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The spectrum radii of free convex sums of projections

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Abstract

We calculate the spectrum radius of a convex sum of a free family of projections using the technique of Gerl and Woess for harmonic analysis of nonisotropic random walks on free groups. We use Voiculescu's R-transform for the additive free convolution to obtain the implicit relation on the resolvent function.

1. Introduction

Voiculescu began studying in [Vo1] the operator algebra free products from the probabilistic point of view. His idea is to look at free products as an analogue of tensor products and to develop a corresponding highly noncommutative probabilistic framework, where freeness is given as the notion of independence (see the monograph [VDN]).

The spectral theory of the infinite graphs such as the homogenous tree or the harmonic analysis on the free groups have been treated in many literatures. The free probability theory can be regarded as an abstract frame work for harmonic analysis on the free groups in the terms of non-commutative probability, which is called, nowadays, the free harmonic analysis.

In this paper, we shall show a kind of free harmonic analysis on a free family of projections, which is closely related to the harmonic analysis of nonisotropic random walk on free groups by Gerl and Woess in [GW]. We treat linear combinations, essentially convex sums, of a free family of projections and calculate their spectrum radii and, hence, their norms are obtained. The computation of the norms of convolution operators in C^*-algebra of the free groups in [AO] and the classical results of Kesten in [Ke] are derived as our special cases. The nice technique about computation of spectra of sums of self-adjoint free operators can be found in [Le], which is originally due to Haagerup. We shall, however, adopt the argument of Gerl and Woess in [GW], which is essentially the same as in [Le] but more graphical and direct way to compute a spectrum radius of self-adjoint operator from the resolvent function.

The paper is organized as follows: In Section 2, we review the terminologies of the free probability theory, which are used in the paper. In Section 3, we compute the spectrum radius of a free sum of positive scalar multiplicated projections. At the end of the paper, we present important examples of our computation related to the transition operator for the random walk on the free group and Plancherel measure for the free product of cyclic groups.

2. Preliminaries of the free probability theory

Recall that a usual probability space is a triple (Ω, Σ, ν), where Ω is a base space, Σ is a σ-algebra and ν is a probability measure that is positive and satisfies $\nu(\Omega) = 1$. A random variable is a measurable function $f : \Omega \to \mathbb{C}$, and if f is integrable then its expectation $E(f)$ is given by

$$E(f) = \int_{\omega \in \Omega} f d\nu(\omega).$$

We can consider a noncommutative probability space in a purely algebraic frame as an analogue of the above usual probability space.

Definition 2.1. A *noncommutative probability space* is $(\mathcal{A}, \phi)$, where $\mathcal{A}$ is a unital algebra and $\phi : \mathcal{A} \to \mathbb{C}$ is a linear functional with $\phi(1) = 1$. We say that $(\mathcal{A}, \phi)$ is a *C^*-probability space* when, in addition, $\mathcal{A}$ is a C^*-algebra and ϕ is a state.

One can define independence in a noncommutative probability space as generalization of the usual definition, which is based on the tensor product of algebras. Instead of the tensor product we use the reduced free product. Then we can introduce a much more noncommutative independence called free, which is due to Voiculescu and explained below.

Definition 2.2. Let $(\mathcal{A}, \phi)$ be a noncommutative probability space and $\mathcal{A}_i$ be subalgebra of $\mathcal{A}$ containing the identity element of $\mathcal{A}$, $1 \in \mathcal{A}_i \subset \mathcal{A}$, for $i \in I$. We say that the family $(\mathcal{A}_i)_{i \in I}$ is *free* if

$$\phi(x_1 x_2 \cdots x_n) = 0$$

whenever $x_j \in \mathcal{A}_{i_j}$ and $i_1 \neq i_2 \neq \cdots \neq i_n$ and $\phi(x_j) = 0$ for all j.

A family of subsets $X_i \subset \mathcal{A}$ (resp. elements $x_i \in \mathcal{A}$) will be called *free* if the family of subalgebras $\mathcal{A}_i$ generated by $\{1\} \cup X_i$ (resp. $\{1, x_i\}$) is free.

Example 2.3. For a discrete group G, consider the left regular representation of G on $\ell^2(G)$, given by $g \mapsto \lambda_G(g)$, where $(\lambda_G(g)\xi)(h) = \xi(g^{-1}h)$ for $g, h \in G$ and $\xi \in \ell^2(G)$. The reduced group C^*-algebra of G is

$$C_r^*(G) = \overline{\text{span}}^{\|\ \|}\{\lambda_G(g) : g \in G\ \},$$

the operator norm closed $*$-algebra generated by $\{\lambda_G(g) : g \in G\}$. It has the canonical faithful tracial state $\tau_G(\cdot) = \langle \cdot\ \delta_e | \delta_e \rangle$ where δ_e is the characteristic function of the identity e of G.

If G is the free product of a family $\{G_j\}_{j=1}^k$ of discrete groups and

$$\mathcal{A}_j = \overline{\text{span}}^{\|\ \|}\{\lambda_G(g) : g \in G_j\ \}$$

then $(\mathcal{A}_j)$ is free in the C^*-probability space $(C_r^*(G), \tau_G)$.

Definition 2.4. Let $(\mathcal{A}, \phi)$ be a noncommutative probability space. A *random variable* is an element $x \in \mathcal{A}$. The *distribution* of x is the linear functional ν_x on $\mathbb{C}[X]$ (the algebra of complex polynomials in the variable X), defined by

$$\nu_x(P(X)) = \phi(P(x)), \quad \text{for all } P \in \mathbb{C}[X].$$

Note that the distribution of a random variable $x \in \mathcal{A}$ is nothing more than a way of describing the simple moments of the random variable.

Remark 2.5. In a C^*-probability space $(\mathcal{A}, \phi)$, if x is a self-adjoint element of $\mathcal{A}$ then the distribution of x, ν_x, extends to a compactly supported measure on $\mathbb{R}$, namely there exists a unique probability measure $d\nu_x$ on $\mathbb{R}$ such that

$$\int_{t \in \mathbb{R}} P(t) d\nu_x(t) = \phi(P(x)).$$

Thus, for a self-adjoint element x, we simply call ν_x the probability measure of x in this paper.

Definition 2.6. If x_1 and x_2 are free random variables with distributions ν_{x_1} and ν_{x_2} then the distribution $\nu_{x_1+x_2}$ (which depends only on ν_{x_1} and ν_{x_2}) is called *the (additive) free convolution* of ν_{x_1} and ν_{x_2}, and denoted by $\nu_{x_1} \boxplus \nu_{x_2}$.

As a tool for computing the additive free convolution, Voiculescu introduced the R-transform in [Vo2], which makes the free convolution linearlize, and its definition goes in terms of a certain family of formal Toeplitz operators. Here we should note that the machinery of the R-transform was found independently and about the same time by Woess in [Wo2] and by Cartwright and Soardi in [CS1], [CS2], and [So] from the studies of the random walks on free product groups to obtain the walk generating function or the Plancherel measures.

Definition 2.7. For a distribution ν, we consider the formal power series in ζ^{-1}

$$G_\nu(\zeta) = \zeta^{-1} + \sum_{k=1}^{\infty} \nu(X^k)\zeta^{-k-1},$$

which is called *the G-series of ν*. Furthermore, we put the formal power series $M_\nu(z)$ by

$$M_\nu(z) = \sum_{k=0}^{\infty} \nu(X^k) z^k,$$

and call it *the moments series of ν*. Of course, the moments series is related to the G-series by the formula,

$$M_\nu(z) = \frac{1}{z} G_\nu\Big(\frac{1}{z}\Big).$$

Remark 2.8. If ν_x is the distribution of an element, $x \in A$, of a C^*-probability space $(\mathcal{A}, \phi)$, then for $|\zeta| > ||x||$,

$$G_{\nu_x}(\zeta) = \phi\left((\zeta \mathbf{1} - x)^{-1}\right).$$

If, in addition, x is a self-adjoint element, then ν_x is a probability measure compactly supported on $\sigma(x) \subset \mathbb{R}$, the spectrum of x, and thus

$$G_{\nu_x}(\zeta) = \int_{\sigma(a)} \frac{d\nu_x(t)}{\zeta - t}$$

is precisely the Cauchy transform of the probability measure ν_x of x, which is defined in a neighbourhood of infinity and analytic for $\zeta > ||x||_{sp}$.

Definition 2.9. *The R-transform $R_\nu(z)$ of the distribution ν* is defined by the functional equation,

$$\frac{1}{G_\nu(z)} = z - R_\nu\big(G_\nu(z)\big).$$

The most important property of the R-transform is that, for free random variables x_1 and x_2, we have

$$R_{\nu_{x_1} \boxplus \nu_{x_2}}(z) = R_{\nu_{x_1}}(z) + R_{\nu_{x_2}}(z),$$

hence, the R-transform can be regarded as a free cumulants series. For a dilation and a shift of a random variable, the R-transform behaves as follows:

$$R_{\nu_{\alpha x}}(z) = \alpha R_{\nu_x}(\alpha z),$$
$$R_{\nu_{x+m\mathbf{1}}}(z) = R_{\nu_x}(z) + m.$$

In the rest of the paper, for a random variable x, we shall simply denote G_{ν_x}, M_{ν_x}, and R_{ν_x} by G_x, M_x, and R_x, respectively, where ν_x means the distribution of x.

3. Free convex sums of projections

Let p be a projections in a C^*-probability space $(\mathcal{A}, \phi)$ with $\phi(p) = \tau\,(0 < \tau < 1)$. Then the G-series of the projection p can be given by

$$G_p(\zeta) = \frac{1-\tau}{\zeta} + \frac{\tau}{\zeta - 1} = \frac{\zeta + \tau - 1}{\zeta(\zeta-1)}.$$

After some small calculation, we obtain that

$$R_p(z) = \frac{1}{2z}\left\{(z-1) + \sqrt{(z-1)^2 + 4\tau z}\right\}$$

where the analytic square root is chosen as $\lim_{z\to 0} R_p(z) = \tau$.

Let $\{p_i\}_{i=1}^n$ be a free family of projections in a C^*-probability space $(\mathcal{A}, \phi)$ with $\phi(p_i) = \tau_i$, where $0 < \tau_i < 1$. We consider the sum of positive scalar multiplicated projections of the form

$$L = \sum_{i=1}^n \alpha_i p_i,$$

where $\alpha_i > 0$, hence if we normalize, L can be regarded as a convex sum of $\{p_i\}_{i=1}^n$. Using the properties of the R-transform in the previous section, we can find that the R-transform of the element L is given by

$$R_L(z) = \sum_{i=1}^n \frac{1}{2z}\left\{(\alpha_i z - 1) + \sqrt{(\alpha_i z - 1)^2 + 4\tau_i\alpha_i z}\right\}.$$

By the definition of the R-transform, we have an implicit relation on the Cauchy transform $G_L(z)$ of the probability measure of the random variable L that

$$\frac{1}{G_L(z)} = z - \sum_{i=1}^n \frac{1}{2G_L(z)}\left\{(\alpha_i G_L(z) - 1) + \sqrt{(\alpha_i G_L(z) - 1)^2 + 4\tau_i\alpha_i G_L(z)}\right\}.$$

Remark 3.1. If we could solve the above equation in $G_L(z)$, explicitly, then the spectral measure of L would be obtained with the helps of the Stieltjes inversion formula. It is immediately seen that $G_L(z)$ is an algebraic function, but it can not be solved in radicals, generally. We can solve in radicals only in the special cases and, in such cases, the corresponding spectral measures are determined (see [AY]).

As we mentioned before, there is the relation between the Cauchy transform G_L and the moment series M_L that

$$M_L(z) = \frac{1}{z}G_L\Big(\frac{1}{z}\Big),$$

we can obtain the following implicit relation on M_L:

Proposition 3.2. *We denote the function*

$$P(t) = 1 + \frac{1}{2}\sum_{i=1}^n \Big(\alpha_i t - 1 + \sqrt{\alpha_i^2 t^2 + 2(2\tau_i - 1)\alpha_i t + 1}\Big).$$

Then the moment series $M_L(z)$ satisfies the relation,

$$M_L(z) = P(zM_L(z)).$$

Now we shall give the spectrum radius of the element L by calculating the radius of convergence r of the series $M_L(z)$ by using the argument of Gerl and Woess in [GW] for the harmonic analysis of the nonisotropic random walks on the free groups.

At the first, we shall see the shape of the curve $y = P(t)$.

Lemma 3.3. *For $t \geq 0$, the curve $y = P(t)$ is strictly increasing and convex.*

Proof. The first derivative of $P(t)$ becomes

$$\begin{aligned} P'(t) &= \frac{1}{2}\sum_{i=1}^{n}\left(\alpha_i + \frac{\alpha_i^2 t + (2\tau_i - 1)\alpha_i}{\sqrt{\alpha_i^2 t^2 + 2(2\tau_i - 1)\alpha_i t + 1}}\right) \\ &= \frac{1}{2}\sum_{i=1}^{n}\alpha_i\left(1 + \frac{\alpha_i t + (2\tau_i - 1)}{\sqrt{\alpha_i^2 t^2 + 2(2\tau_i - 1)\alpha_i t + 1}}\right) \end{aligned}$$

and we have the inequality,

$$\left|\alpha_i t + (2\tau_i - 1)\right| < \sqrt{\alpha_i^2 t^2 + 2(2\tau_i - 1)\alpha_i t + 1},$$

because $-1 < 2\tau_i - 1 < 1$. Hence, $P'(t) > 0$ and the function $P(t)$ is strictly increasing.

The second derivative of $P(t)$ is given by

$$P''(t) = 2\sum_{i=1}^{n}\frac{\alpha_i^2(1-\tau_i)\tau_i}{\left(\alpha_i^2 t^2 + 2(2\tau_i - 1)\alpha_i t + 1\right)^{3/2}}.$$

Since $0 < \tau_i < 1$, we have $P''(t) > 0$ which means the function $P(t)$ is convex. □

Remark 3.4. We can see a little more about the shape of the curve $y = P(t)$ that

$$P(0) = 1 \ \text{ and } \ P'(0) = \sum_{i=1}^{n}\alpha_i\tau_i,$$

and the curve $y = P(t)$ approaches the asymptote

$$y = \left(\sum_{i=1}^{n}\alpha_i\right)t + \left(1 - \sum_{i=1}^{n}(1-\tau_i)\right)$$

which passes below the origin if and only if

$$\sum_{i=1}^{n}\tau_i < n - 1.$$

Proposition 3.5. *The radius of convergence r of $M_L(z)$ is given by the formula,*

$$\frac{1}{r} = \inf\left\{\frac{P(t)}{t} \,\middle|\, t > 0\right\}.$$

Proof. This can be proved by the same argument as for the proof of Proposition 3 in [GW] (see also Corollary 1 in [Wo1]).

The series $M_L(z)$ has positive coefficients because they are given by the moments of a positive random variable L. Thus what we are looking for is the smallest positive singularity of $M_L(z)$, which will give the radius of convergence r. By Proposition 3.2, it follows that, for positive real $z < r$, $t = zM_L(z)$ is also positive and $M_L(z)$ can be illustrated as the y-coordinate of the point of intersection of the line $y = \dfrac{1}{z}t$ with $y = P(t)$ in the (t, y)-plane.

Here we divide the proof into the two cases depending on whether the asymptote of the curve $y = P(t)$ passes below the origin or not.

Case 1 : $\sum_{i=1}^n \tau_i < n - 1$.

In this case the asymptote passes below the origin. For positive $z < \left(\sum_{i=1}^n \alpha_i\right)^{-1}$, there is exactly one point of intersection. For larger z, there are two points of intersection (by continuity of $M_L(z)$, we should take the one closer to the origin) until we reach the line through the origin that is tangent to the curve $y = P(t)$. We note that this tangent line is uniquely determined by the shape of the curve and the tangent point $(\theta, P(\theta))$ is given by unique positive solution θ of the equation $tP'(t) = P(t)$. For z still larger, the equation $\dfrac{1}{z}t = P(t)$ does not have a real solution any longer.

If we write the function

$$\mathcal{F}(z, w) = P(z\,w) - w$$

then we have $\mathcal{F}(z, M_L(z)) = 0$. For positive $z < \dfrac{1}{P'(\theta)}$, it follows that

$$\mathcal{F}_w(z, M_L(z)) = zP'(zM_L(z)) - 1 \neq 0,$$

as otherwise $zM_L(z)P'(zM_L(z)) = M_L(z) = P(zM_L(z))$ which means that $zM_L(z) = \theta$ and contradicts with $z < \dfrac{1}{P'(\theta)}$. By applying the theorem on implicit function to $\mathcal{F}$, we can find the analytic function $M_L(z)$ for $z < \dfrac{1}{P'(\theta)}$. On the other hand, for $z = \dfrac{1}{P'(\theta)}$ we have

$$\mathcal{F}_w(z, M_L(z)) = 0,$$

which yields our desired singularity. Hence we have

$$\frac{1}{r} = P'(\theta) = \frac{P(\theta)}{\theta} = \min\left\{ \frac{P(t)}{t} \;\middle|\; t > 0 \right\}.$$

Case 2 : $\sum_{i=1}^n \tau_i \geq n - 1$.

In this case the asymptote intercepts the y-axis on the interval $[0, 1)$ and, for each positive $z < \left(\sum_{i=1}^n \alpha_i\right)^{-1}$, we can find exactly one solution. Applying the theorem on implicit function again to $\mathcal{F}$, the solution $M_L(z)$ is analytic at z. For larger z, there is no real solution at all, hence, we have

$$\frac{1}{r} = \sum_{i=1}^n \alpha_i = \lim_{t\to\infty} \frac{P(t)}{t} = \inf\left\{ \frac{P(t)}{t} \;\middle|\; t > 0 \right\}.$$

$\square$

Remark 3.6. Replace t by s^{-1} in the expression of the infimum in the previous proposition, the result can be written as

$$\frac{1}{r} = \min_{s\geq 0}\left\{ s + \frac{1}{2}\sum_{i=1}^n \left(\alpha_i - s + \sqrt{s^2 + 2(2\tau_i - 1)\alpha_i s + \alpha_i^2}\right) \right\}$$

where the infimum as $t \to \infty$ in the previous proposition is, of course, attained at $s = 0$.

Since the moment series M_L is related to the G-series by

$$G_L(\zeta) = \frac{1}{\zeta} M_L\Big(\frac{1}{\zeta}\Big),$$

if the series $M_L(z)$ has the radius of convergence r then the function $G_L(\zeta)$ is analytic in $|\zeta| > \frac{1}{r}$. As we have noted in Remark 2.8, the G-series should be analytic in resolvent of L. The spectrum of L lies only on the non-negative real line, because L is positive, hence, $z = r$ is the only sigularity on the circle of convergence for the moment series $M_L(z)$.

Consequently, we obtain the following theorem on the spectrum radius of the operator L:

Theorem 3.7. *Let $\{p_i\}_{i=1}^n$ be a free family of projections in a C^*-probability space $(\mathcal{A}, \phi)$ with $\phi(p_i) = \tau_i$. Let L be the sum of positive scalar multiplicated projections of the form*

$$L = \sum_{i=1}^n \alpha_i p_i$$

where $\alpha_i > 0$. Then the spectrum radius, hence the norm, of the operator L can be given by the formula

$$||L|| = ||L||_{sp} = \frac{1}{2}\Big(\sum_{i=1}^n \alpha_i\Big) + \min_{s \geq 0}\Big\{ s + \frac{1}{2}\sum_{i=1}^n \Big(\sqrt{s^2 + 2(2\tau_i - 1)\alpha_i s + \alpha_i^2} - s\Big)\Big\}.$$

Remark 3.8. A complete description of the atoms of the free convolution $\mu \boxplus \nu$ of two probability measures μ and ν on $\mathbb{R}$ has been investigated by Bercovici and Voiculescu in [BV]. Namely, the free convolution $\mu \boxplus \nu$ has atoms only under the following circumstances:

$$\begin{gathered}(\mu \boxplus \nu)(\{c\}) = \tau > 0 \text{ if and only if there are } a, b \in \mathbb{R}, \\ \text{so that } a + b = c \text{ and } \mu(\{a\}) + \nu(\{b\}) = 1 + \tau.\end{gathered}$$

In our situation, it is obvious that the probability measure ν_L of the random variable L is nothing else than the free convolutions of the dilated Bernoulli measures,

$$\nu_L = \boxplus_{i=1}^n \big((1 - \tau_i)\delta_0 + \tau_i \delta_{\alpha_i}\big).$$

Applying the above criterion repeatedly, we can claim that if $\sum_{i=1}^n \tau_i > (n-1)$ then the probability measure ν_L has an atom that

$$\nu_L\Big(\{\textstyle\sum_{i=1}^n \alpha_i\}\Big) = \big(\textstyle\sum_{i=1}^n \tau_i\big) - (n-1),$$

which yields $||L||_{sp} = \sum_{i=1}^n \alpha_i$ (see the second case in the proof of Proposition 3.5).

Example 3.9. Let $\mathbb{F}_n$ be the free group on n generators $g_1, g_2, \dots, g_n$. We denote the left regular representation of $\mathbb{F}_n$ on $\ell^2(\mathbb{F}_n)$ by $\lambda_{\mathbb{F}_n}$. Akermann and Ostrand gave the exact formula of the norm of the convolution operator

$$T_0 = \sum_{i=1}^n \beta_i\big(\lambda_{\mathbb{F}_n}(g_i) + \lambda_{\mathbb{F}_n}(g_i)^*\big)$$

in [AO] (see also [Wol]). This operator T_0 can be regarded as the transition operator for the symmetric random walk on the free group $\mathbb{F}_n$ (see [GW] and [Wol]) and, hence, on the infinite graph of the $2n$ homogenous tree. From this point of view, the operator T_0 can be interpreted as follows: Let G be the free product of $2n$ copies of $\mathbb{Z}_2$,

$$G = \underbrace{\mathbb{Z}_2 * \mathbb{Z}_2 * \cdots * \mathbb{Z}_2}_{2n},$$

and let u_i ($i = 1, 2, ..., 2n$) be the unitary generator of order 2 corresponding to each cyclic group $\mathbb{Z}_2$ in $C_r^*(G)$. Then the operator T_0 can be identified with the operator

$$T_1 = \sum_{i=1}^{n} \beta_i (u_{2i-1} + u_{2i})$$

on $\ell^2(G)$ as the transition operator for the random walk on the infinite graph of the $2n$ homogeneous tree. As we mentioned in Example 2.3, the unitaries $\{u_i\}_{i=1}^{2n}$ are free. Namely, if we put $p_i = (u_i + 1)/2$ then $\{p_i\}_{i=1}^{2n}$ becomes a free family of projections with $\tau_G(p_i) = 1/2$ in the C^*-probability space $(C_r^*(G), \tau_G)$. Since the operator T_1 can be rewritten as

$$T_1 = \sum_{i=1}^{n} 2\beta_i (p_{2i-1} + p_{2i}) - 2\Big(\sum_{i=1}^{n} \alpha_i\Big)\mathbf{1},$$

by putting

$$L = \sum_{i=1}^{n} 2\beta_i (p_{2i-1} + p_{2i}),$$

our formula is applicable with $\alpha_i = \beta_{[(i-1)/2]+1}$ and $\tau_i = 1/2$ for $i = 1, 2, \ldots, 2n$. Finally, we obtain

$$||T_1|| = \min_{s \geq 0} \Big\{ s + \sum_{i=1}^{n} \Big(\sqrt{s^2 + 4\beta_i^2} - s \Big) \Big\}.$$

Example 3.10. Let us show another important example of our computation. Let G be the free product group of the cyclic groups,

$$G = \mathbb{Z}_{m_1} * \mathbb{Z}_{m_2} * \cdots * \mathbb{Z}_{m_n},$$

where $m_j \geq 2$. The (word) length for the elements of G can be naturally defined (see, for instance, [CS1]). We denote the characteristic function on the elements of length 1 by χ_1. Let T_{χ_1} denote the convolution operator $f \longmapsto \chi_1 * f$ on $\ell^2(G)$. We can regard the operator norm closed $*$-algebra $\mathcal{A}^*$ generated by the operator T_{χ_1} and the identity as a convolution algebra of functions in $\ell^2(G)$. Since T_{χ_1} is self-adjoint, A^* is an abelian C^*-algebra and we may consider the Gelfand isomorphism

$$\widehat{\ } : \mathcal{A}^* \longrightarrow C(X),$$

where $X \subset \mathbb{R}$ is the spectrum of the operator T_{χ_1} and $C(X)$ is the algebra of all the continuous functions on X. Then the Plancherel measure μ that is the unique Borel probability measure on $\mathbb{R}$ with support X, can be defined by the formula

$$f(e) = \int_{t \in X} \widehat{f}(t) d\mu(t)$$

for all $f \in A^*$, where e stands for the unit of G.

Here we call the quantity

$$||\mu||_{Pl} = \sup \Big\{ |\lambda| \,\Big|\, \lambda \in \mathrm{supp}(\mu) \Big\}$$

the Plancherel radius, which will be covered by our computation.

Indeed, the Plancherel radius coincides with the spectrum radius of the operator T_{χ_1} and it can be rewritten as follows: Let $u_i\,(i=1,2,\ldots,n)$ be the unitary generator of the cyclic group $\mathbb{Z}_{m_i}$ in the reduced C^*-algebra $C_r^*(G)$. Then it can be seen that the operator T_{χ_1} is in the form

$$T_{\chi_1}=\sum_{i=1}^{n}\Big(\sum_{k=1}^{m_i-1}u_i^k\Big).$$

On the other hand, it follows that the sum of $\{u_i^k\}_{i=1}^{m_i-1}$ can be given as

$$\sum_{k=1}^{m_i-1}u_i^k=m_ip_i-\mathbf{1},$$

where p_i is a projection of $\tau_G(p_i)=1/m_i$ because the unitary u_i is of order m_i. Hence we obtain that

$$T_{\chi_1}=\Big(\sum_{i=1}^{n}m_ip_i\Big)-n\,\mathbf{1}$$

and $\{p_i\}_{i=1}^n$ is, of course, a free family of projections. Now our result of computation is applicable with $\alpha_i=m_i$ and $\tau_i=1/m_i$.

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