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WEB GEOMETRY of SOLUTIONS of HOLONOMIC FIRST ORDER PDEs

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1 Affine Connection of 3-webs

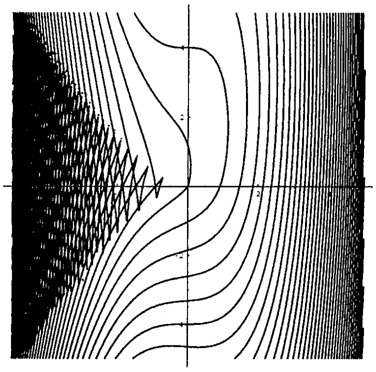
A first order ordinary differential equation of one valuable is

$$f(x,y,y') = 0 \qquad (*)$$

where y' stands for the derivative dy/dx. Here f is a real of complex analytic function. One can solve the equation locally in terms of y' as

$$y' = f_i(x,y)\ ,\ i = 1, \ldots, d$$

using implicit functions f_i. The solutions of each explicit differential equation form a germ of foliation, hence the solutions of the equation $(*)$ form a configuration of foliations of codimension one. Such a structure is called a *d-WEB,* and have been long studied by differential geometers such as Cartan, Blaschke (see c.f. [1,2,3]). The following figure shows a tipical singular 3-web structure.



Wave front 3-web

This is a generic member of a versal deformation family of the equation.

$$y'^3 + yy' - x = 0 \qquad \text{(symmetric wave front)}$$

(For the notion of the versal family, see [5].) One of basic ideas to extract geometric invariants is to extend Bott connection of these foliations (if possible) to an equal affine connection ∇ of the xy-space. In the case $d = 3$, such a connection is called Chern connection. This connection is defined on the complement of the discriminant of the equation, and extends meromorphically to the discriminant ([4]). The singuarity of the connection depends on that of the equation in general. So one may expect to classify the equations in terms of affine connectoin, and in some case only by their curvature forms.

To introduce such an affine connection let

$$\omega_i = U_i\ (dy - f_i\ dx), \quad i = 1, \dots, d$$

with functions $U_i \neq 0$. By integrability of ω_i and Frobenius Theorem or Division theorem,

$$d\omega_i = \theta_i \wedge \omega_i$$

holds with one forms θ_i. Specializing the i-th ω_i, the other equation

$$d \begin{bmatrix} \omega_1 \\ \omega_2 \\ \dots \\ \omega_{n+1} \end{bmatrix} = \begin{bmatrix} \theta_i & 0 & \cdots & 0 \\ 0 & \theta_i & \cdots & 0 \\ & & & \\ 0 & \cdots & 0 & \theta_i \end{bmatrix} \wedge \begin{bmatrix} \omega_1 \\ \omega_2 \\ \dots \\ \omega_{n+1} \end{bmatrix} + T \qquad (**)$$

with coframe $\omega_1, \dots, \hat{\omega}_i, \dots, \omega_{n+1}$, torsion term T and the connection form $\theta_i \cdot I$, I being the identity $n \times n$-matrix. The affine connection ∇_i defined here is a Bott connection of the i-th foliation.

In the case $d = 3$, impose the *normalization condition*

$$\tilde{\omega}_1 + \tilde{\omega}_2 + \tilde{\omega}_3 = 0 \quad .$$

Then there exists a unique $\theta = \theta_i$ independent of i such that

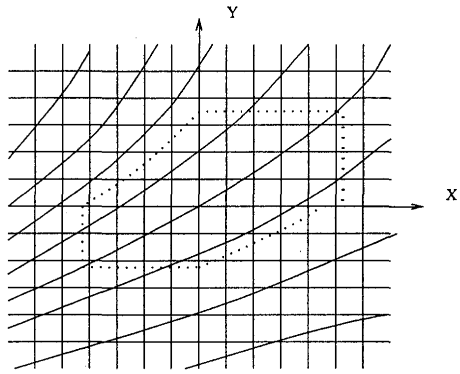
$$d \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \theta \end{bmatrix} \wedge \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \qquad (***)$$

Forgetting one of 3 lines this equation defines an affine connection without torsion which we denote by ∇_i. The above $(***)$ tells that ∇_i is independent of i, which is the Chern connection of the 3-web of $\omega_1, \omega_2, \omega_3$.

Let S denote the variety in the xyp-space (i.e. the 1-jet space $J^1(R, R)$) defined by $f(x, y, p) = 0$. The S is locally identified with the xy-plane via the natural projection. The above method is generalized to define an affine connection on (the smooth part of) S which is an extention of the Bott connection of the foliation on S defined by the contact form.

2 The flat 3-web and non flat 3-webs

A *non singular 3-web* of the plane is a configuration of 3 non singular curvilinear foliations, which are mutually in general position. Portrait of such a generic 3-web is as follows.



A non hexagonal 3-web

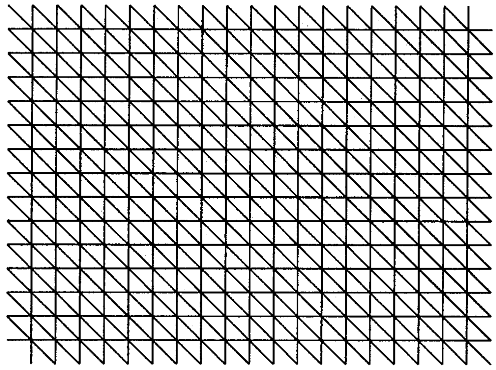
A 3-web is *flat* , or in other words, *hexagonal* if its Chern connection is flat, in other words, the curvature 2-form is identically 0: $d\theta = 0$. The following fact is classically known (see c.f. [3]).

Theorem 2.1 *Let $W = (F_1, F_2.F_3)$ be a germ of 3-web on the plane: $F_i = \{\omega_i = 0\}, i = 1, 2, 3$. Assume W is flat. Then W is diffeomorphic to the linear 3-web defined by $dx, dy, -(dx + dy)$: there exists a germ of diffeomorphism h of the plane such that*

$$\omega_1 = U_1\ h^*dx, \omega_2 = U_2\ h^*dy, \omega_3 = U_3\ h^* - (dx + dy)$$

where U_1, U_2, U_3 are non zero function germs.

The following figure shows the linear 3-web structure.



The linear 3-web

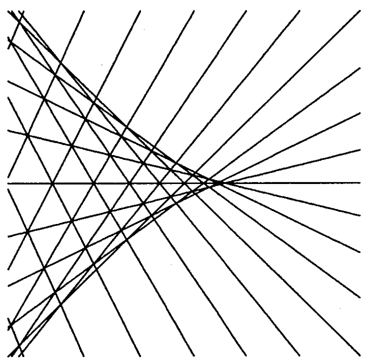
3 Flat Defferential equations

Theorem 3.1 (Lins, Nakai[4]) *Assume the variety S in x, y, p-space is smooth at $(0,0,0)$, the projection of S onto the xy-plane is Whitney cusp mapping, and the 3-web is flat. Then the 3-web is equivalent, via a coordinate change of the xy-plane, to one of the following 2 germs of differential equations at the origin.*

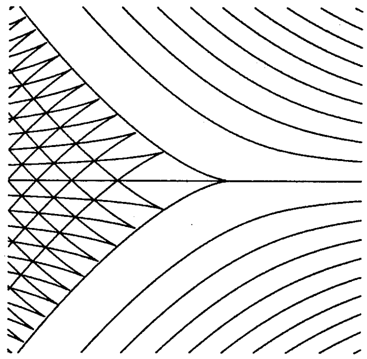
$$y'^3 + xy' - y = 0, \tag{1}$$

$$y'^3 + \frac{1}{4}xy' + \frac{1}{8}y = 0\ . \tag{2}$$

The flat 3-webs of these equations are as follows.



(1): Clairaut 3-web



(2): Rectangular 3-web

The reader may fing the affine (linear) structure on the non singular part of the above webs.

4 Dual 3-web

The *dual 3-line configuration* of a 3-line configuration $L = L_1 \cup L_2 \cup L_3$ of lines in the plane passing through the origin is the invariant configuration (different from L) of the group generated by three involutions respecting the line L_i respecting L. The *dual 3-web* W^* of a 3-web W is defined by integrating the dual 3-line configuration of the tangent 3-line fields of W.

Theorem 4.1 *The bi-duality holds: $W^{**} = W$, and W and W^* have the same Chern connection.*

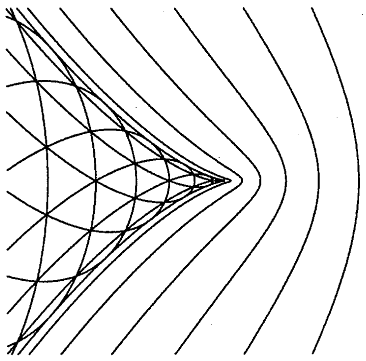
Corollary 4.1 *A 3-web W is flat if and only if its dual W^* is flat.*

The dual equations of the above (1), (2) are respectively

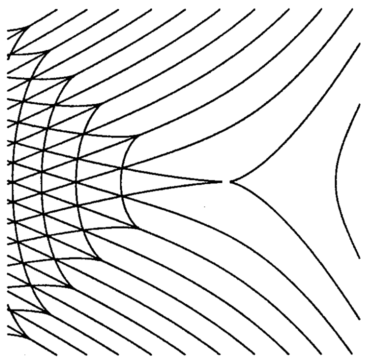
$$dy^3 + \frac{2x^2}{3y}\, dxdy^2 - x\, dx^2dy + \frac{2x^3 + 27y^2}{27y}\, dx^3 = 0, \tag{3}$$

$$dy^3 - \frac{x^2}{3y}\, dxdy^2 - \frac{x}{4}\, dx^2dy - \frac{2x^3 + 27y^2}{216y}\, dx^3 = 0 \tag{4}$$

The 3-web structure of these equations are as follows.



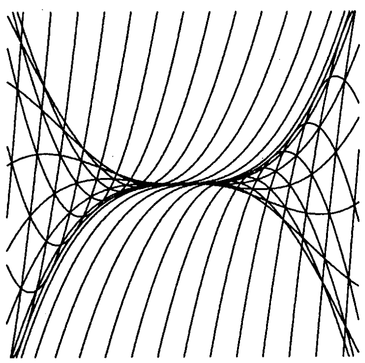
(3): Dual 3-web of (1)



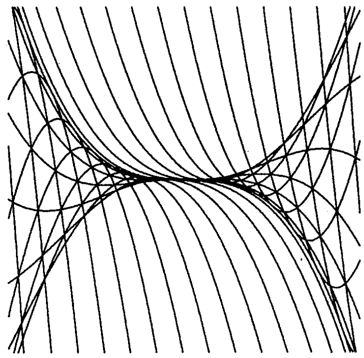
(4): Dual 3-web of (2)

5 A Self-dual flat 3-web

Consider the following flat 3-web obtained by folding the linear 3-web (1) by the antipodal involution of the y-coordinate, $y \to -y$.



The dual of this web is the following



Dual of the Self-dual flat 3-web

It is senn that the dual web is the rotation of the origina web.

References

[1] Blaschke,W. " *Einführung in die Geometrie der Waben* ," Birkhäuser-Verlag, Basel-Stuttgart (1955)

[2] Blaschke,W., Bol, G. " *Geometrie der Gewebe* ," Grundlehren der Mathematischen Wissenschaften, Springer, (1938)

[3] Chern, S.S. " *Web geometry* ," Bull A.M.S. vol. 6,(1982) 1–8

[4] Lins, A., Nakai, I. " *Curvature of first order ordinary differential equations* ," in preparation

[5] Nakai, I. " *Web geometry and the equivalence problem of the first order partial differential equations* ," in Web Theory, ed. Grifone, Salem, Kluwer Publ. 2001