A REMARK ON THE ASYMPTOTIC BEHAVIOR OF SUBORDINATORS

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Abstract

Let $X_1(t)$ and $X_2(t)$ be independent subordinators and let $X_2^{-1}(t)$ be the right-continuous inverse of X_2. The asymptotic behavior of $P\big[X_1(X_2^{-1}(t)) \leq x\big]$ as $x \to 0+$ for every fixed $t > 0$ is studied. It is shown that the infinitesimal order is determined by the exponent of X_1 and the constant, which depends on t, is determined by the Lévy measure of X_2. The problem is motivated by a generalized arc-sine law for one-dimensional diffusion processes.

1. Introduction

Let $X(t)$ be a subordinator, that is, $X(t)$ is a right-continuous, increasing process with stationary independent increments such that $X(0) = 0$. Let Ψ be the class of functions $\psi(\lambda)$ of $\lambda > 0$ which have the form

$$\psi(\lambda) = c\lambda + \int_0^\infty (1 - e^{-\lambda u}) n(du) \tag{1.1}$$

where $c \geq 0$ and $n(du)$ is a nonnegative Radon measure on $(0, \infty)$ with

$$\int_0^\infty \frac{u}{1+u} n(du) < \infty.$$

The law of the process $X(t)$ is uniquely determined by $\psi \in \Psi$ by the relation

$$E[e^{-\lambda X(t)}] = e^{-t\psi(\lambda)}, \qquad 0 \leq t < \infty,\ \lambda > 0.$$

The process $X(t)$ will be referred to as the subordinator determined by the exponent ψ. The measure $n(du)$ in (1.1) is called the Lévy measure of $X(t)$.

Let X_i $(i = 1, 2)$ be two independent subordinators determined by exponents

$$\begin{aligned}\psi_i(\lambda) &= c_i\lambda + \int_0^\infty (1 - e^{-\lambda x}) n_i(dx) \\ &= \lambda \left\{ c_i + \int_0^\infty e^{-\lambda x} n_i(x, \infty) dx \right\},\end{aligned}$$

and let X_i^{-1} be the right-continuous inverses of X_i so that

$$X_i^{-1}(t) = \inf\{s : X_i(s) > t\}, \qquad 0 \leq t < \infty.$$

The main result of this note is the following:

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Theorem 1.1. *If $\psi_1(\lambda)$ varies regularly at ∞ with exponent $0<\alpha<1$, then*

$$P\big[X_1(X_2^{-1}(t)) \le x\big] \sim \frac{1}{\Gamma(1+\alpha)} n_2(t,\infty)\frac{1}{\psi_1(1/x)}, \quad x\to 0+ \tag{1.2}$$

for every continuity point t of $n_2(\cdot,\infty)$.

This result is motivated by the following problem. Let $\{X_t, P^x\}$ be a diffusion process on $(-\infty,\infty)$ and let $\Gamma_+(t)=\int_0^t 1_{\{X_s>0\}}ds$. Thus $\Gamma_+(t)$ is the sojourn time of $\{X_t\}$ on the half line $(0,\infty)$. Let $l_\pm(t)$ be the local times at 0 of the processes which are obtained by the sum of positive or negative excursions of X_t, respectively. Then the right-continuous inverses $l_\pm^{-1}$ of $l_\pm$ are mutually independent subordinators determined by the certain exponents $\psi_\pm$ and we have the following S. Watanabe's formula, which is essentially due to D. Williams:

$$\begin{aligned} P^0(\Gamma_+(t)\le x) &= P\big[l_-(t-x)\le l_+(x)\big] \\ &= P\big[l_+^{-1}(l_-(t-x))\le x\big], \qquad x>0,\ t>0 \end{aligned}$$

(see Corollary 1 of [2]). Therefore, the asymptotic behavior of the distribution function of $\Gamma_+(t)$ as $x\to 0+$ may be understood by studying relation between two independent subordinators. In [1], we studied the asymptotic behavior of $P^0(\Gamma_+(t)\le x)$ as $x\to 0+$ for every fixed $t>0$, however, the analytical proof in [1] did not provide sufficient probabilistic explanations. Our main result in this note gives more probabilistic explanation of that problem, which is based on excursion theory.

2. Proof

For the proof of Theorem 1.1, we prepare the following lemma:

Lemma 2.1. *For $\lambda>0$, $\mu>0$,*

$$\int_0^\infty e^{-\mu t}E\big[e^{-\lambda X_1(X_2^{-1}(t))}\big]dt = \frac{\psi_2(\mu)}{\mu(\psi_1(\lambda)+\psi_2(\mu))}. \tag{2.1}$$

Proof. Note that $1=P\big[X(x)\le t\big]+P\big[X^{-1}(t)<x\big]$. Integrating with respect to $\lambda e^{-\lambda x}dx\cdot \mu e^{-\mu t}dt$ on both sides, we see

$$\begin{aligned} 1 &= \lambda\int_0^\infty e^{-\lambda x}E\big[e^{-\mu X(x)}\big]dx + \mu\int_0^\infty e^{-\mu t}E\big[e^{-\lambda X^{-1}(t)}\big]dt \\ &= \lambda\int_0^\infty e^{-\lambda x}e^{-x\psi(\mu)}dx + \mu\int_0^\infty e^{-\mu t}E\big[e^{-\lambda X^{-1}(t)}\big]dt \\ &= \frac{\lambda}{\lambda+\psi(\mu)} + \mu\int_0^\infty e^{-\mu t}E\big[e^{-\lambda X^{-1}(t)}\big]dt. \end{aligned}$$

Thus we have

$$\int_0^\infty e^{-\mu t}E\big[e^{-\lambda X^{-1}(t)}\big]dt = \frac{\psi(\mu)}{\mu(\lambda+\psi(\mu))}.$$

Since $E\big[e^{-\lambda X_1(X_2^{-1}(t))}\big]=E\big[e^{-\psi_1(\lambda)X_2^{-1}(t)}\big]$, we have

$$\begin{aligned} \int_0^\infty e^{-\mu t}E\big[e^{-\lambda X_1(X_2^{-1}(t))}\big]dt &= \int_0^\infty e^{-\mu t}E\big[e^{-\psi_1(\lambda)X_2^{-1}(t)}\big]dt \\ &= \frac{\psi_2(\mu)}{\mu(\psi_1(\lambda)+\psi_2(\mu))} \end{aligned}$$

which completes the proof of Lemma 2.1. □

We are now ready to prove Theorem 1.1. By Lemma 2.1, we have

$$\lim_{\lambda\to\infty} \psi_1(\lambda) \int_0^\infty e^{-\mu t} E\big[e^{-\lambda X_1(X_2^{-1}(t))}\big] dt = \frac{\psi_2(\mu)}{\mu}$$
$$= c_2 + \int_0^\infty e^{-\mu x} n_2(x,\infty) dx.$$

By the continuity theorem of Laplace transform (see Lemma 2 of [1]), this implies

$$\psi_1(\lambda) E\big[e^{-\lambda X_1(X_2^{-1}(t))}\big] \to n_2(t,\infty), \quad \lambda \to \infty$$

at all continuity points t of $n_2(\cdot,\infty)$. Since ψ_1 varies regularly at ∞ with exponent $0<\alpha<1$ by our assumption, this together with Karamata's Tauberian theorem implies

$$P\big[X_1\big(X_2^{-1}(t)\big) \le x\big] \sim \frac{1}{\Gamma(1+\alpha)} n_2(t,\infty) \frac{1}{\psi_1(1/x)}, \quad x \to 0+.$$

This completes the proof of Theorem 1.1.

References

[1] Y. Kasahara and Y. Yano, On a generalized arc-sine law for one-dimensional diffusion processes, preprint.

[2] S. Watanabe, Generalized arc-sine laws for one-dimensional diffusion processes and random walks, Stochastic analysis (Ithaca, NY, 1993), 157–172, Proc. Sympos. Pure Math., **57**, Amer. Math. Soc., Providence, RI, 1995.