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Parabolic extension of lateral functions in a cylindrical domain

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Abstract

Let D be a bounded domain in $\mathbf{R}^d$ such that ∂D is a β-set $(d-1 \leq \beta < d)$. We consider a cylindrical domain $\Omega_T = D \times (0,T)$. Using a decomposition into closed parabolic cubes of $(\mathbf{R}^d \backslash \partial D) \times \mathbf{R}$ of Whitney type, we construct an extension operator $\mathcal{E}$ which extends in functions on the lateral boundary S_D of Ω_T to all of $\mathbf{R}^{d+1}$. We also estimate two "norms" of $\mathcal{E}(f)$ by the Besov norm of f on S_D.

1. Introduction

Let D be a bounded domain in $\mathbf{R}^d$ such that ∂D is a β-set $(d-1 \leq \beta < d)$, i.e., there is a positive Radon measure μ satisfying

$$b_1 r^\beta \leq \mu(B(z,r) \cap \partial D) \leq b_2 r^\beta \tag{1.1}$$

for all $r \leq r_0$ for some r_0 and all $z \in \partial D$. Here $B(x,r)$ is a ball of radius r, centered at x.

A. Jonsson and H. Wallin introduced an extension operator which extends functions on ∂D to $\mathbf{R}^d$ and is bounded from a Besov space on ∂D to a ssuitable Besov space on $\mathbf{R}^d$ by using the Whitney decomposition ([JW1], [JW2]).

We consider a cylinderical domain $\Omega_D = D \times (0,T)$ for the above domain D and denote by S_D the lateral boundary $\partial D \times [0,T]$ of Ω_D.

In this paper we shall extend functions on S_D to $\mathbf{R}^{d+1}$. This extension is useful for solving the parabolic boundary value problems.

To do so, we consider the parabolic metric

$$\rho(X,Y) = \sqrt{|x-y|^2 + |t-s|}$$

for $X = (x,t)$, $Y = (y,s)$ and $x,\, y \in \mathbf{R}^d,\ t,\, s \in \mathbf{R}$.

Instead of balls we consider parabolic cylinders. Recall that the parabolic cylinder of radius r, centered at $X = (x,t)$ is defined by

$$C(X,r) = \{Y = (y,s);\, |x-y| < r,\ |t-s| < r^2\}.$$

We may suppose that $\partial D \subset B(0, R/2)$ for some $R \geq 1$ and $r_0 = 3R$ in (1.1). Fix a β-measure μ on ∂D and denote by μ_T the product measure of the β-measure and the 1-dimensional Lebesgue measure restricted to $[0, T]$.

Let $p \geq 1$ and $\alpha > 0$. We denote by $L^p(\mu_T)$ the set of all L^p-functions defined on S_D with respect to μ_T and by $\Lambda^p_\alpha(S_D)$ the space of all functions in $L^p(\mu_T)$ such that

$$\iint \frac{|f(X) - f(Y)|^p}{\rho(X, Y)^{\beta+2+p\alpha}} d\mu_T(X) d\mu_T(Y) < \infty.$$

For $f \in \Lambda^p_\alpha(S_D)$ the Besov norm of f is defined by

$$\|f\|_{\alpha,p} = \Big(\int |f(X)|^p d\mu_T(X)\Big)^{1/p} + \Big(\iint \frac{|f(X) - f(Y)|^p}{\rho(X, Y)^{\beta+2+p\alpha}} d\mu_T(X) d\mu_T(Y)\Big)^{1/p}.$$

Using a decomposition into closed parabolic cubes of $(\mathbf{R}^d \backslash \partial D) \times \mathbf{R}$ of Whitney type, we construct an extension operator $\mathcal{E}$ which extends in functions on S_D to $\mathbf{R}^{d+1}$ in §2 and investigate its properties.

We shall see by Lemma 2.2 that if f is ρ-continuous on S_D, then $\mathcal{E}(f)$ is also ρ-continuous in $\mathbf{R}^d \times [0, T]$.

We shall show in Lemma 2.3 that $\mathcal{E}$ is bounded from $L^p(\mu_T)$ to $L^p(\mathbf{R}^{d+1})$.

Let $Y = (y, s) \in \mathbf{R}^d \times [0, T]$. We denote by $\delta(Y)$ (resp. $\delta(y)$) the distance of Y from S_D with respect to ρ (resp. the Euclidean distance of y from ∂D). We easily see that $\delta(Y) = \delta(y)$ for $Y = (y, s) \in \mathbf{R}^d \times [0, T]$.

For a C^1-function f in $(\mathbf{R}^d \setminus \partial D) \times (0, T)$ we write

$$\nabla f(Y) = \Big(\frac{\partial f}{\partial y_1}(Y), \cdots, \frac{\partial f}{\partial y_d}(Y)\Big).$$

Using a maximal function of h in $L^1(\mu_T \times \mu_T)$ on $(\mathbf{R}^d \setminus \partial D) \times [0, T]$, we shall prove the following theorem in §3.

THEOREM 1. *Let $p > 1$, $f \in \Lambda^p_\alpha(S_D)$ and $p - p\alpha - d + \beta > 0$. Then*

$$\int_{(\mathbf{R}^d \backslash \partial D) \times [0,T]} |\nabla \mathcal{E}(f)(Y)|^p \delta(Y)^{p-p\alpha-d+\beta} dY + \int_{(\mathbf{R}^d \backslash \partial D) \times [0,T]} \Big|\frac{\partial}{\partial s}\mathcal{E}(f)(Y)\Big|^p \delta(Y)^{2p-p\alpha-d+\beta} dY \leq c\|f\|^p_{\alpha,p},$$

where c is a constant independent of f.

We next introduce another maximal function of $g \in L^1(\mu_T)$ on $B(0, R) \times [0, T]$ and prove the following theorem in §4.

THEOREM 2. *Let $p > 1$ and $f \in \Lambda^p_\alpha(S_D)$. Then*

$$\int_{D \times [0,T]} dX \int_{(\mathbf{R}^d \backslash \overline{D}) \times [0,T]} \frac{|\mathcal{E}(f)(X) - \mathcal{E}(f)(Y)|^p}{\rho(X, Y)^{d+2+p\alpha+d-\beta}} dY \leq c\|f\|^p_{\alpha,p},$$

where c is a constant independent of f.

2. Decomposition of an open set into parabolic cubes

In this chapter we decompose an open set in $\mathbf{R}^{d+1}$ into parabolic cubes and extend functions defined on S_D to $\mathbf{R}^{d+1}$.

By a parabolic cube we mean a closed set in $\mathbf{R}^{d+1}$ of the form

$$Q = [a_1, a_1 + r] \times [a_2, a_2 + r] \times \cdots \times [a_d, a_d + r] \times [a_{d+1}, a_{d+1} + r^2].$$

Especially, a k-parabolic cube is a parabolic cube of the form

$$Q = [n_1 2^{-k}, (n_1 + 1)2^{-k}] \times \cdots \times [n_d 2^{-k}, (n_d + 1)2^{-k}] \times [n_{d+1} 2^{-k}, (n_{d+1} + 1)2^{-k}],$$

where $n_1,\ n_2, \cdots, n_d,\ n_{d+1}$ are integers.

Let F be a non-empty closed set in $\mathbf{R}^{d+1}$ and $F \neq \mathbf{R}^{d+1}$. Consider the lattice of k-parabolic cubes in $\mathbf{R}^{d+1}$ and omit all those that touch F or that touch a k-parabolic cube which touches F. Discarding any parabolic cubes that are contained in larger ones, we take the union over k. The final collection $\mathcal{W}_p(\mathbf{R}^{d+1} \setminus F)$ of parabolic cubes is called the Whitney parabolic decomposition of $\mathbf{R}^{d+1} \setminus F$.

For each k-parabolic cube Q, $l(Q)$ (resp. $\mathrm{diam}_\rho Q$) stands for 2^{-k} (resp. $\sup_{X \in Q, Y \in Q} \rho(X, Y) = 2^{-k}\sqrt{d+1}$). We denote by $\mathrm{dist}_\rho(A, B)$ the distance of A and B with respect to ρ for two sets A, $B \subset \mathbf{R^{d+1}}$.

We easily see that it has the following properties (cf. [HN]).

LEMMA 2.1. *Let F be a non-empty closed set in $\mathbf{R}^{d+1}$ such that $F \neq \mathbf{R}^{d+1}$. The Whitney parabolic decomposition $\mathcal{W}_p(\mathbf{R}^{d+1} \setminus F) = \{Q_j\}$ has the following properties.*

(i) $\cup_j Q_j = \mathbf{R}^{d+1} \setminus F$.

(ii) *The interiors of any two parabolic cubes of $\mathcal{W}_p(\mathbf{R}^{d+1} \setminus F)$ are disjoint.*

(iii) $\sqrt{d+1}2^{-k} \leq dist_\rho(Q, F) \leq 4\sqrt{d+1}\,2^{-k}$ *for each k-parabolic cube Q.*

(iv) *If $Q \in \mathcal{W}_p(\mathbf{R}^{d+1} \setminus F)$ and Q is a k-parabolic cube, then each k-parabolic cube touching Q is contained in $\mathbf{R}^{d+1} \setminus F$.*

Using this Whitney parabolic decomposition of $(\mathbf{R}^d \setminus \partial D) \times \mathbf{R}$, we shall extend a function defined on the fractal lateral boundary S_D of Ω_D to all of $\mathbf{R}^{d+1}$. Fix η satisfying $0 < \eta < 1/8$ and let Q_0 denote the closed cube in $\mathbf{R}^d$ of unit length centered at the origin. Fix a C^∞-function ϕ in $\mathbf{R}^d$ such that

$$0 \leq \phi \leq 1, \quad \phi = 1 \text{ on } Q_0, \quad \mathrm{supp}\,\phi \subset (1+\eta)Q_0,$$

where $\mathrm{supp}\,\phi$ stands for the support of ϕ and

$$(1+\eta)Q_0 = \{x = (x_1, x_2, \cdots, x_d); -\frac{1}{2} - \frac{1}{2}\eta \leq x_j \leq \frac{1}{2} + \frac{1}{2}\eta\ (j = 1, \cdots, d)\}.$$

Further let ψ be a C^∞-function on $\mathbf{R}$ such that

$$0 \leq \psi \leq 1, \quad \psi = 1 \text{ on } [-\frac{1}{2}, \frac{1}{2}], \quad \mathrm{supp}\,\phi \subset [-\frac{1}{2} - \frac{1}{2}\eta, \frac{1}{2} + \frac{1}{2}\eta].$$

Let $Q_j \in \mathcal{W}_p((\mathbf{R}^d \setminus \partial D) \times \mathbf{R})$ and set, for $X = (x, t)$,

$$\phi_j(X) = \phi(\frac{x - x^{(j)}}{l_j})\psi(\frac{t - t^{(j)}}{l_j^2}),$$

where $X^{(j)} = (x^{(j)}, t^{(j)})$ is the center of Q_j and $l_j = l(Q_j)$. We note that $\phi_j(X) = 0$ for $X \in Q_i$ if Q_i does not touch Q_j. We also note that

$$|\frac{\partial}{\partial x_i}\phi_j(x)| \le c \operatorname{diam} Q_j$$

for $i = 1, \cdots, d$ and

$$|\frac{\partial}{\partial x_{d+1}}\phi_j(x)| \le c (\operatorname{diam} Q_j)^2,$$

where c is a constant inependent of j. We now define

$$\phi_j^*(X) = \frac{\phi_j(X)}{\Phi(X)},$$

where $\Phi(X) = \sum_j \phi_j(X)$.

It is obvious that

$$\sum_j \phi_j^*(X) = 1 \text{ on } (\mathbf{R}^d \setminus \partial D) \times \mathbf{R}.$$

For each parabolic cube Q_j we fix a point $A_j = A(Q_j) \in S_D$ such that

$$\inf\{\rho(X, Y);\ X \in Q_j,\ Y \in S_D\} = \rho(X_j, A_j),$$

for some $X_j \in Q_j$ and $A_j \in S_D$.

Using these functions and points, we extend a function defined on S_D to $\mathbf{R}^{d+1}$. Let $0 < \eta < \frac{1}{8}$, $f \in L^1(\mu_T)$ and we define, for $X = (x, t)$,

$$\mathcal{E}_0(f)(X) = \begin{cases} f(X) & \text{if } X \in \partial D \times [0, T], \\ 0 & \text{if } X \in \partial D \times (\mathbf{R} \setminus [0, T]), \\ \sum_j \frac{\int_{C(A_j, \eta l_j)\cap S_D} f(Y) d\mu_T(Y)}{\mu_T(C(A_j, \eta l_j)\cap S_D)} \phi_j^*(X) & \text{if } X \in (\mathbf{R}^d \setminus \partial D) \times [0, T]. \end{cases}$$

We remark that

$$\mathcal{E}_0(1) = 1 \quad \text{on } \mathbf{R}^d \times [0, T].$$

Choose a C^∞-function τ in $\mathbf{R}^{d+1}$ such that

$$\tau(X) = 1 \quad \text{on } \overline{B(0, R)} \times [-1, T+1]$$

and

$$0 \le \tau \le 1, \qquad \operatorname{supp} \tau \subset B(0, 2R) \times (-2, T+2),$$

and define, for $f \in L^1(\mu_T)$ and $X \in \mathbf{R}^{d+1}$,

$$\mathcal{E}(f)(X) = \tau(X)\mathcal{E}_0(f)(X).$$

We note that

$$\mathcal{E}(f) = 1 \quad \text{on } \overline{B(0, R)} \times [0, T].$$

LEMMA 2.2. *If f is ρ-continuous on S_D, then $\mathcal{E}(f)$ is also ρ-continuous in $\mathbf{R}^d \times [0, T]$.*

PROOF. We may show that $\mathcal{E}_0(f)$ is ρ-continuous at $X \in S_D = \partial D \times [0,T]$. Let $Y \in Q$, $l = l(Q)$ and $A = A(Q)$. Note that that $\mathcal{E}_0(1) = 1$ and $\phi_j^*(Y) = 0$ if Q does not touch Q_j. We claim that

$$|\mathcal{E}_0(f)(Y) - \mathcal{E}_0(f)(X)| \le c_1 \frac{1}{l^{\beta+2}} \int_{C(A,bl)\cap S_D} |f(Z) - f(X)| d\mu_T(Z), \tag{2.1}$$

where b is a constant independent of l. In fact

$$\begin{aligned} & |\mathcal{E}_0(f)(Y) - \mathcal{E}_0(f)(X)| = |\mathcal{E}_0(f - f(X))(Y)| \\ \le \; & \sum_j \phi_j^*(Y) \frac{1}{\mu_T(C(A_j, \eta l_j) \cap S_D)} \int_{C(A_j,\eta l_j)\cap S_D} |f(Z) - f(X)| d\mu_T(Z). \end{aligned}$$

Suppose that Q_j touches Q and $Z \in C(A_j, \eta l_j)$. We choose $X' \in Q_j \cap Q$. Then

$$\begin{aligned} \rho(Z, A) &\le \rho(Z, A_j) + \rho(A_j, X') + \rho(X', A) \\ &\le \sqrt{2}\eta l_j + 5\sqrt{d+1} l_j + 5\sqrt{d+1} l \\ &\le (2\sqrt{2}\eta + 15\sqrt{d+1}) l = bl. \end{aligned}$$

Hence we see that the claim is true.

If $Z \in C(A, bl)$, then

$$\begin{aligned} \rho(Z, X) &\le \rho(Z, A) + \rho(A, Y) + \rho(Y, X) \\ &< bl + 2\rho(Y, X) \le (b+2)\rho(Y, X). \end{aligned}$$

Since f is ρ-continuous on S_D, we have, for any $\epsilon > 0$, there is $\delta > 0$ such that

$$\rho(U, X) < \delta \text{ implies } |f(U) - f(X)| < \epsilon.$$

If $\rho(Y, X) < \frac{\delta}{b+2}$, then, by (2.1),

$$|\mathcal{E}_0(f)(Y) - \mathcal{E}_0(f)(X)| \le c_2 \epsilon.$$

This shows that $\mathcal{E}_0(f)$ is ρ-continuous at $X \in S_D$ and hence $\mathcal{E}(f)$ is ρ-continuous at $X \in S_D$. Q.E.D.

Let $0 < \lambda \le 1$ and $X \in \mathbf{R}^d \times [0,T]$. By the similar method as in [S, p.174] we see that, if f is λ-Hölder continuous on S_D with respect to ρ, then

$$|\frac{\partial}{\partial x_i} \mathcal{E}_0(f)(X)| \le c \operatorname{dist}_\rho(X, S_D)^{\lambda - 1} \tag{2.2}$$

for $i = 1, \cdots, d$. Using (2.2), we also see that, if f is λ-Hölder on S_D with respect to ρ, then $\mathcal{E}_0(f)$ is λ-Hölder continuous in $\mathbf{R}^d \times [0,T]$ with respect to ρ (cf. [S, Theorem 3, p.174]). Hence $\mathcal{E}(f)$ is also λ-Hölder continuous in $\mathbf{R}^d \times [0,T]$ with respect to ρ.

LEMMA 2.3. *Let $p > 1$ and $f \in L^p(\mu_T)$. Then*

$$\int |\mathcal{E}(f)|^p dY \le c \int |f|^p d\mu_T,$$

where c is a constant independent of f.

PROOF. Denote by $\mathcal{P}_k$ the set of all parabolic k-cubes Q in $\mathcal{W}_p((\mathbf{R}^d \setminus \partial D) \times \mathbf{R}) = \{Q_j\}$ such that $Q \cap \left((\mathbf{R}^d \setminus \partial D) \times (-2, T+2)\right) \neq \emptyset$. For each $Y \in Q \in \mathcal{P}_k$ we deduce from the definition of the extension $\mathcal{E}_0$

$$\begin{aligned} |\mathcal{E}_0(f)(Y)| &\leq c_1 \sum_j \phi_j^*(Y) \frac{1}{l_j^{\beta+2}} \int_{C(A_j, \eta l_j) \cap S_D} |f(Z)| d\mu_T(Z) \\ &\leq c_2 \left(2^{-k}\right)^{-\beta-2} \int_{C(A, b2^{-k}) \cap S_D} |f(Z)| d\mu_T(Z), \end{aligned}$$

where $A = A(Q)$ and b is a constant independent of Q. Hence

$$|\mathcal{E}_0(f)(Y)|^p \leq c_3 \left(2^{-k}\right)^{-\beta-2} \int_{C(A(Q), b2^{-k}) \cap S_D} |f(Z)|^p d\mu_T(Z)$$

for every $Y \in Q \in \mathcal{P}_k$. Consider $\{C(A(Q), b2^{-k})\}_{Q \in \mathcal{P}_k}$. Using a covering lemma of Vitali type (cf. [W2, Lemma 2.1]), we can find a subcovering $\mathcal{P}_{k,0}$ such that $\mathcal{P}_{k,0}$ are mutually disjoint and

$$\sum_{Q \in \mathcal{P}_k} C(A(Q), b2^{-k}) \subset \sum_{Q \in \mathcal{P}_{k,0}} C(A(Q), 3b2^{-k}).$$

Each point X in $\sum_{Q \in \mathcal{P}_{k,0}} C(A(Q), b2^{-k})$ is contained in at most N-many parabolic cubes of $\{C(A(Q), 3b2^{-k})\}_{Q \in \mathcal{P}_{k,0}}$, where N is a constant depending only on the dimension $d+1$. Hence

$$\sum_{Q \in \mathcal{P}_k} \int_Q |\mathcal{E}_0(f)(Y)|^p \leq c_4 \left(2^{-k}\right)^{d-\beta} \int_{S_D} |f(Z)|^p d\mu_T(Z).$$

Consequently we have

$$\sum_{k=k_0}^{\infty} \sum_{Q \in \mathcal{P}_k} \int_Q |\mathcal{E}_0(f)(Y)|^p \leq c_5 \int_{S_D} |f(Z)|^p d\mu_T(Z).$$

Here k_0 is the least integer k such that a point of $B(0, 2R) \times (-2, T+2)$ is contained in a k-parabolic cube. Since $0 \leq \tau(X) \leq 1$ and supp $\tau \subset B(0, 2R) \times (-2, T+2)$, we have the conclusion. Q.E.D.

For $f \in L^1(\mu_T)$ we define the parabolic maximal function of f by

$$\mathcal{M}_{\mu_T, p} f(X) = \sup\{\frac{\int_{C(X,r) \cap S_D} |f(Z)| d\mu_T(Z)}{\mu_T(C(X,r) \cap S_D)}; \ 0 < r < 2R\}.$$

Then we see that $\mathcal{M}_{\mu_T, p} f$ is lower semicontinuous on S_D and satisfies

$$\int |\mathcal{M}_{\mu_T, p} f|^p d\mu_T \leq \int |f|^p d\mu_T$$

for $f \in L^p(\mu_T)$.

PROPOSITION 2.4. *Let $p > 1$ and $f \in L^p(\mu_T)$. For $c > 1$ and $X \in S_D$ we set*

$$\Gamma_c(X) = \{Y = (y, s); \rho(X, Y) < c \inf_{Z \in S_D} \rho(Y, Z) = c\delta(Y)\}.$$

If $\Gamma_c(X)$ is non-empty for each $X \in S_D$, then

$$\int \sup_{Y\in\Gamma_c(X)} |\mathcal{E}(f)(Y)|^p dY \le c_1 \int |f|^p d\mu_T \tag{2.3}$$

and

$$\lim_{Y\to X, Y\in\Gamma_c(X)} \mathcal{E}(f)(Y) = f(X) \tag{2.4}$$

for μ_T-a.e. $X \in S_D$.

PROOF. Let $X \in S_D$ and $Y \in \Gamma_c(X)$. If $Y \in Q \in \mathcal{W}_p((\mathbf{R}^d \setminus \partial D) \times \mathbf{R})$ and $A = A(Q)$, then

$$|\mathcal{E}_0(f)(Y)| \le c_1 \frac{1}{l^{\beta+2}} \int_{C(A,bl)\cap S_D} |f| d\mu_T$$

as in the proof of Lemma 2.2, where b is a constant independent of l. Noting that $Y \in \Gamma_c(X)$, we have, for each $Z' \in C(A, bl)$,

$$\begin{aligned} \rho(Z', X) &\le \rho(Z', A) + \rho(A, Y) + \rho(Y, X) \\ &\le \sqrt{2}bl + 5\sqrt{d+1}l + c\delta(Y) \le b'l, \end{aligned}$$

where b' is a constant independent of l. Hence

$$\begin{aligned} \sup_{Y\in\Gamma_c(X)} |\mathcal{E}_0(f)(Y)| &\le c_1 \frac{1}{l^{\beta+2}} \int_{C(X,b'l)\cap S_D} |f(Z)| d\mu_T(Z) \\ &\le c_2 \mathcal{M}_{\mu_T,p} f(X). \end{aligned}$$

This implies

$$\int \left(\sup_{Y\in\Gamma_c(X)} |\mathcal{E}_0(f)|(Y) \right)^p d\mu_T(X) \le c_3 \int |f(Z)|^p d\mu_T(Z).$$

Since $0 \le \tau(Y) \le 1$, we have (2.3).

Using (2.3) and Lemma 2.2, we can also see (2.4) that by the similar method as in the proof of [S, Theorem 1, p.5]. Q.E.D.

3. Estimate of the extension operator

In this section we estimate the norm of $|\nabla \mathcal{E}(f)|$ by the Besov norm on the fractal lateral boundary S_D.

Let $f \in L^1(\mu_T \times \mu_T)$. We introduce a maximal function $\mathcal{M}(\mu_T \times \mu_T)(f)$ of $f \in L^1(\mu_T \times \mu_T)$ on $(B(0,R) \setminus \partial D) \times [0,T]$ defined by

$$\begin{aligned} &\mathcal{M}(\mu_T \times \mu_T)(f)(X) \\ = \ &\sup\{ \frac{1}{\mu_T(C(X,r)\cap S_D)^2} \int_{C(X,r)\cap S_D} d\mu_T(Y) \int_{C(X,r)\cap S_D} |f(Z,Y)| d\mu_T(Z); \\ &b\delta(X) < r < R\} \end{aligned}$$

for each $X \in (B(0,R)\setminus \partial D)\times[0,T]$. Here b is a fixed real number satisfying $b>1$.

We next define a meaure ν_0 on $\mathbf{R}^{d+1}$ by

$$\nu_0(E)=\int_{(B(0,3R)\setminus\partial D)\times[0,T]\cap E}\delta(Y)^{2\beta+2-d}dY$$

for a Borel measurable set $E\subset \mathbf{R}^{d+1}$.

The measure ν_0 is dominated by $\mu_T\times\mu_T$ for parabolic cubes in the following sense.

LEMMA 3.1. *Fix $b>1$. Let $X=(x,t)\in(B(0,R)\setminus\partial D)\times[0,T]$ and $b\delta(X)<r<3R$. Then*

$$\nu_0(C(X,r))\le c_1r^{2\beta+4}\le c_2\mu_T(C(X,r)\cap S_D)^2. \tag{3.1}$$

PROOF. Let $X=(x,t)\in(\mathbf{R}^d\setminus\partial D)\times[0,T]$ and x' be a point in ∂D satisfying $\delta(x)=|x-x'|$. Putting $X'=(x',t)$, we see that $C(X,r)\subset C(X',2r)$. Then

$$\nu_0(C(X',2r))\le\int_{t-(2r)^2}^{t+(2r)^2}ds\int_{(\mathbf{R}^d\setminus\partial D)\cap B(x',2r)}\delta(y)^{2\beta+2-d}dy.$$

By Lemma 2.2 in [W1], we have

$$\nu_0(C(X',2r))\le c_1(2r)^{2\beta+2}(2r)^2\le c_2r^{2\beta+4}.$$

Hence the first inequality holds.

Since $C(X',(1-1/b)r)\subset C(X,r)$ and ∂D is a β-set, we also have the second inequality of (3.1).

Q.E.D.

Using this, we have the following estimate of the maximal function of f in $L^1(\mu_T\times\mu_T)$ on $(B(0,R)\setminus\partial D)\times[0,T]$.

LEMMA 3.2. (i) *Let $\lambda>0$ and f be a $(\mu_T\times\mu_T)$-integrable function. Put*

$$E_\lambda=\{X\in(B(0,R)\setminus\partial D)\times[0,T];\mathcal{M}(\mu_T\times\mu_T)(f)(X)>\lambda\}.$$

Then

$$\nu_0(E_\lambda)\le\frac{c}{\lambda}\int\int|f(X,Y)|d\mu_T(X)d\mu_T(Y), \tag{3.2}$$

where c is a constant independent of f and λ.

(ii) *If $p>1$ and $f\in L^p(\mu_T\times\mu_T)$, then*

$$\int\mathcal{M}(\mu_T\times\mu_T)(f)(Y)^pd\nu_0(Y)\le\int\int|f(X,Y)|^pd\mu_T(X)d\mu_T(Y). \tag{3.3}$$

PROOF. Let $f\in L^1(\mu_T\times\mu_T)$ and $\lambda>0$. Then we see that E_λ is open as usual. Let K be a compact subset of E_λ. For each $X\in K$ we can find a real number $r_X>0$ such that

$$\mu_T(C(X,r_X)\cap S_D)^{-2}\int_{C(X,r_X)\cap S_D}\int_{C(X,r_X)\cap S_D}|f(Y,Z)|d\mu_T(Y)d\mu_T(Z)>\lambda \tag{3.4}$$

and $\delta(X) < r_X < R$. Then the covering lemma of Vitali type (cf. [W2, Theorem 2.1]) asserts that there is a subfamily $\{(C(X_j, r_{X_j})\}$ of finitely many elements of $\{C(X, r_X)\}_{X\in K}$ such that $\{C(X_j, r_{X_j})\}$ are mutually disjoint and

$$K \subset \cup_j C(X_j, 3r_{X_j}).$$

Then, by Lemma 3.1,

$$\begin{aligned}\nu_0(K) &\le \sum_j \nu_0(C(X_j, 3r_{X_j})) \le c_1 \sum_j (3r_{X_j})^{2\beta+4} \\ &= c_2 \sum_j r_{X_j}^{2\beta+4} \le c_3 \sum_j \mu_T(C(X_j, r_{X_j}) \cap S_D)^2.\end{aligned}$$

The inequality (3.4) implies

$$\nu_0(K) \le c_3 \sum_j \frac{1}{\lambda} \int_{C(X_j, r_{X_j})\cap S_D} \int_{C(X_j, r_{X_j})\cap S_D} |f(Y,Z)| d\mu_T(Y) d\mu_T(Z).$$

Since $\{C(X_j, r_{X_j})\}_j$ are mutually disjoint, we have

$$\nu_0(K) \le \frac{c_3}{\lambda} \int_{S_D} \int_{S_D} |f| d\mu_T d\mu_T.$$

Since $\nu_0(E_\lambda) = \sup\{K; K \textit{ is compact},\ K \subset E_\lambda\}$, we have (3.2).

The inequality (3.3) is deduced from (3.2) by the usual method (e.g.[S, p.7]). Q.E.D.

We now are ready to prove Theorem 1.

Proof of Theorem 1. We write $Y = (y, s)$. Let $\{Q_j\}$ be the Whitney parabolic decomposition of $(\mathbf{R}^d \setminus \partial D) \times \mathbf{R}$. For a parabolic cube $Q_j \in \{Q_j\}$ we set $l_j = l(Q_j)$ and $A_j = A(Q_j)$.

Let $Y \in Q \in \{Q_j\}$ and $Y \in (\mathbf{R}^d \setminus \partial D) \times [0, T]$. Further let $l = l(Q)$ and $A = A(Q)$. Put

$$b = \frac{1}{\mu_T(C(A, \eta l) \cap S_D)} \int_{C(A,\eta l)\cap S_D} f(Z) d\mu_T(Z).$$

Noting that $\mathcal{E}_0$ is a linear operator and $\mathcal{E}_0(1) = 1$, we have

$$\begin{aligned}&|\nabla\mathcal{E}_0(f)(Y)| = |\nabla\mathcal{E}_0(f-b)(Y)| \\ \le\ & c_1 \sum_j \frac{\phi_j^*(Y)}{l_j^{\beta+3} l^{\beta+2}} \int_{C(A_j \eta l_j)\cap S_D} d\mu_T(Z) \int_{C(A,\eta l)\cap S_D} |f(Z) - f(U)| d\mu_T(U).\end{aligned}$$

We set

$$h(Z, U) = \frac{|f(Z) - f(U)|}{\rho(Z, U)^{(\beta+2)/p+\alpha}}.$$

Then

$$|\nabla\mathcal{E}_0(f)(Y)| \le c_2 \frac{l^{-1+(\beta+2)/p+\alpha}}{l^{\beta+2} l^{\beta+2}} \int_{C(A,b'l)\cap S_D} d\mu_T(Z) \int_{C(A,\eta l)\cap S_D} h(Z, U) d\mu_T(U), \tag{3.5}$$

b' is a constant independent of Y.

We first suppose that $Y \in (B(0,R)\setminus\partial D)\times[0,T]$. Since $\rho(Y,A)\le 5\sqrt{d+1}\,l$ and $|\nabla\mathcal{E}_0(f)(Y)| = |\nabla\mathcal{E}(f)(Y)|$ for $Y\in(B(0,R)\setminus\partial D)\times[0,T]$, we have, by (3.5),

$$\begin{aligned} &|\nabla\mathcal{E}(f)(Y)|\delta(Y)^{1-\alpha-(\beta+2)/p} \\ \le\ & c_3\frac{1}{l^{\beta+2}l^{\beta+2}}\int_{C(Y,b''l)\cap S_D} d\mu_T(Z)\int_{(C(Y,b''l)\cap S_D} h(Z,U)d\mu_T(U) \\ \le\ & c_4\mathcal{M}(\mu_T\times\mu_T)(h)(Y), \end{aligned}$$

where b'' is a constant independent of Y.

Using Lemma 3.2, we have

$$\begin{aligned} &\int_0^T ds\int_{B(0,R)}|\nabla\mathcal{E}(f)(Y)|^p\delta(Y)^{p(1-\alpha-(\beta+2)/p)}\delta(Y)^{2\beta+2-d}dy \\ \le\ & c_4\int_0^T ds\int_{B(0,R)}\mathcal{M}(\mu_T\times\mu_T)(h)(Y)^p\delta(y)^{2\beta+2-d}dy \\ \le\ & c_5\iint h(Z,U)^p d\mu_T(Z)d\mu_T(U), \end{aligned}$$

whence

$$\int_0^T ds\int_{B(0,R)}|\nabla\mathcal{E}(f)(Y)|^p\delta(Y)^{p-p\alpha-d+\beta}dy \le c_5\iint h(Z,U)^p d\mu_T(Z)d\mu_T(U). \tag{3.6}$$

Noting that $|\frac{\partial}{\partial s}\phi_j^*|\le c_6 l^{-2}$, we also have

$$|\frac{\partial}{\partial s}\mathcal{E}(f)(Y)| \le c_7\frac{l^{(\beta+2)/p+\alpha}}{l^{\beta+4}l^{\beta+2}}\int_{C(A,b'l)\cap S_D} d\mu_T(Z)\int_{C(A,\eta l)\cap S_D} h(Z,U)d\mu_T(U),$$

whence

$$|\frac{\partial}{\partial s}\mathcal{E}(f)(Y)|\delta(Y)^{2-\alpha-(\beta+2)/p} \le c_8\mathcal{M}(\mu_T\times\mu_T)(h)(Y).$$

Using Lemma 3.2, we have

$$\begin{aligned} &\int_0^T ds\int_{B(0,R)}|\frac{\partial}{\partial s}\mathcal{E}(f)(Y)|^p\delta(Y)^{2p-p\alpha-d+\beta}dy \\ \le\ & c_9\iint h(Z,U)^p d\mu_T(Z)d\mu_T(U). \end{aligned}$$

We next suppose that $Y\in(\mathbf{R}^d\setminus B(0,R))\times[0,T]$ and $Y\in Q$. We note that

$$\begin{aligned} &|\frac{\partial}{\partial y_i}\mathcal{E}(f)(Y)| = |\frac{\partial}{\partial y_i}(\mathcal{E}_0(f)(Y)\tau(Y))| \\ \le\ & |\frac{\partial}{\partial y_i}(\mathcal{E}_0(f)(Y)| + |\mathcal{E}_0(f)(Y)||\frac{\partial}{\partial y_i}\tau(Y)| \end{aligned}$$

and

$$\operatorname{supp}\frac{\partial}{\partial y_i}\mathcal{E}(f)\subset B(0,2R)\times(-2,T+2).$$

Since $\partial D\subset B(0,R/2)$, we have $\delta(y)\ge R/2$. Noting that $4\sqrt{d+1}\,l\ge\delta(y)\ge R/2$, we also have, by (3.5),

$$|\nabla\mathcal{E}_0(Y)|\le c_{10}\iint h(Z,U)d\mu_T(Z)d\mu_T(U),$$

whence

$$\int_0^T ds \int_{B(0,2R)\setminus B(0,R)} |\frac{\partial}{\partial y_i}\mathcal{E}_0(f)(Y)|^p \delta(Y)^{p-p\alpha-d+\beta} dy$$
$$\leq c_{11} \iint h(Z,U)^p d\mu_T(Z) d\mu_T(U).$$

On the other hand we note that

$$\begin{aligned} |\mathcal{E}_0(f)(Y)| &\leq c_{12} \sum_j \frac{\phi_j^*(Y)}{l_j^{\beta+2}} \int_{C(A_j,\eta l_j)} |f(Z)| d\mu_T(Z) \\ &\leq c_{13} \int |f(Z)| d\mu_T(Z), \end{aligned}$$

whence

$$\int_0^T ds \int_{B(0,2R)\setminus B(0,R)} |\mathcal{E}_0(f)(Y)|^p |\frac{\partial}{\partial y_i}\tau(Y)|^p \delta(Y)^{p-p\alpha-d+\beta} dy$$
$$\leq c_{14} \int |f(Z)|^p d\mu_T(Z).$$

From this we deduce

$$\int_0^T ds \int_{B(0,2R)\setminus B(0,R)} |\nabla \mathcal{E}(f)(Y)|^p \delta(Y)^{p-p\alpha-d+\beta} dy \leq c_{15} \int |f(Z)|^p d\mu_T(Z).$$

Similarly we also have

$$\int_0^T ds \int_{B(0,2R)\setminus B(0,R)} |\frac{\partial}{\partial s}\mathcal{E}(f)(Y)|^p \delta(Y)^{2p-p\alpha-d+\beta} dy \leq c_{16} \int |f(Z)|^p d\mu_T(Z).$$

Thus we have the conclusion. Q.E.D.

4. Another property of the extension operator

In this secion we consider a maximal function of f in $L^1(\mu_T)$ on $(B(0,R)\setminus \partial D)\times[0,T]$. Let us begin with the following lemma.

LEMMA 4.1. *Let $b>1$ and $X \in B(0,2R)\times[0,T]$. Further let $Y_0=(y_0,s_0)\in(B(0,R)\setminus\partial D)\times[0,T]$ and $b\delta(y_0)<r<3R$. Then*

$$\int_{C(Y_0,r)\cap\{(B(0,R)\setminus\partial D)\times[0,T]\}} \frac{1}{\rho(Y,X)^{d-\beta}} dY \leq c_1 r^{\beta+2} \leq c_2 \int_{C(Y_0,r)\cap S_D} d\mu_T(Z), \tag{4.1}$$

where c_1 and c_2 are constants independent of r, Y_0 and X.

PROOF. Put

$$\mathcal{B}(Z,\epsilon)=\{Y\in\mathbf{R}^{d+1};\rho(Z,Y)<\epsilon\}$$

for $Z\in\mathbf{R}^{d+1}$ and $\epsilon>0$.

We first assume that $\rho(X, Y_0) < 2r$. Then

$$C(Y_0, r) \subset \mathcal{B}(Y_0, \sqrt{2}r) \subset \mathcal{B}(X, (2+\sqrt{2})r).$$

By the property of ρ in [W2, Lemma 2.5], we have

$$\begin{aligned} &\int_{C(Y_0,r)\cap\{(B(0,R)\setminus\partial D)\times[0,T]\}} \frac{1}{\rho(Y,X)^{d-\beta}} dY \\ \leq\ &\int_{\mathcal{B}(X,(2+\sqrt{2})r)} \frac{1}{\rho(Y,X)^{d-\beta}} dY \leq c_1((2+\sqrt{2})r)^{d+2-d+\beta} = c_2 r^{\beta+2}. \end{aligned}$$

We next assume that $\rho(X, Y_0) \geq 2r$. Then

$$\begin{aligned} &\int_{C(Y_0,r)\cap\{(B(0,R)\setminus\partial D)\times[0,T]\}} \frac{1}{\rho(Y,X)^{d-\beta}} dY \\ \leq\ &\int_{\{\rho(Y,X)\geq(2-\sqrt{2})r\}\cap C(Y_0,r)} \frac{1}{\rho(Y,X)^{d-\beta}} dY \\ \leq\ &\frac{1}{((2-\sqrt{2})r)^{d-\beta}} \int_{C(Y_0,r)} dY \leq c_3 r^{-d+\beta} r^{d+2} = c_4 r^{\beta+2}. \end{aligned}$$

Thus we obtain the first inequality of (4.1).

Noting that ∂D is a β-set, we also have the second inequality of (4.1). Q.E.D.

Fix $b > 1$ and define, for $f \in L^1(\mu_T)$ and $Y \in (B(0,R) \setminus \partial D) \times [0,T]$,

$$\mathcal{M}(\mu_T)(f)(Y) = \sup\{\frac{\int_{C(Y,r)\cap S_D} |f(Z)| d\mu_T(Z)}{\mu_T(C(Y,r)\cap S_D)}; b\delta(Y) < r < 3R\}.$$

Using Lemma 4.1, we can prove the following lemma by the same method as in the proof of Lemma 3.1.

LEMMA 4.2. *Let* $X \in B(0, 2R) \times [0,T]$.

(i) *Set, for* $\lambda > 0$,

$$F_\lambda = \{Y \in (B(0,R) \setminus \partial D) \times [0,T]; \mathcal{M}(\mu_T)(f)(Y) > \lambda\}$$

If $f \in L^1(\mu_T)$, *then*

$$\int_{F_\lambda} \frac{1}{\rho(Y,X)^{d-\beta}} dY \leq \frac{c}{\lambda} \int |f(Z)| d\mu_T(Z).$$

(ii) *If* $1 < p < \infty$ *and* $f \in L^p(\mu_T)$, *then*

$$\int \frac{\mathcal{M}(\mu_T)(f)(Y)^p}{\rho(Y,X)^{d-\beta}} dY \leq c \int |f(Z)|^p d\mu_T(Z).$$

Here c *is a constant independent of* f, λ *and* X.

We now prove Theorem 2.

Proof of Theorem 2. Let $X \in D \times [0,T]$ and $Y \in (B(0,R) \setminus \overline{D}) \times [0,T]$. Further, let $Y \in Q \in \mathcal{W}_\rho(\mathbf{R}^d \setminus \partial D) \times \mathbf{R}) = \{Q_j\}$ and set

$$l = l(Q), \quad l_j = l(Q_j), \quad A = A(Q), \text{ and } A_j = A(Q_j).$$

Since $\mathcal{E}_0(1) = 1$, we have

$$\begin{aligned} I &\equiv |\mathcal{E}(f)(Y) - \mathcal{E}(f)(X)| = |\mathcal{E}_0(f - \mathcal{E}(f)(X))(Y)| \\ &\leq c_1 \sum_j \phi_j^*(Y) \frac{1}{l_j^{\beta+2}} \int_{C(A_j,\eta l_j)\cap S_D} |f(Z) - \mathcal{E}(f)(X)| d\mu_T(Z), \end{aligned}$$

whence

$$I \leq c_2 \frac{1}{l^{\beta+2}} \int_{C(A,bl)\cap S_D} \frac{|f(Z) - \mathcal{E}(f)(X)|}{\rho(Z,X)^{(d+2)/p+\alpha}} \rho(Z,X)^{(d+2)/p+\alpha} d\mu_T(Z). \tag{4.2}$$

Here b is a constant independent of l with $b \geq 5\sqrt{d+1}$.

We consider two cases.

If $\rho(X,A) \leq 3bl$, then, for $Z \in C(A,bl) \cap S_D$,

$$\rho(Z,X) \leq \rho(Z,A) + \rho(A,X) \leq (\sqrt{2} + 3)bl$$

and $l \leq |x - y| \leq \rho(X,Y)$. From (4.2) we deduce

$$I \leq c_3 l^{(d+2)/p+\alpha-\beta-2} \int_{C(Y,b'l)\cap S_D} \frac{|f(Z) - \mathcal{E}(f)(X)|}{\rho(Z,X)^{(d+2)/p+\alpha}} d\mu_T(Z),$$

whence

$$\frac{I}{\rho(X,Y)^{(d+2)/p+\alpha}} \leq c_3 \frac{1}{l^{\beta+2}} \int_{C(Y,b'l)\cap S_D} \frac{|f(Z) - \mathcal{E}(f)(X)|}{\rho(Z,X)^{(d+2)/p+\alpha}} d\mu_T(Z). \tag{4.3}$$

If $\rho(X,A) > 3bl$, then, for $Z \in C(A,bl) \cap S_D$,

$$\rho(X,Z) \leq \rho(X,A) + \rho(A,Z) < \rho(X,A) + \sqrt{2}bl \leq \frac{1}{3}(3 + \sqrt{2})\rho(X,A)$$

and

$$\rho(X,Y) \geq \rho(X,A) - \rho(Y,A) \geq \frac{2}{3}\rho(X,A).$$

Hence

$$\rho(X,Z) < \frac{3+\sqrt{2}}{2}\rho(X,Y).$$

From (4.2) we deduce (4.3).

In each case we have (4.3) and hence

$$\frac{I}{\rho(X,Y)^{(d+2)/p+\alpha}} \leq c_4 \mathcal{M}(\mu_T)\left(\frac{|f(\cdot) - \mathcal{E}(f)(X)|}{\rho(\cdot,X)^{(d+2)/p+\alpha}}\right)(Y).$$

Using Lemma 4.2, we obtain

$$\begin{aligned} &\int_{(B(0,R)\setminus\overline{D})\times[0,T]} \frac{I^p}{\rho(X,Y)^{d+2+\alpha p+d-\beta}} dY \\ \leq\ & c_5 \int_{(B(0,R)\setminus\overline{D})\times[0,T]} \mathcal{M}(\mu_T)\left(\frac{|f(\cdot) - \mathcal{E}(f)(X)|}{\rho(\cdot,X)^{(d+2)/p+\alpha}}\right)(Y)^p \frac{1}{\rho(X.Y)^{d-\beta}} dY \\ \leq\ & c_6 \int_{S_D} \frac{|f(Z) - \mathcal{E}(f)(X)|^p}{\rho(Z,X)^{d+2+\alpha p}} d\mu_T(Z), \end{aligned}$$

whence

$$\begin{aligned} &\int_{D\times[0,T]} dX \int_{(B(0,R)\setminus\overline{D})\times[0,T]} \frac{|\mathcal{E}(f)(Y)-\mathcal{E}(f)(X)|^p}{\rho(X,Y)^{d+2+\alpha p+d-\beta}} dY \\ \le c_6 &\int_{D\times[0,T]} dX \int_{S_D} \frac{|f(Z)-\mathcal{E}(f)(X)|^p}{\rho(Z,X)^{d+2+\alpha p}} d\mu_T(Z). \end{aligned}$$

Similarly we also have

$$\begin{aligned} &\int_{S_D} d\mu_T(Z) \int_{D\times[0,T]} \frac{|\mathcal{E}(f)(X)-f(Z)|^p}{\rho(X,Z)^{d+2+\alpha p}} dX \\ \le\ & c_7 \int_{S_D} d\mu_T(Z) \int_{S_D} \frac{|f(X)-f(Z)|^p}{\rho(X,Z)^{\beta+2+\alpha p}} d\mu_T(X), \end{aligned}$$

whence,

$$\begin{aligned} &\int_{D\times[0,T]} dX \int_{(B(0,R)\setminus\overline{D})\times[0,T]} \frac{|\mathcal{E}(f)(Y)-\mathcal{E}(f)(X)|^p}{\rho(X,Y)^{d+2+\alpha p+d-\beta}} dY \\ \le\ & c_8 \int_{S_D} d\mu_T(Z) \int_{S_D} \frac{|f(X)-f(Z)|^p}{\rho(X,Z)^{\beta+2+\alpha p}} d\mu_T(X). \end{aligned}$$

Since

$$\begin{aligned} &\int_{D\times[0,T]} dX \int_{(\mathbf{R}^d\setminus B(0,R))\times[0,T]} \frac{|\mathcal{E}(f)(Y)-\mathcal{E}(f)(X)|^p}{\rho(X,Y)^{d+2+\alpha p+d-\beta}} dY \\ \le\ & c_9 \int_{(\mathbf{R}^d\setminus B(0,R))\times[0,T]} |\mathcal{E}(f)(Y)|^p dY + c_9 \int_{D\times[0,T]} |\mathcal{E}(f)(X)|^p dX, \end{aligned}$$

we have, by Lemma 2.3,

$$\begin{aligned} &\int_{D\times[0,T]} dX \int_{(\mathbf{R}^d\setminus\overline{D})\times[0,T]} \frac{|\mathcal{E}(f)(Y)-\mathcal{E}(f)(X)|^p}{\rho(X,Y)^{d+2+\alpha p+d-\beta}} dY \\ \le\ & c_{10} \left(\int_{S_D} d\mu_T(X) \int_{S_D} \frac{|f(X)-f(Y)|^p}{\rho(X,Y)^{\beta+2+\alpha p}} d\mu_T(Y) + \int_{S_D} |f(X)|^p d\mu_T(X) \right). \end{aligned}$$

Thus we have the conclusion. Q.E.D.

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