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Estimates of the α-Riesz potentials in metric spaces

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Abstract. We define an α-Riesz potential operator on $L^p(X,\mu)$ in a quasi-metric space X with a doubling measure μ, satisfying a certain lower estimate of the measure of a ball. For this operator we give estimates of weak type.

1. Introduction

Let Ω be a domain in $\mathbf{R}^n$. By means of distribution functions, sharper results than the Sobolev inequality have been obtained by several mathmaticians, O'Neil [6], Peetre [7], Brézis and Wainger [1], Hansson [3] and Maz'ya [5]. The following result is one of them, which follows from results of O'Neil [6] and Peetre [7].

Theorem A. *Let $1 < p < n$ and u be a function in $C_0^\infty(\Omega)$, with $||\nabla u||_{L^p(\Omega)} \leq 1$. Then there exists $C > 0$ such that*

$$\int_0^\infty t^{p-1}|\{|u| > t\}|^{1-p/n}\, dt \leq C, \tag{1.1}$$

where C is a constant independent of u and $|A|$ stands for the n-dimensional Lebesgue measure of a set A.

As the limiting case of Theorem A the following result is also obtained by Brézis and Wainger [1], Hansson [3] and Maz'ya [5].

Theorem B. *Let Ω be a bounded domain in $\mathbf{R}^n$ and u be a function in $C_0^\infty(\Omega)$, with $||\nabla u||_{L^n(\Omega)} \leq 1$. Then*

$$\int_0^\infty \frac{t^{n-1}}{\log^{n-1}(2|\Omega|/|\{|u| > t\}|)}\, dt \leq C, \tag{1.2}$$

where C is a constant independent of u.

Inequalities (1.1) and (1.2) are regarded as estimates in the Sobolev space $W^{1,p}(\Omega)$. On the other hand it is well-known that, if m is a non-negative integer, then $W^{m,p}(\mathbf{R}^n)$ is identical with the Bessel potential space $L^{m,p}$ as a Banach space. Here

$$L^{m,p} = \{G_m * f \ : \ f \in L^p(\mathbf{R}^n)\} \tag{1.3}$$

for the Bessel function G_m of order m and the norm $||G_m * f||_{m,p}$ is defined to be the L^p-norm $||f||_p$. But even if m is not integer, G_m is defined. So, for a non-negative α, $L^{\alpha,p}$ is regarded as a Sobolev space with a fractional order in the case $\Omega = \mathbf{R}^n$. We note that the Bessel function with order α is comparable to the α-Riesz function $|x|^{\alpha-n}$ in a fixed ball.

In 2002 J. Malý and L. Pick [4] considered a quasi-metric space X with $\operatorname{diam} X = R/2$ and a positive Radon measure μ on X with $\mu(X) < \infty$ (See §2). Here $\operatorname{diam} X$ stands for the diameter of X. Further they assumed that μ satisfies the following two conditions;

(μ1) The doubling condition: there exists a positive constant D such that for every $x \in X$ and $r \in (0, \frac{R}{2}]$

$$\mu(B(x,2r)) \leq D\mu(B(x,r)).$$

($\mu 2$) The lower estimate for the measure of a ball: there exists a constant $\gamma > 0$ and a real number $\beta > 1$ such that for every $x \in X$ and $r \in (0, R]$

$$\mu(B(x,r)) \geq \gamma r^{\beta}.$$

In this X, they defined a general Riesz potential $I_1 g$ with order 1 in X by

$$(I_1 g)(x) = \int_0^R \left(\fint_{B(x,t)} g(y)\, d\mu(y) \right) dt, \tag{1.4}$$

where the notation $\fint$ is the integral average of g on a set E with $0 < \mu(E) < \infty$, i.e.,

$$\fint_E g d\mu = \frac{1}{\mu(E)} \int_E g d\mu.$$

and proved the following theorem.

Theorem C. *Let g be a non-negative μ-integrable function. Put*

$$G_{g,t} = \{y \in X \ : \ (I_1 g)(y) > t\}.$$

(i) *Let $1 < p < \beta$. Then there is a constant $C > 0$ such that for every non-negative function $g \in L^p(X,\mu)$ with $||g||_p \leq 1$ we have*

$$\int_0^\infty t^{p-1} \mu(G_{g,t})^{1-p/\beta}\, dt \leq C.$$

(ii) *There exists a constant $C > 0$ such that for every non-negative function $g \in L^\beta(X,\mu)$ with $||g||_\beta \leq 1$ we have*

$$\int_0^\infty t^{\beta-1} \left(\log \frac{2\mu(X)}{\mu(G_{g,t})} \right)^{1-\beta} dt \leq C.$$

In this paper we consider a quasi-metric space (X, ρ) and a positive Radon measure μ on X such that $\mu(X) < \infty$ and μ satisfies ($\mu 1$) and ($\mu 2$). Further for a non-negative μ-integrable function g on X we define a generalized α-Riesz potential operator I_α by

$$(I_\alpha g)(x) = \int_0^R \alpha t^{\alpha-1} \left(\fint_{B(x,t)} g(y)\, d\mu(y) \right) dt \qquad (0 < \alpha < \beta). \tag{1.5}$$

If there exist a real number $\beta > 1$ and constants γ, γ' such that

$$\gamma r^\beta \leq \mu(B(x,r)) \leq \gamma' r^\beta,$$

then the generalized α-Riesz potential $I_\alpha g$ is comparable to the function $x \to \int_X \rho(x,y)^{\alpha-\beta} g(y) d\mu(y)$ (see, Lemma 2.1). To give the estimates of $I_\alpha g$ corresponding to Theorem C, set

$$G_{\alpha,g,t} = G_t = \{y \in X \ : \ (I_\alpha g)(y) > t\} \tag{1.6}$$

for $t > 0$. In §7 we shall prove the followng theorem which extends Theorem C.

Theorem 1. *Let $1 < p < \infty$ and $\alpha > 0$.*

(i) *If $\alpha p < \beta$, then there exists a constant $C > 0$, independent of g, such that for every non-negative function $g \in L^p(X,\mu)$ we have*

$$\int_0^\infty t^{p-1} \mu(G_t)^{1-\alpha p/\beta}\, dt \leq C ||g||_p^p.$$

(ii) *If $\alpha p = \beta$, then there exists a constant $C > 0$, independent of g, such that for every non-negative function $g \in L^p(X, \mu)$ we have*

$$\int_0^\infty t^{p-1} \left(\log \left(\frac{2\mu(X)}{\mu(G_t)} \right) \right)^{1-p} dt \le C||g||_p^p.$$

We also consider the case where $\alpha p > \beta$ and we obtain the following consequence which will be proved in §7.

Theorem 2. *Let $1 < p < \infty$, $\alpha > 0$ and $\alpha p > \beta$. Then there exists a constant $C > 0$, independent of g, such that for every non-negative function $g \in L^p(X, \mu)$ we have*

$$||I_\alpha g||_\infty \le C||g||_p.$$

Remark 1. We see that C depends only on p, β, α, γ and d. Here γ is the constant in (μ2) and d is the one in the definition of a quasi-metric space in §2.

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2. Preliminaries

Recall that (X, ρ) is a quasi-metric space if ρ is a non-negative function on $X \times X$ with the following properties:

(i) $\rho(x, y) = 0$ if and only if $x = y$,

(ii) $\rho(x, y) = \rho(y, x)$ for all x and $y \in X$,

(iii) there is a constant $d \ge 1$ such that $\rho(x, y) \le d\{\rho(x, z) + \rho(z, y)\}$ for all x, y and $z \in X$.

We shall assume that X is bounded and diam $X = R/2$ for some $R \in (0, \infty)$. Denote by $B(x_0, r)$ the open ball centered at x_0 of radius r, i.e.,

$$B(x_0, r) = \{x \in X \ : \ \rho(x, x_0) < r\}.$$

We note that balls $\{B(x, r)\}_{r>0}$ form a basis of neighborhoods of x for the topology induced by the quasi-metric ρ. Further denote by $||g||_p$ the L^p-norm of a function g in $L^p(X, \mu)$ and put $p' = p/(p-1)$ for $1 < p < \infty$.

Now we define an operator I_α^r by

$$(I_\alpha^r g)(x) = \int_0^r \alpha t^{\alpha-1} \left(-\!\!\!\!\!\!\int_{B(x,t)} g(y)\, d\mu(y) \right) dt, \quad r \in (0, R], \quad x \in X.$$

Especially if $r = R$, then we write $I_\alpha^R = I_\alpha$. By the Fubini's theorem we see

$$(I_\alpha^r g)(x) = \int_{B(x,r)} g(y) \left(\int_{\rho(x,y)}^r \frac{\alpha t^{\alpha-1}}{\mu(B(x,t))} dt \right) d\mu(y).$$

Therefore by putting

$$I_\alpha^r(x, y) = \int_{\rho(x,y)}^r \frac{\alpha t^{\alpha-1}}{\mu(B(x,t))} dt,$$

we can also write

$$(I_\alpha^r g)(x) = \int_{B(x,r)} I_\alpha^r(x, y) g(y)\, d\mu(y).$$

If $r = R$, then we write $I_\alpha^R(x,y) = I_\alpha(x,y)$.

Furthermore, given an $r \in (0,R)$ and define an operator E_α^r by

$$(E_\alpha^r g)(x) = \int_r^R \alpha t^{\alpha-1}\left(⨍_{B(x,t)} g(y)d\mu(y)\right)dt.$$

Then $I_\alpha = I_\alpha^r + E_\alpha^r$ and by Fubini's theorem we have

$$(E_\alpha^r g)(x) = \int_X g(y)\left(\int_{\max\{r,\rho(x,y)\}}^R \frac{\alpha t^{\alpha-1}}{\mu(B(x,t))}dt\right)d\mu(y) = \int_X E_\alpha^r(x,y)g(y)d\mu(y),$$

where

$$E_\alpha^r(x,y) = \int_{\max\{r,\rho(x,y)\}}^R \frac{\alpha t^{\alpha-1}}{\mu(B(x,t))}dt.$$

We now consider a quasi-metric space no assuming the upper estimate of the measures of a ball. But, in case we assume the upper estimate of the measure of a ball, the following lemma shows that (1.5) is comparable to a usual α-Riesz potential.

Lemma 2.1. *If X is a quasi-metric space and a measure μ also satisfies the upper and lower estimates of the measure of a ball, that is, there exist constants γ, $\gamma' > 0$ and a real number $\beta > 0$ such that for every $x \in X$ and $r \in (0,R]$*

$$\gamma r^\beta \le \mu(B(x,r)) \le \gamma' r^\beta.$$

Furthermore, let $0 < \alpha < \beta$. Then (1.5) is comparable to the function

$$x \to \int_X \rho(x,y)^{\alpha-\beta} g(y)d\mu(y).$$

Proof. Let $x \in X$ and k_0 be the smallest integer k satysfying $X \subset B(x,2^k)$. We have

$$\begin{aligned}\rho(x,y)^{\alpha-\beta} &\le \sum_{k=-\infty}^{k_0} 2^{(\alpha-\beta)k}(1_{B(x,2^{k+1})}(y) - 1_{B(x,2^k)}(y)) \\ &\le 2^{n-\alpha}\sum_{k=-\infty}^{k_0+1} 2^{(\alpha-\beta)k}1_{B(x,2^k)}(y),\end{aligned}$$

whence

$$\begin{aligned}\int_X \rho(x,y)^{\alpha-\beta}g(y)d\mu(y) &\le 2^{\beta-\alpha}\sum_{k=-\infty}^{k_0+1} 2^{(\alpha-\beta)k}\int_{B(x,2^k)} g(y)d\mu(y) \\ &\le 2^{\beta-\alpha}\gamma'\sum_{k=-\infty}^{k_0+1} 2^{k\alpha} ⨍_{B(x,2^k)} g(y)d\mu(y).\end{aligned}$$

In the same way we have

$$\int_X \rho(x,y)^{\alpha-\beta}g(y)d\mu(y) \ge (1-2^{\alpha-\beta})\gamma\sum_{k=-\infty}^{k_0+1} 2^{k\alpha} ⨍_{B(x,2^k)} g(y)d\mu(y).$$

Hence

$$\int_X \rho(x,y)^{\alpha-\beta}g(y)d\mu(y) \approx \int_0^R ⨍_{B(x,t)} g(y)d\mu(y)dt^\alpha = \int_0^R \alpha t^{\alpha-1} ⨍_{B(x,t)} g(y)d\mu(y)dt.$$

□

The following proposition will be crucial to prove Theorem 1. The proposition will be proved in §6. Recall that a maximal function Mg is defined by

$$(Mg)(x) = \sup \fint_B |g(y)|\, d\mu(y), \quad x \in X, \tag{2.1}$$

for a μ-integrable function g, where the supremum is taken over all balls B containing x.

Proposition. *Let $1 < p \le \beta$. Then there exist a constant $C > 0$ and a positive number a, depending only on D and d, such that, for every non-negative function $g \in L^p(X, \mu)$ and $t > b$,*

$$t^p \le C \left\{ q_t^{\alpha p-\beta} + \left(\alpha \int_{q_t}^R s^{(\alpha-\beta)(p'-1)+(\alpha-1)}\, ds \right)^{p-1} \right\} \int_{G_{t/a} \setminus G_{at}} (Mg)(y)^p\, d\mu(y), \tag{2.2}$$

where

$$b = \lambda R^\alpha \mu(X)^{-1/p} ||g||_p \qquad \text{and} \qquad q_t = \left(\frac{\mu(G_t^i)}{\gamma} \right)^{1/\beta}, \tag{2.3}$$

if we put

$$\delta = \frac{\log D}{\log 2}, \quad \lambda = D(2d)^\delta \tag{2.4}$$

and denote the interior of G_t by G_t^i.

3. Statements of elementary properties

In this section we state comparatively elementary properties without proofs but they are proved by the same methods as in [4].

Lemma 3.1. *Let $x \in X$, $p > 1$, and $0 < r < R$. Then there is a constant $C > 0$ depending only on p, β, α, and γ such that*

$$\int_X E_\alpha^r(x,y)^{p'}\, d\mu(y) \le C \int_r^R \alpha s^{(\alpha-\beta)(p'-1)+(\alpha-1)}\, ds.$$

Moreover, if $\alpha p > \beta$, then

$$\int_X I_\alpha^R(x,y)^{p'}\, d\mu(y) \le C \int_0^R \alpha s^{(\alpha-\beta)(p'-1)+(\alpha-1)}\, ds.$$

Remark 2. By the direct calculation we see that $(I_\alpha^r 1)(y) = r^\alpha$ for each $y \in X$ and $0 < r \le R$. In fact,

$$(I_\alpha^r 1)(y) = \int_0^r \alpha t^{\alpha-1} \left(\fint_{B(y,t)} 1\, d\mu(z) \right) dt = \int_0^r \alpha t^{\alpha-1}\, dt = r^\alpha.$$

The doubling condition leads to the following two lemmas. The first one is Lemma 4.1 in [4].

Lemma D. *For all $x \in X$ and $0 < t \le s \le R$ we have*

$$\frac{\mu(B(x,s))}{\mu(B(x,t))} \le D \left(\frac{s}{t} \right)^\delta.$$

The second one is as follows.

Lemma 3.2. *There is an increasing function φ from $[1,\infty)$ to $[1,\infty)$ such that*

$$\int_{r/A}^R \frac{\alpha t^{\alpha-1}}{\mu(B(y,t))}\, dt \le \varphi(A) \int_r^R \frac{\alpha t^{\alpha-1}}{\mu(B(y,t))}\, dt \quad \text{for each } y \in X \text{ and } r < \frac{R}{2},$$

where φ depends only on D and α.

Lemma 3.3. *Let λ be the number defined in* (2.4). *Then*

$$\frac{1}{\lambda}I_\alpha(x,y) \le I_\alpha(y,x) \le \lambda I_\alpha(x,y)$$

for every x and y.

Hereafter let g be a non-negative function in $L^p(X,\mu)$ and G_t be the set defined in (1.6).

Lemma 3.4. *Let $a \ge 2\lambda^2\varphi(3d)$ and x be a point in G_t. Then*

$$\int_{G^c_{t/a}} I_\alpha(x,y)g(y)\,d\mu(y) \le \frac{t}{2}.$$

Lemma 3.5. *Let x, y, $z \in X$. Asuume that $\rho(y,z) \le A\rho(x,y)$ for some $A \ge 1$. Then*

$$I_\alpha(x,y) \le \lambda^2\varphi(A)I_\alpha(z,y).$$

Lemma 3.6. *Let x be a point in G_t. Suppose that $B = B(z,r) \subset G_t$ for some $z \in X$ and $r > 0$. Further let $y, \xi \in B$. Then*

$$E^{q_t}_\alpha(x,y) \le \varphi(d+2d^2)E^{q_t}_\alpha(x,\xi),$$

where q_t is the number in (2.3).

4. Properties of the set G_t

We see by the following lemma that the complement G^c_t of G_t is nonempty for a sufficiently large t.

Lemma 4.1. *Suppose that $t \ge 2b$ for b in* (2.3). *If $g \ge 0$ on X. Then*

$$\mu(G_t) \le \frac{1}{2}\mu(X).$$

Proof. By the definition of G_t we have

$$\mu(G_t) \le \frac{1}{t}\int_{G_t}(I_\alpha g)(\xi)\,d\mu(\xi) \le \frac{1}{t}\int_{G_t}\int_X I_\alpha(\xi,y)g(y)\,d\mu(y)\,d\mu(\xi)$$

whence, by Lemma 3.3, Remark 2 and (2.3),

$$\begin{aligned}\mu(G_t) &\le \frac{\lambda}{t}\int_X g(y)\left(\int_{G_t} I_\alpha(y,\xi)\,d\mu(\xi)\right)d\mu(y)\\ &\le \frac{\lambda}{t}\int_X g(y)(I_\alpha 1)(y)\,d\mu(y)\\ &= \frac{\lambda}{t}R^\alpha\int_X g(y)\,d\mu(y) \;\le\; \frac{\lambda}{t}R^\alpha\|g\|_p\mu(X)^{1-\frac{1}{p}} \;\le\; \frac{1}{2}\mu(X).\end{aligned}$$

□

Lemma 4.2. *There exists a constant $k \ge 1$ depending only on μ, α and p such that, if g is a non-negative function in $L^p(X,\mu)$, then*

$$k\liminf_{z\to x}(I_\alpha g)(z) \ge (I_\alpha g)(x) \quad \text{for } \mu-\text{almost everywhere } x \in X. \tag{4.1}$$

Moreover, if $t > 0$, then $G_{kt} \subset G^i_t$.

PROOF. Let g be a non-negative function in $L^p(X,\mu)$ and let y, $z \in X$. Since $\mu(\{x \in X : (I_\alpha g)(x) = \infty\}) = 0$ by [2, Theorem 2.1 on p.71], we may suppose that $(I_\alpha g)(x) < \infty$. By the monotone convergence theorem

$$(I_\alpha g)(x) = \lim_{z\to x} \int_{\rho(x,z)\le\rho(x,y)} I_\alpha(x,y)g(y)d\mu(y).$$

Note that

$$\{y \in X : \rho(x,z) \le \rho(x,y)\} \subset \{y \in X : \rho(y,z) \le 2d\rho(x,y)\}$$

and hence

$$I_\alpha(x,y) \le \lambda^2\varphi(2d)I_\alpha(z,y)$$

by Lemma 3.5. Therefore

$$(I_\alpha g)(x) \le \lambda^2\varphi(2d) \liminf_{z\to x} \int_{\rho(x,z)\le d\rho(x,y)} I_\alpha(z,y)g(y)d\mu(y) \le \lambda^2\varphi(2d) \liminf_{z\to x}(I_\alpha g)(z),$$

which is the first assertion with $k = \lambda^2\varphi(2d)$.

For the second assertion, let $x \in G_{kt}$. Then $kt < (I_\alpha g)(x) \le k \liminf_{z\to x}(I_\alpha g)(z)$, so that $I_\alpha g > t$ on some neighborhood of x. Hence $x \in G_t^i$. Thus $G_{kt} \subset G_t^i$. □

5. A covering of Whitney type and the maximal operator

In this section we fix a non-negative function $g \in L^p(X,\mu)$ and $t > 2b$ for b in (2.3).

In general, the Whitney decomposition is known as the one of an open set into a suitable union of cubes. But, in a quasi-metric space X we use its version of a covering by balls. Since $(I_\alpha g)(x)$ is not always lower semicontinuous, we can't use the covering lemma for G_t. Applying Theorem 1.3 on p.70 in [2] to G_t^i, we have following lemma.

LEMMA E. *There exists a countable system of balls $\{B_j\}_{j\in\mathbf{N}}$, $B_j = B(x_j, r_j)$ with the following properties;*

(i) $G_t^i = \cup_{j\in\mathbf{N}} B_j$,

(ii) $\dfrac{1}{C_1} r_j < \mathrm{dist}(y, (G_t^i)^c) < C_1 r_j$ *for every* $y \in B_j$,

(iii) $\sum_{j\in\mathbf{N}} 1_{B_j} \le C_2$.

Here two constants C_1, $C_2 \ge 1$ depend only on β and d.

LEMMA 5.1. *There is a positive constant C_3, depending only on d, D and α, with the following property : For every $j \in \mathbf{N}$ there exists a point $z_j \in (G_t^i)^c$ such that*

$$\int_{B_j} I_\alpha(\xi,y)\, d\mu(\xi) \le C_3\mu(B_j)I_\alpha(z_j,y) \quad \text{for all} \quad y \in X. \tag{5.1}$$

PROOF. Since $(G_t^i)^c$ is nonempty from Lemma 4.1, for each $j = 1, 2, \cdots$, there is a point z_j in $(G_t^i)^c$ satisfying

$$\rho(x_j, z_j) < C_1 r_j.$$

Case 1. $\rho(y, x_j) \ge 2dr_j$. Then $\rho(y,\xi) \ge r_j$ for every $\xi \in B_j$. Therefore

$$r_j \le \frac{\rho(y,x_j)}{2d} \le \frac{1}{2}(\rho(y,\xi) + \rho(\xi,x_j)) \le \frac{1}{2}(\rho(y,\xi) + r_j)$$

and hence

$$r_j \le \rho(y,\xi).$$

Thus

$$\begin{aligned}\rho(z_j,y) &\le d\,(\rho(z_j,x_j)+d\,(\rho(x_j,\xi)+\rho(\xi,y)))\\ &< d\,(C_1r_j+dr_j+d\rho(\xi,y)) \le C\rho(\xi,y)\end{aligned}$$

with $C=d(C_1+2d)$. Since

$$I_\alpha(\xi,y)\le\lambda^2\varphi(C)I_\alpha(z_j,y)$$

by Lemma 3.5, we obtain (5.1).

Case 2. $\rho(y,x_j)<2dr_j$. Then

$$\rho(z_j,y)\le d\,(\rho(z_j,x_j)+\rho(x_j,y))\le Cr_j \tag{5.2}$$

and, by Lemma 3.2 and Lemma 3.3,

$$\int_{r_j}^{R}\frac{\alpha t^{\alpha-1}}{\mu(B(y,t))}dt\le\int_{\frac{\rho(z_j,y)}{C}}^{R}\frac{\alpha t^{\alpha-1}}{\mu(B(y,t))}dt\le\varphi(C)I_\alpha(y,z_j)\le\lambda\varphi(C)I_\alpha(z_j,y),$$

whence

$$\int_{B_j}\left(\int_{r_j}^{R}\frac{\alpha t^{\alpha-1}}{\mu(B(y,t))}dt\right)d\mu(\xi)\le\lambda\varphi(C)\mu(B_j)I_\alpha(z_j,y). \tag{5.3}$$

Let $r_j<t<2r_j$. Then

$$B(y,t)\subset B(x_j,(2d+2d^2)r_j).$$

Hence, by the Lemma D and putting $C'=D(2d+2d^2)^\delta$, we have

$$\mu(B(y,t))\le\mu(B(x_j,(2d+2d^2)r_j))\le C'\mu(B_j).$$

The inequality (5.2) and Lemma 3.2 yield

$$\begin{aligned}\mu(B_j)I_\alpha(y,z_j) &= \mu(B_j)\int_{\rho(y,z_j)}^{R}\frac{\alpha t^{\alpha-1}}{\mu(B(y,t))}dt \ge \mu(B_j)\int_{Cr_j}^{R}\frac{\alpha t^{\alpha-1}}{\mu(B(y,t))}dt\\ &\ge \frac{\mu(B_j)}{\varphi(C)}\int_{r_j}^{R}\frac{\alpha t^{\alpha-1}}{\mu(B(y,t))}dt \ge \frac{\mu(B_j)}{\varphi(C)}\int_{r_j}^{2r_j}\frac{\alpha t^{\alpha-1}}{\mu(B(y,t))}dt\\ &\ge \frac{\mu(B_j)}{\varphi(C)}\int_{r_j}^{2r_j}\frac{\alpha t^{\alpha-1}}{C'\mu(B_j)}dt = \frac{(2^\alpha-1)r_j^\alpha}{C'\varphi(C)}.\end{aligned}$$

Hence, by Remark 2,

$$\int_{B(y,r_j)}\left(\int_{\rho(\xi,y)}^{r_j}\frac{\alpha t^{\alpha-1}}{\mu(B(y,t))}dt\right)d\mu(\xi)=r_j^\alpha\le\frac{C'\lambda}{2^\alpha-1}\varphi(C)\mu(B_j)I_\alpha(z_j,y). \tag{5.4}$$

By using Lemma 3.3 we have

$$\begin{aligned}\int_{B_j}I_\alpha(\xi,y)\,d\mu(\xi) &\le \lambda\int_{B_j}I_\alpha(y,\xi)\,d\mu(\xi)\\ &\le \lambda\left(\int_{B(y,r_j)}\left(\int_{\rho(y,\xi)}^{r_j}\frac{\alpha t^{\alpha-1}}{\mu(B(y,t))}dt\right)d\mu(\xi)+\int_{B_j}\left(\int_{r_j}^{R}\frac{\alpha t^{\alpha-1}}{\mu(B(y,t))}dt\right)d\mu(\xi)\right).\end{aligned}$$

Here we used that $\int_{\rho(y,\xi)}^{r_j}\frac{\alpha t^{\alpha-1}}{\mu(B(y,t))}dt\le 0$ for $\rho(y,\xi)\ge r_j$ and $\{\xi\in B_j:\rho(y,\xi)<r_j\}\subset B(y,r_j)$. By (5.4) and (5.3) we have

$$\begin{aligned}\int_{B_j}I_\alpha(\xi,y)\,d\mu(\xi) &\le \lambda\left(\frac{C'\lambda}{2^\alpha-1}\varphi(C)\mu(B_j)I_\alpha(z_j,y)+\lambda\varphi(C)\mu(B_j)I_\alpha(z_j,y)\right)\\ &= \lambda^2\varphi(C)\left(\frac{C'}{2^\alpha-1}+1\right)\mu(B_j)I_\alpha(z_j,y).\end{aligned}$$

Putting $C_3=\max\{\lambda^2\varphi(C),\lambda^2\varphi(C)(\frac{C'}{2^\alpha-1}+1)\}$, we have the conclusion. □

Lemma 5.2. *Let C_2 in Lemma E and suppose that $a \geq 2C_3k$ for C_3 in (5.1) and k in (4.1). Then*

(i) $\mu(B_j \cap G_{at}) \leq \dfrac{\mu(B_j)}{2}$ *for every $j \in \mathbf{N}$,*

(ii) $\mu(G_t^i) \leq 2C_2\mu(G_t^i \backslash G_{at})$.

Proof. (i) From the definition of G_{at} and Fubini's theorem it follows that

$$at\mu(B_j \cap G_{at}) \leq \int_{B_j \cap G_{at}} (I_\alpha g)(\xi)\,d\mu(\xi) \leq \int_X g(y)\left(\int_{B_j} I_\alpha(\xi,y)\,d\mu(\xi)\right)d\mu(y)$$

By Lemma 5.1 there is a point $z_j \in (G_t^i)^c$ such that for all $y \in X$

$$\int_{B_j} I_\alpha(\xi,y)\,d\mu(\xi) \leq C_3\mu(B_j)I_\alpha(z_j,y),$$

whence

$$\mu(B_j \cap G_{at}) \leq \frac{C_3}{at}\mu(B_j)(I_\alpha g)(z_j).$$

Since that $(I_\alpha g)(z_j) \leq kt$ by Lemma 4.2, we obtain (i).

(ii) By (i) we have, for each $j = 1, 2, \cdots$,

$$\mu(B_j) \leq \mu(B_j \cap G_{at}) + \mu(B_j \backslash G_{at}) \leq \frac{1}{2}\mu(B_j) + \mu(B_j \backslash G_{at}),$$

whence

$$\mu(B_j) \leq 2\mu(B_j \backslash G_{at}).$$

By (i) and (iii) in Lemma E we have

$$\mu(G_t^i) \leq \sum_{j \in \mathbf{N}} \mu(B_j) \leq 2\sum_{j \in \mathbf{N}} \mu(B_j \backslash G_{at}) \leq 2C_2\mu(G_t^i \backslash G_{at}).$$

□

By the same method as in [4, Lemma 5.5] we can show the following lemma.

Lemma 5.3. *Let $a \geq 2\lambda^2\varphi(3d)$ and x be a point in G_t^i. Then*

$$\int_{G_{t/a}^c} I_\alpha(x,y)g(y)\,d\mu(y) \leq \frac{t}{2}.$$

The following lemma is the last one as the preparation of the proof of Lemma 2.2.

Lemma 5.4. *Let $a \geq 2\max\{C_3k,\ \lambda^2\varphi(3d)\}$ for C_3 in (5.1) and k in (4.1) and x be a point in G_t^i. Furthermore, let q_t be the number defined by (2.4). Then there is a constant $C_4 > 0$ such that*

$$(E_\alpha^{q_t} g)(x) \leq \frac{t}{2} + C_4\int_{G_{t/a} \backslash Gat} E_\alpha^{q_t}(x,y)(Mg)(y)\,d\mu(y).$$

Here C_4 depends only on d, D and α.

Proof. Fix $x \in G_t^i$. Then we have from Lemma 3.6 and (2.1) that

$$\begin{aligned}\int_{B_j} E_\alpha^{q_t}(x,y)g(y)\,d\mu(y) &\leq \varphi(d+2d^2)E_\alpha^{q_t}(x,\xi)\int_{B_j} g(y)\,d\mu(y)\\ &\leq \varphi(d+2d^2)E_\alpha^{q_t}(x,\xi)\mu(B_j)(Mg)(\xi).\end{aligned}$$

for every $\xi \in B_j$. Integrating over $B_j \backslash G_{at}$ with respect to ξ and using Lemma 5.2, (i), we have

$$\begin{aligned}\int_{B_j} E_\alpha^{q_t}(x,y)g(y)\,d\mu(y) &\le \varphi(d+2d^2)\frac{\mu(B_j)}{\mu(B_j\backslash G_{at})}\int_{B_j\backslash G_{at}} E_\alpha^{q_t}(x,\xi)(Mg)(\xi)\,d\mu(\xi)\\ &\le 2\varphi(d+2d^2)\int_{B_j\backslash G_{at}} E_\alpha^{q_t}(x,\xi)(Mg)(\xi)\,d\mu(\xi),\end{aligned}$$

whence, by Lemma E,

$$\begin{aligned}\int_{G_t^i} E_\alpha^{q_t}(x,y)g(y)\,d\mu(y) &\le \sum_{j\in\mathbf{N}}\int_{B_j} E_\alpha^{q_t}(x,y)g(y)d\mu(y)\\ &\le 2\varphi(d+2d^2)\sum_{j\in\mathbf{N}}\int_{B_j\backslash G_{at}} E_\alpha^{q_t}(x,\xi)(Mg)(\xi)\,d\mu(\xi)\\ &\le 2\varphi(d+2d^2)C_2\int_{G_t^i\backslash G_{at}} E_\alpha^{q_t}(x,\xi)(Mg)(\xi)\,d\mu(\xi).\end{aligned}$$

Since, for μ-a.e. x,

$$⨍_{B(x,r)} g(y)d\mu(y) \to g(x) \quad \text{as} \quad r\to 0,$$

we see that $g \le Mg$ μ-a.e. and

$$\int_{G_{t/a}\backslash G_t^i} E_\alpha^{q_t}(x,y)g(y)\,d\mu(y) \le \int_{G_{t/a}\backslash G_t^i} E_\alpha^{q_t}(x,y)(Mg)(y)\,d\mu(y).$$

On the other hand we also have, by Lemma 5.3

$$\int_{G_{t/a}^c} E_\alpha^{q_t}(x,y)g(y)\,d\mu(y) \le \int_{G_{t/a}^c} I_\alpha(x,y)g(y)\,d\mu(y) \le \frac{t}{2}.$$

Consequently

$$\begin{aligned}&(E_\alpha^{q_t}g)(x) = \int_X E_\alpha^{q_t}(x,y)g(y)\,d\mu(y)\\ =\;& \int_{G_{t/a}^c} E_\alpha^{q_t}(x,y)g(y)d\mu(y) + \int_{G_{t/a}\backslash G_t^i} E_\alpha^{q_t}(x,y)g(y)d\mu(y) + \int_{G_t^i} E_\alpha^{q_t}(x,y)g(y)d\mu(y)\\ \le\;& \frac{t}{2} + \int_{G_{t/a}\backslash G_t^i} E_\alpha^{q_t}(x,y)(Mg)(y)d\mu(y) + 2\varphi(d+2d^2)C_2\int_{G_t^i\backslash G_{at}} E_\alpha^{q_t}(x,y)(Mg)(y)\mu(y)\\ \le\;& \frac{t}{2} + C_4\int_{G_{t/a}\backslash G_{at}} E_\alpha^{q_t}(x,y)(Mg)(y)\,d\mu(y),\end{aligned}$$

where $C_4 = 1 + 2\varphi(d+2d^2)C_2$. □

6. Proof of Proposition

In this section we give the proof of Proposition.

Proof of Proposition. Let g be a non-negative function in $L^p(X,\mu)$ and let $a \ge 2\max\{C_3k,\ \lambda^2\varphi(3d)\}$. Recall that for each $r \in (0,R)$

$$(I_\alpha g)(x) = (I_\alpha^r g)(x) + (E_\alpha^r g)(x).$$

Let $t > b$. For $x \in G_t^i$

$$t < (I_\alpha^{q_t} g)(x) + (E_\alpha^{q_t} g)(x),$$

where q_t is defined in (2.3). Since

$$(I_\alpha^{q_t} g)(x) = \int_0^{q_t} \alpha t^{\alpha-1} \left(\fint_{B(x,t)} g(y)\, d\mu(y) \right) dt \le (Mg)(x) \int_0^{q_t} \alpha t^{\alpha-1}\, dt = q_t^\alpha (Mg)(x),$$

we have, by Lemma 5.4,

$$t < 2q_t^\alpha (Mg)(x) + 2C_4 \int_{G_{t/a}\setminus G_{at}} E_\alpha^{q_t}(x,y)(Mg)(y)\, d\mu(y). \tag{6.1}$$

By Hölder's inequality and Lemma 3.1 we have

$$\begin{aligned} &\int_{G_{t/a}\setminus G_{at}} E_\alpha^{q_t}(x,y)(Mg)(y)\, d\mu(y) \qquad (6.2)\\ \le\ & \left(\int_{G_{t/a}\setminus G_{at}} (Mg)(y)^p\, d\mu(y) \right)^{1/p} \left(\int_X E_\alpha^{q_t}(x,y)^{p'}\, d\mu(y) \right)^{1/p'} \\ \le\ & C_5^{\frac{1}{p'}} \left(\int_{G_{t/a}\setminus G_{at}} (Mg)(y)^p\, d\mu(y) \right)^{1/p} \left(\int_{q_t}^R \alpha s^{(\alpha-n)(p'-1)+(\alpha-1)}\, ds \right)^{1/p'}, \end{aligned}$$

where C_5 is the constant in Lemma 3.1. Therefore, by (6.1) and (6.2),

$$\begin{aligned} t^p &\le 2^{2p-1}\{q_t^{\alpha p}(Mg)(x)^p + C_4^p \left(\int_{G_{t/a}\setminus G_{at}} E_\alpha^{q_t}(x,y)(Mg)(y)\, d\mu(y) \right)^p \} \\ &\le 2^{2p-1}\{q_t^{\alpha p-\beta} q_t^\beta (Mg)(x)^p \\ &\quad + C_4^p C_5^{p-1} \int_{G_{t/a}\setminus G_{at}} (Mg)(y)^p\, d\mu(y) \left(\int_{q_t}^R \alpha s^{(\alpha-\beta)(p'-1)+(\alpha-1)}\, ds \right)^{p-1} \}. \end{aligned}$$

Noting that $q_t^\beta = \dfrac{\mu(G_t^i)}{\gamma}$, we have, by Lemma 5.2, (ii),

$$\begin{aligned} t^p &\le 2^{2p-1}\{q_t^{\alpha p-\beta} \frac{2C_2\mu(G_t^i\setminus G_{at})}{\gamma} (Mg)(x)^p \\ &\quad + C_4^p C_5^{p-1} \int_{G_{t/a}\setminus G_{at}} (Mg)(y)^p\, d\mu(y) \left(\int_{q_t}^R \alpha s^{(\alpha-\beta)(p'-1)+(\alpha-1)}\, ds \right)^{p-1} \}, \end{aligned}$$

whence

$$t^p \le C\{q_t^{\alpha p-\beta} + \left(\int_{q_t}^R \alpha s^{(\alpha-\beta)(p'-1)+(\alpha-1)}\, ds \right)^{p-1} \} \int_{G_{t/a}\setminus G_{at}} (Mg)(y)^p\, d\mu(y),$$

where $C = 2^{2p-1}(2\gamma^{-1}C_2 + C_4^p C_5^{p-1})$. □

7. Proof of Theorem 1 and Theorem 2

In this section we prove Theorem 1 and Theorem 2.

Proof of Theorem 1. Let b be in (2.3). Then we have, by the assumption ($\mu 2$) for the measure,

$$\int_0^b t^{p-1}\mu(G_t)^{1-\alpha p/\beta} dt \le \mu(X)^{1-\alpha p/\beta}\frac{b^p}{p} = \frac{\lambda^p}{p}\left(\frac{R^\beta}{\mu(X)}\right)^{\alpha p/\beta} ||g||_p^p \le \frac{\lambda^p}{p}\gamma^{-\alpha p/\beta}||g||_p^p. \tag{7.1}$$

On the other hand Proposition yields

$$\int_b^\infty \frac{t^{p-1}\,dt}{q_t^{\alpha p-\beta}+(\int_{q_t}^R \alpha s^{(\alpha-\beta)(p'-1)+(\alpha-1)}ds)^{p-1}} \tag{7.2}$$
$$\le C\int_b^\infty \frac{1}{t}\int_{G_{t/a}\setminus G_{at}} (Mg)(y)^p\,d\mu(y)\,dt \le C\int_X (Mg)(y)^p \int_{(I_\alpha g)(y)/a}^{a(I_\alpha g)(y)} \frac{dt}{t}d\mu(y)$$
$$= 2C\log a\int_X (Mg)(y)^p\,d\mu(y) \le 2CC_6(\log a)||g||_p^p.$$

Case 1. $\alpha p<\beta$. Then

$$\int_{q_t}^R \alpha s^{(\alpha-\beta)(p'-1)+(\alpha-1)}\,ds \le \frac{\alpha(p-1)}{\beta-\alpha p}q_t^{(\alpha p-\beta)/(p-1)},$$

whence, by (2.3) and Lemma 4.2,

$$q_t^{\alpha p-\beta}+\left(\int_{q_t}^R \alpha s^{(\alpha-\beta)(p'-1)+(\alpha-1)}ds\right)^{p-1} \le \{1+\left(\frac{\alpha(p-1)}{\beta-\alpha p}\right)^{p-1}\}\left(\frac{\mu(G_t^i)}{\gamma}\right)^{\alpha p/\beta-1}$$
$$= \{1+\left(\frac{\alpha(p-1)}{\beta-\alpha p}\right)^{p-1}\}\left(\frac{\mu(G_{kt})}{\gamma}\right)^{\alpha p/\beta-1}.$$

Therefore by (7.2),

$$\int_b^\infty t^{p-1}\mu(G_{kt})^{1-\alpha p/\beta}\,dt \le 2CC_6\left(1+\left(\frac{\alpha(p-1)}{\beta-\alpha p}\right)^{p-1}\right)\gamma^{1-\alpha p/\beta}(\log a)||g||_p^p. \tag{7.3}$$

Hence, by (7.1) and (7.3),

$$\int_0^\infty t^{p-1}\mu(G_{kt})^{1-\alpha p/\beta}\,dt \le \{\frac{\lambda^p}{p}\gamma^{-\frac{\alpha p}{\beta}}+2CC_6\left(1+\left(\frac{\alpha(p-1)}{\beta-\alpha p}\right)^{p-1}\right)\gamma^{1-\alpha p/\beta}\log a\}||g||_p^p.$$

By change of variables we have the conclusion (i).

Case 2. $\alpha p=\beta$. Then, noting that

$$\left(\log\frac{2\mu(X)}{\mu(G_t^i)}\right)^{1-p} \le (\log 2)^{1-p},$$

for $t>b$ we have, by (2.3),

$$q_t^{\alpha p-\beta}+\left(\int_{q_t}^R \alpha s^{(\alpha-\beta)(p'-1)+(\alpha-1)}ds\right)^{p-1} = 1+\alpha^{p-1}\left(\log\frac{R}{q_t}\right)^{p-1}$$
$$= 1+\alpha^{p-1}\beta^{1-p}\left(\log\frac{\gamma R^\beta}{\gamma q_t^\beta}\right)^{p-1} \le 1+\alpha^{p-1}\beta^{1-p}\left(\log\frac{2\mu(X)}{\mu(G_t^i)}\right)^{p-1}$$
$$\le \{(\log 2)^{1-p}+\alpha^{p-1}\beta^{1-p}\}\left(\log\frac{2\mu(X)}{\mu(G_t^i)}\right)^{p-1},$$

whence, by (7.2),

$$\int_b^\infty t^{p-1}\left(\log\frac{2\mu(X)}{\mu(G_t^i)}\right)^{1-p}dt \le \{2CC_6((\log 2)^{1-p}+\alpha^{p-1}\beta^{1-p})\log a\}||g||_p^p. \tag{7.4}$$

We have, by (7.4) and Lemma 4.2,

$$\int_0^\infty t^{p-1}\left(\log\frac{2\mu(X)}{\mu(G_{kt})}\right)^{1-p}dt \le (\log 2)^{1-p}\frac{b^p}{p}+C_7||g||_p^p \le ((\log 2)^{1-p}\frac{(\lambda R^\alpha)^p}{p\mu(X)}+C_7)||g||_p^p,$$

where $C_7 = 2CC_6((\log 2)^{1-p}+\alpha^{p-1}\beta^{1-p})\log a$. This leads to the conclusion (ii). □

Proof of Theorem 2. Let $x \in X$. Then

$$|I_\alpha g(x)| \le (\int_X I_\alpha^R(x,y)^{p'}d\mu(y))^{1/p'}||g||_p.$$

Noting $\alpha p > \beta$ and using Lemma 3.1 we have

$$\int_X I_\alpha^R(x,y)^{p'}d\mu(y) \le C\int_0^R \alpha s^{(\alpha-\beta)(p'-1)+(\alpha-1)}ds = C\frac{\alpha(p-1)}{\alpha p-\beta}R^{(\alpha p-\beta)/(p-1)}.$$

Hence, for any $x \in X$,

$$|I_\alpha g(x)| \le C_8||g||_p,$$

where $C_8 = \left(C\frac{\alpha(p-1)}{\alpha p-\beta}\right)^{1/p'}$. Thus we have

$$||I_\alpha g||_\infty \le C_8||g||_p.$$

□

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