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Numerical solutions of the bifurcation problem of interfacial progressive water waves

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Abstract: We consider progressive water-waves on the interface between two fluids of different densities. Such interfacial waves problem is a generalization of surface waves problem, which we have studied extensively in [2,3]. By this generalization, our numerical results on the bifurcation problem of the surface waves can be explained mathematically. We showed in [2] the ratio of the propagation speeds between the upper and lower fluids plays an important role in governing bifurcation structures. And it is meaningful to see changes in bifurcation structures as the key parameter varies. We show here our numerical results.

1. Introduction and formulation

The problem discussed here is to consider interfacial progressive waves between two fluids of different densities. Both of the fluids are assumed to be incompressible and inviscid. The flows are assumed to be two-dimensional and irrotational. Let m_u be the mass density of the upper fluid and m_l be of the lower fluid. The constants c_u and c_l are mean speeds of the upper fluid and the lower fluid, respectively. We take a coordinate system (x-y), moving together with the interfacial wave. Namely the wave profiles are stationary when we observe them in this coordinate system. Let $y = h(x)$ represent the interface. The problem is to determine the shape of the interface, that is to say, it is a free boundary problem. The depths of fluids may be finite or infinite. The gravity and the surface tension are taken into account. Let g and T be the gravity acceleration and the surface tension coefficient, respectively. We further assume that the wave profile is periodic in x with a period L and is symmetric with respect to the y-axis.

By these assumptions, the fluid motion can be described by using the velocity potential and the stream function. One of the mathematical difficulties of the problem is that the boundary is not known in advance. The difficulty can be overcome considering the problem described by the velocity potential and the stream function. However here, modifying Kotchin [1], we use the more convenient formulation, which will be shown as follows :

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Problem

Find 2π-periodic functions $\theta = \theta(\sigma)$ and $S = S(\sigma)$ such that

$$\begin{aligned} F_1 \equiv\ & \frac{1}{2}\frac{d}{d\sigma}\left(\Gamma_l^2 e^{2H\theta} - b\Gamma_u^2 e^{-2(\tilde{H}\tilde{\theta})\circ\widetilde{S}}\right) \\ & -\frac{p}{\Gamma_l}e^{-H\theta}\sin(\theta) + q\Gamma_l\frac{d}{d\sigma}\left(e^{H\theta}\frac{d\theta}{d\sigma}\right) = 0, \\ F_2 \equiv\ & \Gamma_l\frac{d\widetilde{S}(\sigma)}{d\sigma} - \Gamma_u\exp(-(\tilde{H}\tilde{\theta})\circ\widetilde{S} - H\theta) = 0, \end{aligned}$$

where

$$\widetilde{S}(\sigma) = \sigma + S(\sigma), \quad \tilde{\theta} = \theta\circ\widetilde{S}^{-1}$$

and

$$\begin{aligned} \Gamma_l &= \frac{1}{2\pi}\int_0^{2\pi} e^{-(H\theta)(\sigma)}\cos\theta(\sigma)d\sigma, \\ \Gamma_u &= \frac{1}{2\pi}\int_0^{2\pi} e^{(\tilde{H}\tilde{\theta})(\sigma)}\cos\tilde{\theta}(\sigma)d\sigma. \end{aligned}$$

H and $\tilde{H}$ are linear operators defined through the Fourier series as follows:

$$\begin{aligned} & H\left(\sum_{n=1}^{\infty}\left(a_n\sin n\sigma + b_n\cos n\sigma\right)\right) \\ & \quad = \sum_{n=1}^{\infty}\frac{1+\eta_l^{2n}}{1-\eta_l^{2n}}\left(-a_n\cos n\sigma + b_n\sin n\sigma\right), \\ & \tilde{H}\left(\sum_{n=1}^{\infty}\left(a_n\sin n\sigma + b_n\cos n\sigma\right)\right) \\ & \quad = \sum_{n=1}^{\infty}\frac{1+\eta_u^{2n}}{1-\eta_u^{2n}}\left(-a_n\cos n\sigma + b_n\sin n\sigma\right). \end{aligned}$$

This problem has five dimensionless parameters b, p, q, η_l and η_u. We can see it is a bifurcation problem with these five bifurcation parameters. b, p and q are parameters concerning with the average energy densities, the gravity acceleration and the surface tension, respectively, given as follows:

$$b = \frac{m_u c_u^2}{m_l c_l^2}, \quad p = \frac{gL(m_l - m_u)}{2\pi c_l^2 m_l}, \quad q = \frac{2\pi T}{m_l c_l^2 L}.$$

η_l and η_u are parameters concerning with the depth of fluids such as $0 \le \eta_l,\ \eta_u < 1$. $\eta_l = 0$ or $\eta_u = 0$ corresponds to the infinite depth of the lower fluid or the upper fluid, respectively.

Our task is to solve $F(\theta(\sigma), S(\sigma)) \equiv (F_1,\ F_2) = 0$, for given b, p, q, η_u and η_l. When the parameter b is equal to zero the above formulation reduces to the surface wave problem. Namely if the fluid motion above the free boundary is neglected, the problem reduces to surface progressive waves. Note that if $b = 0$ then the equation $F_1 = 0$ contains unknown θ only.

Our concern is to see the bifurcation structures globally as b changes from 0.

2. Numerical method and results

Here we will show only the results of which the depths of both fluids are infinite since the depths play little role in the bifurcation. Consequently η_u and η_l are fixed to be zero hereafter.

Since the profiles of wave are assumed to be symmetric with respect to y-axis, $\theta(\sigma)$ and $S(\sigma)$ are odd functions. Then using the spectral-collocation method, we look for approximate solutions of the following form:

$$\theta_N(\sigma) = \sum_{j=1}^{N} a_j \sin n\sigma, \qquad S_N(\sigma) = \sum_{j=1}^{N} b_j \sin j\sigma,$$

where a_j and b_j $(1 \le j \le N)$ are unknowns. Consider A_k and B_k defined as

$$\begin{aligned} A_k &= F_1(b\ ,p\ ,q\ ;\ \theta_N,\ S_N)|_{\sigma=\pi k/(N+1)}, \\ B_k &= F_2(b\ ,p\ ,q\ ;\ \theta_N,\ S_N)|_{\sigma=\pi k/(N+1)} \end{aligned}$$

for $k = 1, 2, \cdots, N$. This defines the following mapping

$$\begin{aligned} G\ :\ &(b, p, q; a_1, \cdots, a_N, b_1, \cdots, b_N) \\ &\to (A_1, \cdots, A_N, B_1, \cdots, B_N) \end{aligned}$$

which is denoted by $G\ :\ \mathbf{R}^3 \times \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}^N \times \mathbf{R}^N$. Consequently we look for $(b, p, q; a, b) \in \mathbf{R}^3 \times \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}^N$ such that $G(b, p, q; a, b) = 0$.

It can be easily seen we have a trivial solution $(a, b) = (a_1, \cdots, a_N, b_1, \cdots, b_N) = (0, 0)$ whatever (b, p, q) may be. And our aim is to find $(a, b) \ne (0, 0)$. Regarding the bifurcation, we can see the followings details of which is written in [3]:

- The Fréchet derivative $D_{a,b}G$ has a nontrivial null space if and only if (b, p, q) satisfies
$$(1 + b)n = p + n^2 q$$
for some $n = 1, 2, \cdots, N$. In this case, the vector $\Sigma = \left(\sin n\sigma, \dfrac{2}{n}\sin n\sigma\right)$ is a null vector; $D_{a,b}G(\Sigma) = 0$.
This means existences of a primary bifurcation of mode n $(n = 1, 2, \cdots)$.

- $(b, p_0, q_0; 0, 0)$ is a double bifurcation point of mode (m, n), where $p_0 = mn(1 + b)/(m + n)$ and $q_0 = (1 + b)/(m + n)$ for $0 < m < n$.

If $\theta(\sigma)$ and $S(\sigma)$ are obtained, then the interface $\{(x(\sigma), y(\sigma))\ ;\ 0 \le \sigma < 2\pi\}$ is given by

$$\begin{aligned} x(\sigma) &= \frac{L}{2\pi}\int_0^{\sigma} e^{-\tau_l(1,\sigma')} \cos\theta_l(1, \sigma')d\sigma', \\ y(\sigma) &= -\frac{L}{2\pi}\int_0^{\sigma} e^{-\tau_l(1,\sigma')} \sin\theta_l(1, \sigma')d\sigma'. \end{aligned}$$

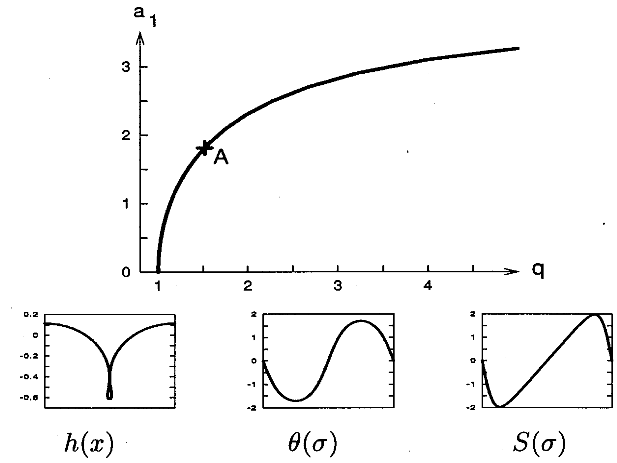

Figure 1: The bifurcation diagram and plots of $h(x), \theta(\sigma)$ and $S(\sigma)$ at **A** on the Crapper's waves of mode 1 branch; $p = 0$ and $b = 0$. In the diagram he horizontal axis represents q and the vertical axis represents a_1, the first Fourier coefficient of θ. **A** shows the onset of solutions which have self-intersections. Profiles of solution are drawn in one wavelength.

2.1 Pure Capillary Waves

We put $p = 0$. This is the case of pure capillary waves. For surface waves of deep water, Crapper ('57) gave the exact solutions, which have explicit expressions in terms of elementary functions.

Figure 1 shows Crapper's waves simulated by our algorithm. The bifurcation diagram is drawn in a_1 versus q and the profiles of $h(x), \theta(\sigma)$ and $S(\sigma)$ are drawn in $-\pi \le \sigma \le \pi$. Though there is shown a diagram in $a_1 \ge 0$, there is in $a_1 < 0$ symmetrically with respect to the x-axis. It is a pitchfork bifurcation. The wave profile $h(x)$ of the solution at the point **A** has a contact point as is shown in the Figure. The solutions to the right of **A** contain self-intersections in their profiles, which are unphysical waves but are considered to be valid solutions mathematically. The computations are performed with $N = 127$. The simulated solutions correspond well to the exact solutions.

It can be proved mathematically that there is no secondary bifurcation from the branches of Crapper's waves. Then is Crapper's wave a unique solution? We have to check whether there might be a family of solutions, which are disconnected from Crapper's waves. Concerning the question, we conjecture numerically that there is no solution other than Crapper's waves for $b = 0$. Then how about for $b \ne 0$?

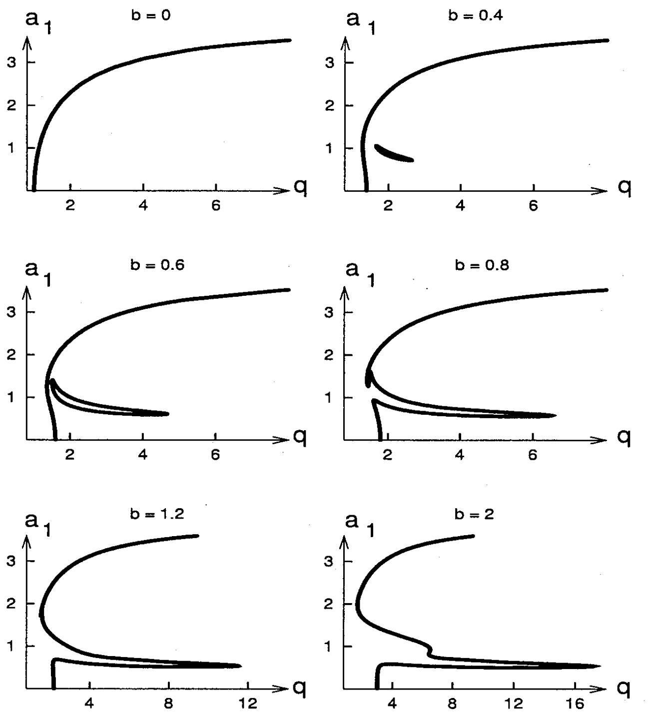

Figure 2: The bifurcation diagrams of pure capillary waves when b varies. The branches are symmetrically in $a_1 < 0$ with respect to the x-axis.

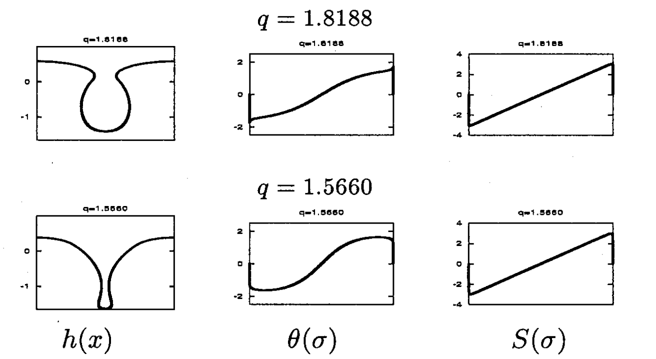

Figure 3: The profiles of solution of pure capillary waves when $b = 1$ and $p = 0$. These are solutions located near turning points. $\theta(\sigma)$ and $S(\sigma)$ have sharp inclinations at $\sigma = \pm\pi$.

Figure 2 shows the bifurcation diagrams when b varies. The figure shows that there is no solution other than a primary branch when $b < b_0 \approx 0.364\cdots$. If $b = b_0$ there appears a looped branch disconnected from a primary branch. And the looped branch touches the primary branch when $b = b_1 \approx 0.767\cdots$. Namely, for $b_0 \leq b < b_1$, we can see another bifurcation solutions than a primary bifurcation.

Examples of solutions are shown in Figure 3 for $b = 1$. They are located on a wedge part of the branch. The computations have been performed with $N = 2047$, but it needs to take larger N or another scheme in order to simulate accurately. $\theta(\sigma)$ and $S(\sigma)$ have sharp inclinations at $\sigma = \pm\pi$ as are shown in the figure.

2.2 Gravity Waves

We put $q = 0$. This is the case of gravity waves.

Figure 4 shows bifurcation diagram of gravity waves of mode 2, which is drawn in a_2 versus p. The primary bifurcation of mode 2 is a pitchfork bifurcation. But a secondary bifurcation branch exists only in $a_2 > 0$. The secondary bifurcation is a pitchfork and is supercritical in p. The secondary branch ends when $h(x)$ has one sharp crest at $x = 0$ and the crest forms a corner of angle $2\pi/3$, namely $\lim_{\sigma\to 0} |\theta(\sigma)| = \pi/6$. The bottom profiles are plots of nearly highest wave on the secondary branch.

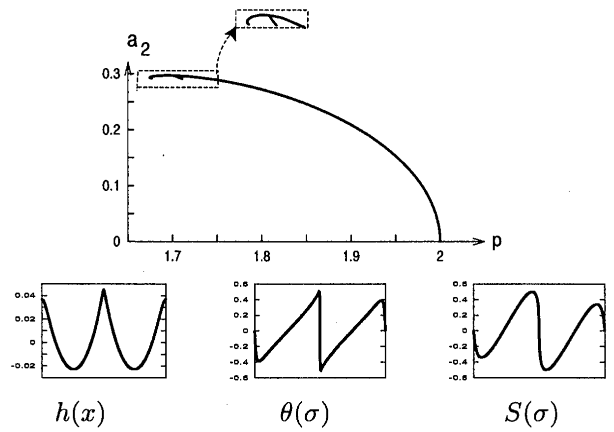

Figure 4: The bifurcation diagram of mode (1,2) and the wave profile of the nearly highest wave on the secondary branch when $b = 0$ and $q = 0$. The primary branch is symmetrically in $a_2 < 0$ with respect to the x-axis, but the secondary branch exists only in $a_2 > 0$.

Figure 5 shows the case of $b = 0.005$. In this case there does not appear sharp crest but a mushroom shaped wave. We guess the secondary branch may continue to be a nearly touching wave profile, but we could not examine it although we increased N up to 3071. As b is larger, it develops an S shape.

2.3 Capillary-Gravity Waves

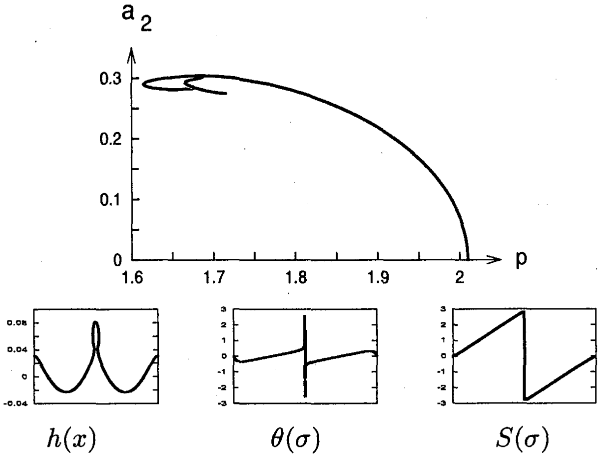

Figure 5: The bifurcation diagram of mode (1,2) and the wave profile of the nearly highest wave on the secondary branch when $b = 0.005$ and $q = 0$.

Here we consider the cases of $p \neq 0$ and $q \neq 0$. In this case there exist double bifurcation points of mode (m, n) which arise from the interaction between the mth and nth harmonics of the motion for $m \neq n$.

Around the double bifurcation, we can see very complicated and interesting structures. In [2,3] we show many kinds of bifurcations around the double bifurcation for the surface waves problem. Applying a bifurcation theory, we calculated the normal form of mode (1,2) and confirmed our numerical results mathematically. We proved in [2] that in the interfacial waves problem, b is the key parameter to give a degenerate bifurcation point of some coefficient in the normal form. When the degenerate bifurcation point appears, much more complicated structures may be possible.

The numerical results of $b = 0$ and $b = 1$ are shown in Figures 6 and 7, respectively. The right diagrams in the figures are views from the q-axis. In the case of $\eta_l = \eta_u = 0$, a degenerate bifurcation point appears when $b = 1$. We see more complicated structures in Figure 7 than in Figure 6. For $b = 0$, $(p, q) = (2/3, 1/3)$ is the double bifurcation point of mode (1,2) and the middle in Figure 6 represents the diagram at the double bifurcation point. When $p = 0.6$, a secondary branch emanates from the upper part of mode 2 primary branch. It constitutes a closed loop to meet the mode 2 branch again. When $p = 0.7$, a secondary branch emanates from the lower part of mode 2 primary branch. Here the mode 1 branch constitutes a closed loop to meet the mode 2 branch.

In Figure 7 there is no closed branch as is in the case of $b = 0$. For $b = 1$, $(p, q) = (4/3, 2/3)$ is the double bifurcation of mode (1,2) and the middle of Figure 7 represents the diagram at the double bifurcation point. When $p = 1.5$ and $p = 1.333$, the primary mode 1 branch forms closed loop but it meets mode 4 branch. When $p = 1.26$, the secondary branch emanating from the lower part of mode 2 branch forms closed loop but it meets mode 3 branch.

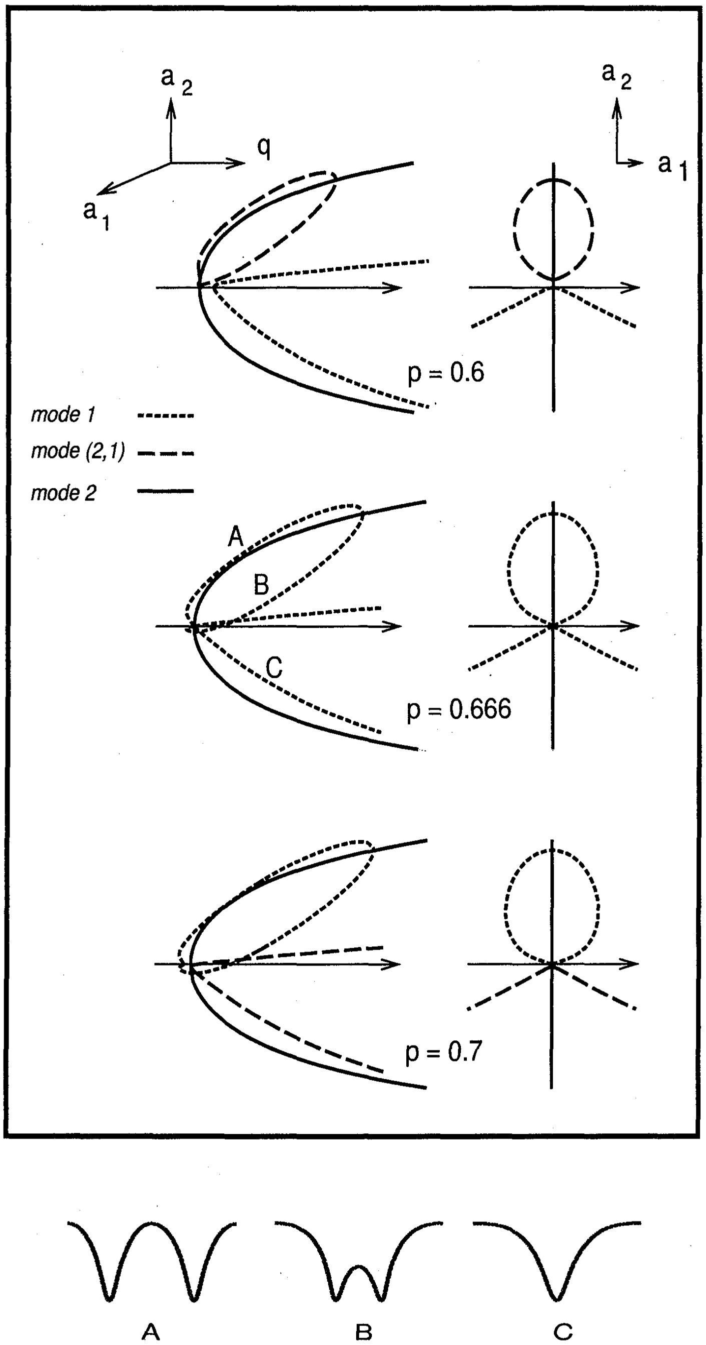

Figure 6: The bifurcation diagrams around the double bifurcation of mode (1,2) when $b = 0$. When $p = 0.6$, a secondary bifurcation branch emanates from the upper part of the mode 2 primary branch. When $p = 0.7$, a secondary bifurcation branch emanates from the lower part of the mode 2 primary branch. The bottom figures are some wave profiles for $p = 0.666$.

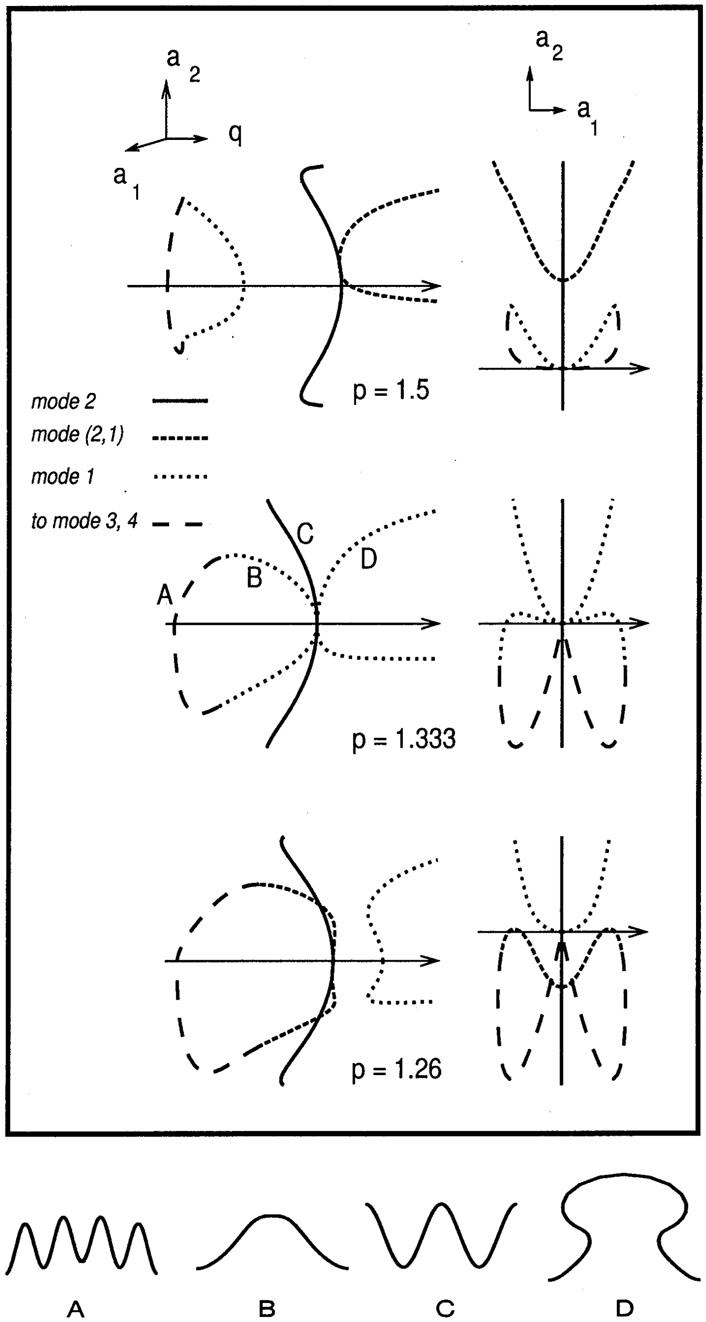

Figure 7: The bifurcation diagrams around the double bifurcation of mode (1,2) when $b = 1$. When $p = 1.5$ and $p = 1.333$, the left branch meets mode 4 branch. When $p = 1.26$, the left branch meets mode 3 branch. The bottom figures are some wave profiles for $p = 1.5$.

3. Concluding remarks

We have examined bifurcation structures varying either or both of η_u and η_l in $[0, 1)$. Our computation shows that the aspect ratio of the flow, i.e., the depths of the upper and the lower fluid, play little role in the bifurcation structure qualitatively.

It is proved that there exist triple bifurcation points in the interfacial waves problem, but they all exist in unphysical range $b < 0$. However the problem has an equally valid mathematical meaning even for negative b. We think it is important to study bifurcation structures around triple bifurcation points. Because Zufiria [4,5] obtained numerically non-symmetric surface waves which arise from the interaction of three modes. They seem to strongly suggest the existence of a triple bifurcation point of mode (1,3,6), though it was proven there exist no triple bifurcation point in the surface wave problem. It might be possible to interpret Zufiria's bifurcations as the effect of the triple bifurcation of interfacial waves, since the surface wave problem is embedded in the interfacial problem. It would be interesting to study structures around the triple bifurcation.

[References]

[1] N. Kotchine: "Détermination rigoroureuse de ondes permanentes d'ampleur finie à la surface de séparation deux liquides de profondeur finie", Math. Ann., vol 98 (1928), pp. 582–615.

[2] H.Okamoto and M.Shōji: "The mathematical theory of permanent progressive water-waves", Advanced Series in Nonlinear Dynamics vol.20, World Scientific, 2001.

[3] H.Okamoto and M.Shōji: "The resonance of modes in the problem of capillary gravity waves", Physica D. vol.95, 336–350, 1996.

[4] J.A.Zufiria: "Weakly nonlinear non-symmetric gravity waves on water of finite depth", J. Fluid Mech. vol. 180, 371–385, 1987.

[5] J.A.Zufiria: "Non-symmetric gravity waves on water of infinite depth", J. Fluid Mech. vol. 181, 17–39, 1987.