpha-\beta)p'}d\mu(x)\right)^{1-p}, 2^{(\beta-\alpha)p}C_5^{1-p}r^{\beta-\alpha p}\} \qquad (4.1)\\ \le\ C_{\alpha,p,\mu}(B(x_0,r)) \le 2^{(\beta-\alpha)p}\left(\int_{d(x_0,x)\ge 2r} d(x_0,x)^{(\alpha-\beta)p'}d\mu(x)\right)^{1-p}. \end{aligned}$$

Proof. We first show the upper estimate of $C_{\alpha,p,\mu}$. To do this, let $x_0\in X$ and assume that $||K_\alpha\nu||_{L^{p'}(\mu)}\le 1$ for a measure ν in $\mathcal{M}^+(\overline{B(x_0,r)})$. Then

$$\begin{aligned} 1 &\ge ||K_\alpha\nu||^{p'}_{L^{p'}(\mu)} \ge \int_{d(x_0,x)\ge 2r}\left(\int_{\overline{B(x_0,r)}} d(x,y)^{\alpha-\beta}d\nu(y)\right)^{p'} d\mu(x)\\ &\ge \int_{d(x_0,x)\ge 2r}\left(\int_{\overline{B(x_0,r)}} (2d(x_0,x))^{\alpha-\beta}d\nu(y)\right)^{p'} d\mu(x)\\ &= 2^{(\alpha-\beta)p'}\nu(\overline{B(x_0,r)})^{p'}\int_{d(x_0,x)\ge 2r} d(x_0,x)^{(\alpha-\beta)p'}d\mu(x) \end{aligned}$$

whence

$$\nu(\overline{B(x_0,r)}) \le 2^{\beta-\alpha}\left(\int_{d(x_0,x)\ge 2r} d(x_0,x)^{(\alpha-\beta)p'}d\mu(x)\right)^{-1/p'}.$$

Taking the supremum over such measures ν, we have, by Proposition 3.2

$$C_{\alpha,p,\mu}(\overline{B(x_0,r)})^{\frac{1}{p}} \le 2^{\beta-\alpha}\left(\int_{d(x_0,x)\ge 2r} d(x_0,x)^{(\alpha-\beta)p'}d\mu(x)\right)^{-1/p'},$$

which leads to the right-hand inequality of (4.1).

In order to prove the lower estimate of $C_{\alpha,p,\mu}$, let ν_r be the measure μ restricted to $B(x_0,r-\varepsilon)$ and normalized, i.e.,

$$\int f(y)d\nu_r(y) = \frac{1}{\mu(B(x_0,r-\varepsilon))}\int_{B(x_0,r-\varepsilon)} f(y)d\mu(y),$$

for a sufficient small $\varepsilon>0$ ($\varepsilon<\frac{r}{2}$). Then we have

$$\begin{aligned} &\int_{d(x_0,x)\ge 2r}\left(\int_{B(x_0,r-\varepsilon)} d(x,y)^{\alpha-\beta}d\nu_r(y)\right)^{p'} d\mu(x)\\ \le\ &\int_{d(x_0,x)\ge 2r}\left(\int_{B(x_0,r-\varepsilon)} (\frac{1}{2}d(x_0,x))^{\alpha-\beta}d\nu_r(y)\right)^{p'} d\mu(x)\\ =\ &2^{(\beta-\alpha)p'}\int_{d(x_0,x)\ge 2r} d(x_0,x)^{(\alpha-\beta)p'}d\mu(x) \end{aligned}$$

and, by (1.1) and Proposition 4.2, (i),

$$\begin{aligned}
&\int_{d(x_0,x)<2r}\left(\int_{B(x_0,r-\varepsilon)} d(x,y)^{\alpha-\beta}d\nu_r(y)\right)^{p'} d\mu(x)\\
\leq\ & \mu(B(x_0,\frac{r}{2}))^{-p'}\int_{d(x_0,x)<2r}\left(\int_{B(x,3r)} d(x,y)^{\alpha-\beta}d\mu(y)\right)^{p'} d\mu(x)\\
\leq\ & (b_1 2^{-\beta}r^{\beta})^{-p'}(C_1(3r)^{\alpha})^{p'}b_2(2r)^{\beta} = C_5 r^{\frac{\alpha p-\beta}{p-1}},
\end{aligned}$$

where $C_5 = (3b_1^{-1}2^{\beta}C_1)^{p'}b_2 2^{\beta}$. Therefore

$$||K_\alpha\nu_r||^{p'}_{L^{p'}(\mu)} \leq 2\max\{2^{(\beta-\alpha)p'}\int_{d(x_0,x)\geq 2r} d(x_0,x)^{(\alpha-\beta)p'}d\mu(x),\ C_5 r^{\frac{\alpha p-\beta}{p-1}}\},$$

whence

$$1\geq ||K_\alpha\nu_r||_{L^{p'}(\mu)}2^{-1/p'}\min\{2^{\alpha-\beta}\left(\int_{d(x_0,x)\geq 2r} d(x_0,x)^{(\alpha-\beta)p'}d\mu(x)\right)^{-\frac{1}{p'}},\ C_5^{-\frac{1}{p'}}r^{\frac{\beta-\alpha p}{p'(p-1)}}\}.$$

Thus, again by Proposition 3.2 we have

$$\begin{aligned}
&C_{\alpha,p,\mu}(B(x_0,r))^{1/p}\geq C_{\alpha,p,\mu}(\overline{B(x_0,r-\varepsilon)})^{1/p}\\
\geq\ & \nu_r(B(x_0,r-\varepsilon))2^{-1/p'}\min\{2^{\alpha-\beta}\left(\int_{d(x_0,x)\geq 2r} d(x_0,x)^{(\alpha-\beta)p'}d\mu(x)\right)^{-1/p'},C_5^{-1/p'}r^{\frac{\beta-\alpha p}{p'(p-1)}}\}\\
=\ & 2^{-1/p'}2^{\alpha-\beta}\min\{\left(\int_{d(x_0,x)\geq 2r} d(x_0,x)^{(\alpha-\beta)p'}d\mu(x)\right)^{-\frac{1}{p'}},2^{\beta-\alpha}C_5^{-\frac{1}{p'}}r^{\frac{\beta-\alpha p}{p'(p-1)}}\},
\end{aligned}$$

which leads to the left-hand inequality of (4.1). □

Proof of Theorem 1. Let $x_0\in X$ and $0<r\leq R$. We first show the left-hand inequality. Noting that $(\alpha-\beta)p'+\beta<0$ and using Proposition 4.2, (ii), we have

$$\left(\int_{d(x_0,x)\geq 2r} d(x_0,x)^{(\alpha-\beta)p'}d\mu(x)\right)^{1-p}\geq (C_2(2r)^{(\alpha-\beta)p'+\beta})^{1-p} = C_2^{1-p}2^{\beta-\alpha p}r^{\beta-\alpha p},$$

where C_2 is the number in Proposition 4.2, (ii). Hence, by Proposition 4.4, we have

$$C_{\alpha,p,\mu}(B(x_0,r))\geq 2^{1-p}2^{(\alpha-\beta)p}\min\{C_2^{1-p}2^{\beta-\alpha p},\ 2^{(\beta-\alpha)p}C_5^{1-p}\}r^{\beta-\alpha p}, \tag{4.2}$$

where C_5 is the number in Proposition 4.4.

We next show the right-hand inequality. Let $||K_\alpha\nu||_{L^{p'}(\mu)}\leq 1$ for a measure ν in $\mathcal{M}^+(\overline{B(x_0,r)})$. If $x\in B(x_0,r)$ and $y\in\overline{B(x_0,r)}$ then $d(x,y)<2r$. Hence, by (1.1),

$$\begin{aligned}
1\geq ||K_\alpha\nu||^{p'}_{L^{p'}(\mu)} &\geq \int_{B(x_0,r)}\left(\int_{\overline{B(x_0,r)}}(2r)^{\alpha-\beta}d\nu(x)\right)^{p'}d\mu(y)\\
&= 2^{(\alpha-\beta)p'}r^{(\alpha-\beta)p'}\nu(\overline{B(x_0,r)})^{p'}\mu(B(x_0,r))\\
&\geq 2^{(\alpha-\beta)p'}b_1 r^{\frac{\alpha p-\beta}{p-1}}\nu(\overline{B(x_0,r)})^{p'}.
\end{aligned}$$

Thus

$$\nu(\overline{B(x_0,r)})\leq 2^{\beta-\alpha}b_1^{\frac{1-p}{p}}r^{\frac{\beta-\alpha p}{p}}.$$

Taking the supremum over such measures ν, we have, by Proposition 3.2,

$$C_{\alpha,p,\mu}(\overline{B(x_0,r)})^{\frac{1}{p}} \leq 2^{\beta-\alpha} b_1^{\frac{1-p}{p}} r^{\frac{\beta-\alpha p}{p}},$$

whence

$$C_{\alpha,p,\mu}(B(x_0,r)) \leq 2^{(\beta-\alpha)p} b_1^{1-p} r^{\beta-\alpha p}. \tag{4.3}$$

By (4.2) and (4.3) we have the conclusion. □

Proof of Theorem 2. Note that $r \leq \zeta R$ implies $2\eta_1 r \leq R$. By Proposition 4.4 and Proposition 4.3, (i) we have

$$C_{\alpha,p,\mu}(B(x_0,r)) \leq 2^{(\beta-\alpha)p} \left(\int_{d(x_0,x)\geq 2r} d(x_0,x)^{-\beta} d\mu(y) \right)^{1-p} \leq 2^{(\beta-\alpha)p} C_3^{1-p} (\log \frac{R}{2r})^{1-p},$$

where C_3 is the number in Proposition 4.3.

Similarly, by Proposition 4.4 and Proposition 4.3, (ii) we have

$$\begin{aligned} C_{\alpha,p,\mu}(B(x_0,r)) &\geq 2^{1-p} 2^{(\alpha-\beta)p} \min\{ \left(\int_{d(x_0,x)\geq 2r} d(x_0,x)^{-\beta} d\mu(x) \right)^{1-p}, \ 2^{(\beta-\alpha)p} C_5^{1-p} \} \\ &\geq 2^{1-p} 2^{\alpha-\beta} \min\{ (C_4 \log \frac{R}{2r})^{1-p}, \ 2^{(\beta-\alpha)p} C_5^{1-p} \}, \end{aligned}$$

where C_5 is the number in Proposition 4.4. The inequality $r \leq \frac{R}{2e}$ implies $(\log \frac{R}{2r})^{1-p} \leq 1$. Hence

$$C_{\alpha,p,\mu}(B(x_0,r)) \geq 2^{1-p} 2^{(\alpha-\beta)p} \min\{ C_4^{1-p}, \ 2^{(\beta-\alpha)p} C_5^{1-p} \} (\log \frac{R}{2r})^{1-p}.$$

□

By Theorem 1 and Theorem 2 we can obtain the following consequence.

Corollary 4.5. *Let $0 < \alpha < \beta$. Then*

(i) *If $1 < p < \frac{\beta}{\alpha}$, then $C_{\alpha,p,\mu}(X) < \infty$ and $C_{\alpha,p,\mu}(B(x,r)) > 0$ for every $x \in X$ and $0 < r \leq R$.*

(ii) *If $\alpha p = \beta$, then $C_{\alpha,p,\mu}(B(x,r)) > 0$ for every $x \in X$ and $0 < r \leq \zeta R$ for ζ in (1.2).*

5. Hausdorff measures and (α, p, μ)-capacities

In this section we give upper and lower estimates, Theorem 3 and Theorem 4 for α-Riesz capacities in term of s-Hausdorff measures in §1. To prove Theorem 3 we prepare the following lemma.

Lemma 5.1. *Let $0 < \alpha < \beta$, $1 < p < \frac{\beta}{\alpha}$, $p' = \frac{p}{p-1}$ and ν be a measure in $\mathcal{M}^+(X)$ such that $K_\alpha \nu \in L^{p'}(\mu)$. Then there exists a non-negative, lower semi-continuous, non-increasing function k_0 on $\mathbf{R}^+$ such that $K_0 \nu \in L^{p'}(\mu)$ for the kernel $K_0(x,y) = k_0(d(x,y))$ and*

$$\lim_{r\to 0} \left(\sup_{x\in X} \frac{C_{K_0,p,\mu}(B(x,r))}{C_{\alpha,p,\mu}(B(x,r))} \right) = 0. \tag{5.1}$$

Proof. For each integer j we put

$$\varphi_j(y) = \int_{2^{-j} \leq d(x,y) < 2^{-j+1}} d(x,y)^{\alpha-\beta} d\nu(x).$$

Then we note that $K_\alpha \nu(y) = \sum_{j=-\infty}^{\infty} \varphi_j(y)$.

Step 1. We first claim that there is a non-decreasing sequence $\{a_j\}$ such that $a_j \geq 1$, $a_j \to \infty$ as $j \to \infty$ and $\sum_{j=-\infty}^{\infty} a_j\varphi_j \in L^{p'}(\mu)$. In fact, by the Lebesgue convergence theorem we have $\lim_{k\to\infty} ||\sum_{j=k}^{\infty}\varphi_j||_{L^{p'}(\mu)} = 0$, and hence there is an increasing sequence $\{k_i\} \subset \mathbf{N}$ such that $||\sum_{j=k_i}^{\infty}\varphi_j||_{L^{p'}(\mu)} < 2^{-i}$. Now, set $a_j = i$ for $k_i \leq j < k_{i+1}$ and $a_j = 1$ for $j < k_1$. Then we have

$$\begin{aligned}
||\sum_{j=-\infty}^{\infty} a_j\varphi_j||_{L^{p'}(\mu)} &\leq ||\sum_{j<k_1} a_j\varphi_j||_{L^{p'}(\mu)} + ||\sum_{i=1}^{\infty}\sum_{j=k_i}^{k_{i+1}-1} a_j\varphi_j||_{L^{p'}(\mu)} \\
&\leq ||\sum_{j<k_1}\varphi_j||_{L^{p'}(\mu)} + \sum_{i=1}^{\infty} i||\sum_{j=k_i}^{\infty}\varphi_j||_{L^{p'}(\mu)} \\
&\leq ||K_\alpha\nu||_{L^{p'}(\mu)} + \sum_{i=1}^{\infty} i2^{-i} < \infty.
\end{aligned}$$

Step 2. For $t \in \mathbf{R}^+$ we put $k_0(t) = a_j t^{\alpha-\beta}$ for $2^{-j} \leq t < 2^{-j+1}$ and $k_0(t) = t^{\alpha-\beta}$ for $t \geq 1$. Then k_0 is a non-negative, lower semi-continuous, non-increasing function on $\mathbf{R}^+$. And for each $y \in X$ we have

$$\begin{aligned}
K_0\nu(y) &= \sum_{j=1}^{\infty}\int_{2^{-j}\leq d(x,y)<2^{-j+1}} a_j d(x,y)^{\alpha-\beta}d\nu(x) + \sum_{j<1}\int_{2^{-j}\leq d(x,y)<2^{-j+1}} d(x,y)^{\alpha-\beta}d\nu(x) \\
&= \sum_{j=1}^{\infty} a_j\varphi_j(y) + \sum_{j<1}\varphi_j(y) = \sum_{j=-\infty}^{\infty} a_j\varphi_j(y).
\end{aligned}$$

Hence $K_0\nu \in L^{p'}(\mu)$.

Step 3. We show that (5.1) holds for K_0. Let $x_0 \in X$, $0 < r < \min\{\frac{1}{2}, R\}$ and η be a measure in $\mathcal{M}^+(\overline{B(x_0,r)})$ such that $||K_0\eta||_{L^{p'}(\mu)} \leq 1$. By Hölder's inequality we have

$$\int_{B(x_0,r)} K_0\eta d\mu \leq \mu(B(x_0,r))^{1/p}||K_0\eta||_{L^{p'}(\mu)}.$$

If $x \in B(x_0,r)$ and $y \in \overline{B(x_0,r)}$, then $d(x,y) < 2r$. Hence

$$\begin{aligned}
1 &\geq ||K_0\eta||_{L^{p'}(\mu)} \geq \mu(B(x_0,r))^{-1/p}\int_{B(x_0,r)} K_0\eta d\mu \\
&= \mu(B(x_0,r))^{-1/p}\int_{B(x_0,r)}\left(\int_{\overline{B(x_0,r)}} k_0(d(x,y))d\eta(x)\right)d\mu(y) \\
&\geq \mu(B(x_0,r))^{-1/p}k_0(2r)\eta(\overline{B(x_0,r)})\mu(B(x_0,r)) \\
&\geq (b_1r^\beta)^{1-1/p}k_0(2r)\eta(\overline{B(x_0,r)}),
\end{aligned}$$

whence

$$\eta(\overline{B(x_0,r)}) \leq b_1^{(1-p)/p}r^{(\beta-\beta p)/p}k_0(2r)^{-1}.$$

Thus, by taking the supremum over such measures η and Proposition 3.2, we have

$$C_{K_0,p,\mu}(\overline{B(x_0,r)})^{1/p} \leq b_1^{(1-p)/p}r^{(\beta-\beta p)/p}k_0(2r)^{-1}.$$

Since $0 < 2r < 1$, we can find a natural number j_r such that $2^{-j_r} \leq 2r < 2^{-j_r+1}$. Hence, by Theorem 1, we have

$$C_{K_0,p,\mu}(B(x_0,r)) \leq b_1^{1-p}a_{j_r}^{-p}2^{(\beta-\alpha)p}CC_{\alpha,p,\mu}(B(x_0,r)),$$

which leads to (5.1). □

The Hausdorff measure has the following property (cf. [4]).

Lemma H . *For every set E in X there is a Borel set G of X such that $E \subset G$ and $\Lambda_h(E) = \Lambda_h(G)$.*

Proof of Theorem 3. We first show (1.3). We may assume that $\Lambda_h^\delta(E) < \infty$. Let $\varepsilon > 0$ be arbitrary and let $\{B(x_j, r_j)\}$ be a countable covering of the set E such that $0 < r_j \leq \delta$ and

$$\sum_j h(r_j) < \Lambda_h^\delta(E) + \varepsilon.$$

If $\alpha p < \beta$, then it follows from Theorem 1 that

$$C_{\alpha,p,\mu}(E) \leq \sum_j C_{\alpha,p,\mu}(B(x_j, r_j)) \leq \sum_j C r_j^{\beta-\alpha p} \leq C(\Lambda_h^\delta(E) + \varepsilon).$$

If $\alpha p = \beta$, then, by Theorem 2,

$$C_{\alpha,p,\mu}(E) \leq \sum_j C_{\alpha,p,\mu}(B(x_j, r_j)) \leq \sum_j C(\log \frac{R}{2r_j})^{1-p} < C(\Lambda_h^\delta(E) + \varepsilon).$$

Thus the inequality (1.3) holds.

We next prove the second assertion. To prove it, we show that $C_{\alpha,p,\mu}(E) > 0$ implies $\Lambda_h(E) = \infty$. If $C_{\alpha,p,\mu}(E) = \infty$, then by (1.3), $\Lambda_h(E) \geq \Lambda_h^\delta(E) = \infty$. So we may assume that $0 < C_{\alpha,p,\mu}(E) < \infty$.

Let E be a compact set. By Proposition 3.3 there is a measure $\nu_E \in \mathcal{M}^+(E)$ such that

$$\nu_E(E) = \int_X (K_\alpha \nu_E)^{p'} d\mu = C_{\alpha,p,\mu}(E).$$

Since $K_\alpha \nu_E \in L^{p'}(\mu)$, by Lemma 5.1 there exists a non-negative, lower semi-continuous, non-increasing function k_0 on $\mathbf{R}^+$ such that $K_0 \nu_E \in L^{p'}(\mu)$ for the carnel $K_0(x,y) = k_0(d(x,y))$ and

$$\lim_{r\to 0}(\sup_{x\in X} \frac{C_{K_0,p,\mu}(B(x,r))}{C_{\alpha,p,\mu}(B(x,r))}) = 0.$$

Then, for $\varepsilon > 0$, there is a number δ_0, $0 < \delta_0 < R$, such that

$$\sup_{x\in X} \frac{C_{K_0,p,\mu}(B(x,r))}{C_{\alpha,p,\mu}(B(x,r))} < \varepsilon \quad \text{for } 0 < r < \delta_0. \tag{5.2}$$

Let $\{B(x_j, r_j)\}$ be a countable covering of the set E satisfying $0 < r_j \leq \delta_0$. Then, by Theorem 1 and (5.2) we have

$$\begin{aligned} C_{K_0,p,\mu}(E) &\leq \sum_j C_{K_0,p,\mu}(B(x_j, r_j)) \\ &\leq \sup_j \frac{C_{K_0,p,\mu}(B(x_j,r_j))}{C_{\alpha,p,\mu}(B(x_j,r_j))} (\sum_j C_{\alpha,p,\mu}(B(x_j, r_j))) \\ &\leq C\varepsilon \sum_j h(r_j). \end{aligned}$$

Therefore

$$\Lambda_h(E) \geq \Lambda_h^{\delta_0}(E) \geq \frac{1}{C\varepsilon} C_{K_0,p,\mu}(E).$$

Since $\frac{\nu_E}{||K_0\nu_E||_{L^{p'}(\mu)}} \in \mathcal{M}^+(E)$ and $||K_0(\frac{\nu_E}{||K_0\nu_E||_{L^{p'}(\mu)}})||_{L^{p'}(\mu)} \leq 1$, by Proposition 3.2,

$$C_{K_0,p,\mu}(E)^{\frac{1}{p}} \geq \frac{\nu_E(E)}{||K_0\nu_E||_{L^{p'}(\mu)}} > 0.$$

Thus we see that $\Lambda_h(E) = \infty$ as $\varepsilon \to 0$.

To prove the second assertion of Theorem 3 for a general set E, it is enough to prove it when E is a Borel set. Because, for a general set E, by Lemma H there is a Borel set G such that $E \subset G$, $\Lambda_h(E) = \Lambda(G)$ and $C_{\alpha,p,\mu}(G) \geq C_{\alpha,p,\mu}(E) > 0$. If E is a Borel set, then, by Theorem G, we have

$$C_{\alpha,p,\mu}(E) = \sup\{C_{\alpha,p,\mu}(F) : F \subset E,\ F \text{ compact}\}.$$

Since $C_{\alpha,p,\mu}(E) > 0$, there is a compact set $F \subset E$ such that $C_{\alpha,p,\mu}(F) > 0$. Thus

$$\Lambda_h(E) \geq \Lambda_h(F) = \infty$$

which proved the second assertion. □

By Lemma 4.1, (i) and (1.1), we see that the following lemma holds.

Lemma 5.2. *Let $\nu \in \mathcal{M}^+(X)$ and $0 < \lambda < \beta$. Then there exists a constant $C > 0$ such that*

$$\int_{B(x,r)} d(x,y)^{\lambda-\beta} d\nu(y) \leq C r^{\lambda} M\nu(x) \quad \text{for } 0 < r \leq R \text{ and } x \in X.$$

Proposition 5.3. *Let $\nu \in \mathcal{M}^+(X)$ and $0 < \alpha < \beta$. For any $\lambda > 0$ there exists a constant $C > 0$ such that*

$$\mu(\{x : K_\alpha\nu(x) > \lambda\}) \leq C(\frac{1}{\lambda}\int_X d\nu)^{\frac{\beta}{\beta-\alpha}}.$$

Proof. We note that

$$\int_{d(x,y)\geq r} d(x,y)^{\alpha-\beta} d\nu(y) \leq r^{\alpha-\beta}\nu(X).$$

We claim that

$$K_\alpha\nu(x) \leq (C+1)\nu(X)^{\frac{\alpha}{\beta}} M\nu(x)^{\frac{\beta-\alpha}{\beta}}. \tag{5.3}$$

Indeed, if $M\nu(x) = +\infty$, then (5.3) holds. So we assume that $M\nu(x) < \infty$. Put $r = (\frac{\nu(X)}{M\nu(x)})^{1/\beta}$. The above inequality and Lemma 5.2 yield

$$\begin{aligned} K_\alpha\nu(x) &= \int_{d(x,y)<r} d(x,y)^{\alpha-\beta} d\nu(y) + \int_{d(x,y)\geq r} d(x,y)^{\alpha-\beta} d\nu(y) \\ &\leq C(\frac{\nu(X)}{M\nu(x)})^{\alpha/\beta} M\nu(x) + (\frac{\nu(X)}{M\nu(x)})^{(\alpha-\beta)/\beta}\nu(X) \\ &= (C+1)\nu(X)^{\alpha/\beta} M\nu(x)^{(\beta-\alpha)/\beta}. \end{aligned}$$

Thus we see that (5.3) holds. Using Proposition 2.2, (iii), we have

$$\begin{aligned} \mu(\{x : K_\alpha\nu(x) > \lambda\}) &\leq \mu(\{x : (C+1)\nu(X)^{\alpha/\beta} M\nu(x)^{(\beta-\alpha)/\beta} > \lambda\}) \\ &= \mu(\{x : M\nu(x) > (\frac{\lambda}{(C+1)\nu(X)^{\alpha/\beta}})^{\beta/(\beta-\alpha)}\}) \\ &\leq C'(\frac{(C+1)\nu(X)^{\alpha/\beta}}{\lambda})^{\beta/(\beta-\alpha)} \int_X d\nu = C'(C+1)^{\beta/(\beta-\alpha)}(\frac{1}{\lambda}\int_X d\nu)^{\beta/(\beta-\alpha)}, \end{aligned}$$

which shows the proposition. □

We prove the following fundamental property by using the covering lemma of Whitney type (cf. [3, Theorem (1.3) on p.70]).

Proposition 5.4. *Let $\nu \in \mathcal{M}^+(X)$. Then there exist constants $a > 1$ and $b > 0$ such that*

$$\mu(\{x : K_\alpha\nu(x) > a\lambda\}) \leq b\varepsilon^{\frac{\beta}{\beta-\alpha}}\mu(\{x : K_\alpha\nu(x) > \lambda\}) + \mu(\{x : M_\alpha\nu(x) > \varepsilon\lambda\})$$

for every $\lambda > 0$ and $0 < \varepsilon \leq 1$.

Proof. By the lower semi-continuity of $K_\alpha\nu$, the set $\{x : K_\alpha\nu(x) > \lambda\}$ is open. Then, by the covering lemma in [3, Theorem (1.3) on p.70] there exists a countable family of balls $\{B_j\}_{j\in\mathbf{N}}$, $B_j = B(x_j, r_j)$ with the following properties;

(i) $\{x : K_\alpha\nu(x) > \lambda\} = \cup_{j\in\mathbf{N}} B_j$,

(ii) $\sum_{j\in\mathbf{N}} 1_{B_j} \leq h_1$ for some $h_1 \in \mathbf{N}$,

(iii) $B(x_j, h_2 r_j) \cap \{x : K_\alpha\nu(x) \leq \lambda\} \neq \emptyset$, for some $h_2 > 2$ and every $j \in \mathbf{N}$.

Let $a > 1$ and $0 < \varepsilon \leq 1$. We decompose $\{B_j\}$ into Λ_1 and Λ_2. If B_j intersects the set $\{x \in B_j : M_\alpha\nu(x) \leq \varepsilon\lambda\}$, then $B_j \in \Lambda_1$, otherwise $B \in \Lambda_2$.

First, suppose that $B = B(z, r) \in \Lambda_1$ and $x_0 \in B$ such that $M_\alpha\nu(x_0) \leq \varepsilon\lambda$. Denote the restriction of ν to $B(z, h_2 r)$ by ν_1, and set $\nu_2 = \nu - \nu_1$. Let $B(x_0) = B(x_0, 2h_2 r)$. Then we have

$$\int_X d\nu_1 \leq \int_{B(x_0)} d\nu = \mu(B(x_0))^{\frac{\beta-\alpha}{\beta}} M_\alpha\nu(x_0) \leq \varepsilon\lambda\mu(B(x_0))^{\frac{\beta-\alpha}{\beta}},$$

whence

$$\left(\frac{2}{a\lambda}\int_X d\nu_1\right)^{\frac{\beta}{\beta-\alpha}} \leq \left(\frac{2\varepsilon}{a}\right)^{\frac{\beta}{\beta-\alpha}} \mu(B(x_0)).$$

This and Proposition 5.3 yield

$$\mu(\{x : K_\alpha\nu_1(x) > \frac{a\lambda}{2}\}) \leq C\left(\frac{2\varepsilon}{a}\right)^{\frac{\beta}{\beta-\alpha}} \mu(B(x_0)). \tag{5.4}$$

On the other hand, the assertion (iii) shows that there is a point $x_1 \in X$ such that $d(x_1, z) \leq h_2 r$ and $K_\alpha\nu(x_1) \leq \lambda$. If $y \in B(z, h_2 r)^c$ and $x \in B$, then

$$d(x, y) \geq d(z, y) - d(z, x) > h_2 r - r$$

and

$$\begin{aligned} d(y, x_1) &\leq d(x, y) + d(x, x_1) < d(x, y) + r + h_2 r \\ &\leq d(x, y) + (1 + h_2)\frac{d(x, y)}{h_2 - 1} < \frac{2h_2 d(x, y)}{h_2 - 1}, \end{aligned}$$

whence

$$K_\alpha\nu_2(x) \leq \left(\frac{2h_2}{h_2-1}\right)^{\beta-\alpha} \int_X d(y, x_1)^{\alpha-\beta} d\nu_2(y) \leq \left(\frac{2h_2}{h_2-1}\right)^{\beta-\alpha} K_\alpha\nu_2(x_1) \leq \left(\frac{2h_2}{h_2-1}\right)^{\beta-\alpha} \lambda.$$

Thus, if a is choosen so that $a = 2(\frac{2h_2}{h_2-1})^{\beta-\alpha}$, then $K_\alpha\nu_2(x) \leq \frac{a\lambda}{2}$. Hence

$$\{x \in B : K_\alpha\nu(x) > a\lambda\} \subset \{x \in B : K_\alpha\nu_1(x) > \frac{a\lambda}{2}\}.$$

In this case it follows from (5.4) and (1.1) that

$$\mu(\{x \in B : K_\alpha\nu(x) > a\lambda\}) \leq C\left(\frac{2\varepsilon}{a}\right)^{\frac{\beta}{\beta-\alpha}} \mu(B(x_0)) \leq C'\varepsilon^{\frac{\beta}{\beta-\alpha}} \mu(B),$$

where $C' = Ca^{\frac{\beta-\alpha}{\beta}} \frac{b_2}{b_1}(2h_2)^\beta$.

We next suppose that $B \in \Lambda_2$. Then $B \subset \{x : M_\alpha\nu(x) > \varepsilon\lambda\}$. By the assertion (ii) we obtain

$$\begin{aligned} \mu(\{x : K_\alpha\nu(x) > a\lambda\}) &\leq \sum_{B\in\Lambda_1} C'\varepsilon^{\frac{\beta}{\beta-\alpha}} \mu(B) + \mu(\{x : M_\alpha\nu(x) > \varepsilon\lambda\}) \\ &\leq C' h_1 \varepsilon^{\frac{\beta}{\beta-\alpha}} \mu(\{x : K_\alpha\nu(x) > \lambda\}) + \mu(\{x : M_\alpha\nu(x) > \varepsilon\lambda\}). \end{aligned}$$

This is the desired inequality. □

We shall show the equivalence of the L^p-norm of $K_\alpha\nu$ and $M_\alpha\nu$.

Proposition 5.5. *Let $1 < p < \infty$ and $0 < \alpha < \beta$. Then there exists a constant $C > 0$ such that*

$$C^{-1}||M_\alpha\nu||_{L^p(\mu)} \leq ||K_\alpha\nu||_{L^p(\mu)} \leq C||M_\alpha\nu||_{L^p(\mu)} \tag{5.5}$$

for all $\nu \in \mathcal{M}^+(X)$.

Proof. Noting that, for each r satisfying $0 < r \leq R$,

$$K_\alpha\nu(x) \geq r^{\alpha-\beta}\int_{d(x,y)<r} d\nu(y) \geq \frac{b_1^{\frac{\beta-\alpha}{\beta}}}{\mu(B(x,r))^{\frac{\beta-\alpha}{\beta}}}\int_{B(x,r)} d\nu(y),$$

we have

$$||K_\alpha\nu||_{L^p(\mu)} \geq b_1^{\frac{\beta-\alpha}{\beta}}||M_\alpha\nu||_{L^p(\mu)},$$

which is the left-hand inequality of (5.5).

The right-hand inequality of (5.5) follows from Proposition 5.4. In fact, we obtain, for any $t > 0$,

$$\begin{aligned}
&\int_0^t \mu(\{x : K_\alpha\nu(x) > a\lambda\})\lambda^{p-1}d\lambda \\
\leq\ & b\varepsilon^{\frac{\beta}{\beta-\alpha}}\int_0^t \mu(\{x : K_\alpha\nu(x) > \lambda\})\lambda^{p-1}d\lambda + \int_0^t \mu(\{x : M_\alpha\nu(x) > \varepsilon\lambda\})\lambda^{p-1}d\lambda,
\end{aligned}$$

whence

$$\begin{aligned}
&a^{-p}\int_0^{at} \mu(\{x : K_\alpha\nu(x) > \lambda\})\lambda^{p-1}d\lambda \\
\leq\ & b\varepsilon^{\frac{\beta}{\beta-\alpha}}\int_0^t \mu(\{x : K_\alpha\nu(x) > \lambda\})\lambda^{p-1}d\lambda + \varepsilon^{-p}\int_0^{\varepsilon t} \mu(\{x : M_\alpha\nu(x) > \lambda\})\lambda^{p-1}d\lambda.
\end{aligned}$$

If ε is chosen so small that $b\varepsilon^{\frac{\beta}{\beta-\alpha}} \leq \frac{1}{2}a^{-p}$, then

$$\begin{aligned}
&a^{-p}\int_0^{at} \mu(\{x : K_\alpha\nu(x) > \lambda\})\lambda^{p-1}d\lambda \\
\leq\ & \frac{1}{2}a^{-p}\int_0^t \mu(\{x : K_\alpha\nu(x) > \lambda\})\lambda^{p-1}d\lambda + \varepsilon^{-p}\int_0^{\varepsilon t} \mu(\{x : M_\alpha\nu(x) > \lambda\})\lambda^{p-1}d\lambda.
\end{aligned}$$

Since these integrals are finite, we have

$$a^{-p}\int_0^{at} \mu(\{x : K_\alpha\nu(x) > \lambda\})\lambda^{p-1}d\lambda \leq 2\varepsilon^{-p}\int_0^{\varepsilon t} \mu(\{x : M_\alpha\nu(x) > \lambda\})\lambda^{p-1}d\lambda,$$

whence $||K_\alpha\nu||_{L^p(\mu)} \leq 2^{1/p}\frac{a}{\varepsilon}||M_\alpha\nu||_{L^p(\mu)}$. □

We shall now look at Proposition 5.5 from slightly different point of view. Let $n \in \mathbf{Z}$ and E be a subset of X. We will use the notation $\operatorname{diam} E$ that stands for the diameter of E. Further we denote by $B_n(x)$ the open ball of radius 2^{-n} centered at x. We let n_0 be the smallest integer such that $diam\, X \leq 2^{-n_0}$. We recall that $L^p(l^q)$-norm for a sequence $\{f_n\}$ in l^q is defined by

$$||\{f_n\}||^p_{L^p(l^q)} = \int\left(\sum_n |f_n(x)|^q\right)^{\frac{p}{q}} d\mu(x).$$

Then we see that Proposition 5.5 gives the following corollary.

Corollary 5.6. *Let $0 < p < \infty$, $1 < q < \infty$ and $0 < \alpha < \beta$. Then there are constants $C, C', C'' > 0$ such that for all $\nu \in \mathcal{M}^+(X)$*

$$\begin{aligned}
||K_\alpha\nu||_{L^p(\mu)} &\leq C||M_\alpha\nu||_{L^p(\mu)} \leq C'||\{2^{-n(\alpha-\beta)}\nu(B_n(x))\}_{n_0}^\infty||_{L^p(l^\infty)} \\
&\leq C'||\{2^{-n(\alpha-\beta)}\nu(B_n(x))\}_{n_0}^\infty||_{L^p(l^q)} \leq C'||\{2^{-n(\alpha-\beta)}\nu(B_n(x))\}_{n_0}^\infty||_{L^p(l^1)} \\
&\leq C''||K_\alpha\nu||_{L^p(\mu)}.
\end{aligned}$$

Proof. For each $x \in X$ we have

$$\begin{aligned} d(x,y)^{\alpha-\beta} &\leq \sum_{n=n_0}^{\infty} 2^{-(n+1)(\alpha-\beta)}(1_{B_n(x)}(y) - 1_{B_{n+1}(x)}(y)) \\ &\leq 2^{\beta-\alpha} \sum_{n=n_0}^{\infty} 2^{-n(\alpha-\beta)} 1_{B_n(x)}(y), \end{aligned}$$

whence

$$K_\alpha \nu(x) \leq 2^{\beta-\alpha} \sum_{n=n_0}^{\infty} 2^{-n(\alpha-\beta)} \nu(B_n(x)).$$

Similarly we have

$$K_\alpha \nu(x) \geq (1 - 2^{\alpha-\beta}) \sum_{n=n_0}^{\infty} 2^{-n(\alpha-\beta)} \nu(B_n(x)).$$

In other words,

$$2^{\alpha-\beta} K_\alpha \nu(x) \leq ||\{2^{-n(\alpha-\beta)}\nu(B_n(x))\}_{n_0}^{\infty}||_{l^1} \leq \frac{1}{1-2^{\alpha-\beta}} K_\alpha \nu(x). \tag{5.6}$$

Next, for $0 < r \leq R$ we choose an integer n_r satisfying $2^{-n_r} < r \leq 2^{-n_r+1}$. Then

$$\begin{aligned} \frac{1}{\mu(B(x,r))^{\frac{\beta-\alpha}{\beta}}} \int_{B(x,r)} d\nu(y) &\leq b_1^{\frac{\alpha-\beta}{\beta}} 2^{-n_r(\alpha-\beta)} \nu(B_{n_r-1}(x)) \\ &\leq b_1^{\frac{\alpha-\beta}{\beta}} ||\{2^{-n(\alpha-\beta)}\nu(B_n(x))\}_{n_0}^{\infty}||_{l^\infty}. \end{aligned}$$

Hence

$$M_\alpha \nu(x) \leq b_1^{\frac{\alpha-\beta}{\beta}} ||\{2^{-n(\alpha-\beta)}\nu(B_n(x))\}_{n_0}^{\infty}||_{l^\infty}. \tag{5.7}$$

We recall that for each sequence $\{a_n\}$ and $1 < q < \infty$ we have

$$||\{a_n\}||_{l^\infty} \leq ||\{a_n\}||_{l^q} \leq ||\{a_n\}||_{l^1} \tag{5.8}$$

Thus, by (5.6), (5.7) and (5.8) we obtain the desired estimates. □

The following lemma is the result of Frostman type in a compact metric space. P. Mattila proved it by using the Hahn-Banach extension theorem (cf. [6, 8.17 Theorem]).

Lemma I . *Let $0 < \delta \leq \infty$ and $0 \leq s < \infty$. There is a positive Radon measure ν on X such that $\nu(X) = \lambda_\delta^s(X)$ and*

$$\nu(E) \leq (\operatorname{diam} E)^s \quad \textit{for all } E \subset X \textit{ with } \operatorname{diam} E < \delta. \tag{5.9}$$

In particular, if $\mathcal{H}(X) > 0$, then there exist $\delta > 0$ and ν satisfying (5.9) *and* $\nu(X) > 0$. *Here*

$$\lambda_\delta^s(X) = \inf\{\sum_j c_j(\operatorname{diam} E_j)^s \ : \ \sum_j c_j 1_{E_j} \geq 1 \textit{ on } X,\ c_j > 0,\ (\operatorname{diam} E_j) \leq \delta\}.$$

Remark 5.1. *Let k, $\delta > 0$. We also define, for $k > 0$ and $\delta > 0$,*

$$\tilde{\mathcal{H}}_\delta^k(E) = \inf\{\sum_j (\operatorname{diam} E_j)^k \ : \ \operatorname{diam} E_j < \delta,\ E \subset \cup_j E_j\}.$$

Then we see that $\tilde{\mathcal{H}}_\delta^k$ is comparable to $\mathcal{H}_\delta^k$.

We next define Wolff's potentials.

Definition 5.1. *Let* $1 < p < \infty$, $p' = \frac{p}{p-1}$, $0 < \alpha p \leq \beta$ *and* $\nu \in \mathcal{M}^+(X)$. *Wolff's potential* $W^\nu_{\alpha,p}$ *is defined as follows:*

$$W^\nu_{\alpha,p}(x) = \sum_{n=n_0}^{\infty} (2^{n(\beta-\alpha p)}\nu(B_n(x)))^{p'-1}. \tag{5.10}$$

Remark 5.2. *Wolff's potential* $W^\nu_{\alpha,p}(x)$ *is comparable to*

$$\int_0^{2^{-n_0}} (\frac{\nu(B(x,t))}{t^{\beta-\alpha p}})^{p'-1}\frac{dt}{t}. \tag{5.11}$$

We can use (5.11) instead of (5.10).

Proposition 5.7. *Let* $1 < p < \infty$ *and* $0 < \alpha p \leq \beta$. *Then there is a constant* A *such that*

$$\int_X K_\alpha\nu(x)^{p'}d\mu(x) \leq A\int_X W^\nu_{\alpha,p}(x)d\nu(x) \quad for\, each\, \nu \in \mathcal{M}^+(X).$$

Proof. Corollary 5.6 yields

$$\begin{aligned}
\int_X K_\alpha\nu(x)^{p'}d\mu(x) &\leq C\|\{2^{-n(\alpha-\beta)}\nu(B_n(x))\}_{n_0}^{\infty}\|^{p'}_{L^{p'}(l^{p'})} \\
&= C\int_X\sum_{n=n_0}^{\infty}(2^{-n(\alpha-\beta)}\nu(B_n(x)))^{p'}d\mu(x) \\
&= C\sum_{n=n_0}^{\infty}2^{-n(\alpha-\beta)p'}\int_X\left(\int_{B_n(x)}\nu(B_n(x))^{p'-1}d\nu(y)\right)d\mu(x) \\
&= C\sum_{n=n_0}^{\infty}2^{-n(\alpha-\beta)p'}\int_X\left(\int_{B_n(y)}\nu(B_n(x))^{p'-1}d\mu(x)\right)d\nu(y)
\end{aligned}$$

Noting that $x \in B_n(y)$ implies $B_n(x) \subset B_{n-1}(y)$, we have, by (5.10),

$$\begin{aligned}
\int_X K_\alpha\nu(x)^{p'}d\mu(x) &\leq C\sum_{n=n_0}^{\infty}2^{-n(\alpha-\beta)p'}\int_X\nu(B_{n-1}(y))^{p'-1}\mu(B_n(y))d\nu(y) \\
&\leq Cb_2\int_X\sum_{n=n_0}^{\infty}2^{-n(\alpha-\beta)p'}2^{-n\beta}\nu(B_{n-1}(y))^{p'-1}d\nu(y) \\
&\leq C'b_2\int_X W^\nu_{\alpha,p}(x)d\nu(x).
\end{aligned}$$

□

Proof of Theorem 4. Put $k = \beta - \alpha p + \varepsilon$. Let $E \subset X$ be a compact set. Assume that $\mathcal{H}^k(E) > 0$. Then, by Lemma I there is a measure ν in $\mathcal{M}^+(E)$ such that $\nu(B(x,r)) \leq (2r)^k$ for all balls $B(x,r)$ and $\nu(E) > 0$. By Proposition 5.7 we have

$$\int_X(K_\alpha\nu(x))^{p'}d\mu(x) \leq A_1\int_X W^\nu_{\alpha,p}(x)d\nu(x),$$

where A_1 is a constant independent of ν. From Remark 5.2 and $k = \beta - \alpha p + \varepsilon$ it follows that

$$W^\nu_{\alpha,p}(x) \leq A_2\int_0^{2^{-n_0}}(\frac{\nu(B(x,t))}{t^{\beta-\alpha p}})^{p'-1}\frac{dt}{t} \leq A_3\int_0^{2^{-n_0}}(\frac{t^k}{t^{\beta-\alpha p}})^{p'-1}\frac{dt}{t} = A_4.$$

Thus

$$\|K_\alpha\nu\|_{L^{p'}} \leq \left(A_1\int_X A_4 d\nu(x)\right)^{\frac{1}{p'}} = A_5\nu(E)^{\frac{1}{p'}}.$$

Using Proposition 3.2, we have

$$C_{\alpha,p,\mu}(E)^{\frac{1}{p}} \geq \frac{\nu(E)}{A_5\nu(E)^{\frac{1}{p'}}} = A_5^{-1}\nu(E)^{\frac{1}{p}}.$$

Therefore we have $C_{\alpha,p,\mu}(E) > 0$. Thus we have the conclusion. □

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