ESTIMATES OF THE BESOV NORMS ON FRACTAL LATERAL BOUNDARY BY VOLUME INTEGRALS

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ABSTRACT. Consider a bounded domain D with fractal boundary in $\mathbf{R}^d$ such that ∂D is a β-set ($d-1 \leq \beta < d$). Under an additional condition we estimate the L^p-norm and the Besov norms with respect to the parabolic metric on the lateral boundary of the cylinder $D \times (0,T)$ by the L^p-norm and the Besov norms defined by the volume integrals, respectively.

1. Introduction.

Let D be a domain in $\mathbf{R}^d$ such that the boundary ∂D is compact and assume that ∂D is a β-set ($d-1 \leq \beta < d$), i.e., there exist a positive Radon measure μ on ∂D and positive real numbers b_1, b_2, r_0 such that

$$b_1 r^\beta \leq \mu(B(z,r) \cap \partial D) \leq b_2 r^\beta \tag{1.1}$$

for all points $z \in \partial D$ and all positive real numbers $r \leq r_0$, where $B(z,r)$ stands for the open ball in $\mathbf{R}^d$ with centered at z and radius r. Such a measure μ is called a β-measure.

We fix a β-measure μ on ∂D and consider the cylinder with respect to the domain D

$$\Omega_D = D \times (0,T)$$

and the lateral boundary

$$S_D = \partial D \times [0,T].$$

The parabolic metric is very useful to estimate the norms of operators with respect the heat kernel in this cylinder. Recall that the parabolic metric $\rho(X,Y)$ between two points $X=(x,t)$ ($x \in \mathbf{R}^d$, $t \in \mathbf{R}$) and $Y=(y,s)$ ($y \in \mathbf{R}^d$, $s \in \mathbf{R}$) is defined by

$$\rho(X,Y) = (|x-y|^2 + |t-s|)^{1/2}$$

and the heat kernel

$$W(X-Y) = \begin{cases} \frac{\exp\left(-\frac{|x-y|^2}{4(t-s)}\right)}{(4\pi(t-s))^{d/2}} & \text{if } t-s>0 \\ 0 & \text{otherwise.} \end{cases}$$

We detote by μ_T the product measure of the β-measure μ and the 1-dimensional Lebesgue measure restricted to [0,T]. Let $p \geq 1$, $\alpha > 0$. We denote by $L^p(\mu_T)$ the set of all L^p-functions defined on S_D with respect to μ_T and by $\Lambda^p_\alpha(S_D)$ the Banach space of all functions in $L^p(\mu_T)$ such that

$$\iint \frac{|f(X)-f(Y)|^p}{\rho(X,Y)^{\beta+2+p\alpha}} d\mu_T(X) d\mu_T(Y) < \infty.$$

For $f \in \Lambda^p_\alpha(S_D)$ a Besov norm is defined by

$$\|f\|_{\alpha,p} = \left(\int |f(X)|^p d\mu_T(X)\right)^{1/p} + \left(\iint \frac{|f(X)-f(Y)|^p}{|\rho(X,Y)|^{\beta+2+p\alpha}} d\mu_T(X)d\mu_T(Y)\right)^{1/p}.$$

For such a cylinder Ω_D volume integrals are more easy to deal with than integrals on S_D, if f is defined on $\overline{D} \times [0,T]$. It seems that, if we find a norm defined by a volume integral "equivalent" to the L^p-norm or the Besov norm on the fractal boundary, it is useful for us to prove that operators are bounded on $L^p(\mu_T)$ or $\Lambda^p_\alpha(S_D)$.

By the same method of A. Jonsson and H. Wallin [JW1] we can extend the function f on S_D to $\mathcal{E}(f)$ on $\mathbf{R}^d \times [0,T]$ such that the function $X \mapsto \mathcal{E}(f)(X)$ is a C^1-function on $\mathbf{R}^{d+1} \setminus (\partial D \times \mathbf{R})$ by using a Whitney decomposition of a parabolic type..

But, what type of a volume integral in $D \times (0,T)$ is equivalent to the L^p-norm or the Besov norm on S_D? In this paper we consider this problem.

Hereafter we suppose that $\partial D \subset B(0,R/2)$ with $R \geq 1$. We may assume that (1.1) holds for all points $z \in \partial D$ and all positive real numbers $r \leq 3R$.

To consider the above problem, we need add a condition to D. We say that a set G satisfies *the condition (b)* if there exist positive real numbers c and $r_1 > 0$ such that

$$|B(z,r) \cap G| \geq cr^d \tag{1.2}$$

for each point $z \in \partial G$ and each positive real number $r \leq r_1$, where $|A|$ stands for the d-dimensional (resp. the $(d+1)$-dimensional Lebesgue measure) if $A \subset \mathbf{R}^d$ (resp. $A \subset \mathbf{R}^{d+1}$).

If D satisfies the condition (b), then (1.2) holds for every $r \leq 3R$ by replacing with another constant c.

For $y \in D$ (resp. $Y \in D \times (0,T)$) we denote by $\delta(y)$ (resp. $\delta(Y)$) the d- dimensional Euclidean distance from y to ∂D (resp. the $(d+1)$- dimensional Euclidean distance from Y to S_D). Note that $\delta(y) = \delta(Y)$ for $Y = (y,s)$.

Put, for $r > 0$ and $0 < c < 1$,

$$F_{r,c} = \{Y = (y,s) \in (D \cap B(0,R)) \times (0,T); cr \leq \delta(y) < r\}. \tag{1.3}$$

THEOREM 1. *Let D be a domain in $\mathbf{R}^d$ such that ∂D is a compact β-set $(d-1 \leq \beta < d)$ and satisfies the condition* (b). *Then there is a constant $c_0 < 1$ such that, if f is a nonnegative, uniformly continuous function on $\overline{D \cap B(0,R)} \times [0,T]$ with respect to the parabolic metric ρ,*

$$c_1 \limsup_{r\to 0} r^{\beta-d} \int_{F_{r,c_0}} f(Y)dY \leq \int_{S_D} f(Z)d\mu_T(Z) \leq c_2 \liminf_{r\to 0} r^{\beta-d} \int_{F_{r,c_0}} f(Y)dY, \tag{1.4}$$

where c_1 and c_2 are constants independent of f.

THEOREM 2. *Suppose that D satisfies the same conditions as in Theorem 1. Let $1 \leq p < \infty$, $p - p\alpha - d + \beta > 0$ and $\alpha + (d-\beta)/p < \lambda < 1$. If f is λ-Hölder continuous on $\overline{D \cap B(0,R)} \times [0,T]$ with respect to the parabolic metric, then, for the positive number c_0 in Theorem 1,*

$$\begin{aligned} &c_1 \limsup_{r\to 0} \int_{F_{r,c_0}} \int_{F_{r,c_0}} \frac{|f(X)-f(Y)|^p}{\rho(X,Y)^{d+2+p\alpha+d-\beta}} dXdY \\ &\leq \int_{S_D} \int_{S_D} \frac{|f(X)-f(Y)|^p}{\rho(X,Y)^{\beta+2+p\alpha}} d\mu_T(X)d\mu_T(Y) \\ &\leq c_2 \liminf_{r\to 0} \int_{F_{r,c_0}} \int_{F_{r,c_0}} \frac{|f(X)-f(Y)|^p}{\rho(X,Y)^{d+2+p\alpha+d-\beta}} dXdY, \end{aligned} \tag{1.5}$$

and

$$c_3 \limsup_{r\to 0} \int_{F_{r,c_0}} \int_{S_D} \frac{|f(X)-f(Y)|^p}{\rho(X,Y)^{d+2+p\alpha}} d\mu_T(Y)dX \tag{1.6}$$
$$\leq \int_{S_D}\int_{S_D} \frac{|f(X)-f(Y)|^p}{\rho(X,Y)^{\beta+2+p\alpha}} d\mu_T(X)d\mu_T(Y)$$
$$\leq c_4 \liminf_{r\to 0} \int_{F_{r,c_0}} \lim_{S_D} \frac{|f(X)-f(Y)|^p}{\rho(X,Y)^{d+2+p\alpha}} d\mu_T(Y)dX,$$

where c_1, c_2, c_3 and c_4 are constants independent of f.

2. Fundamental lemmas

Hereafter we assume that D is a domain in $\mathbf{R}^d$ such that ∂D is a compact β-set satisfying $d-1 \leq \beta < d$ and $\partial D \subset B(0,R/2)$ $(R \geq 1)$ and $T \leq R$. Note that we may assume that $r_0 \geq 3R$ in (1.1) by taking other constants b_1 and b_2.

In this section we prepare several fundamental lemmas.

Denote by the parabolic ball $\mathcal{B}(X,r)$ with centered at X and radius r defined by

$$\mathcal{B}(X,r) = \{Y \in \mathbf{R}^{d+1};\, \rho(X,Y) < r\}$$

and the parabolic cylinder $C(X,r)$ with centered at $X=(x,t)$ and radius r defined by

$$C(X,r) = \{Y=(y,s); |x-y| < r,\ |t-s| < r^2\}.$$

Note that

$$C(X, \frac{\sqrt{2}r}{2}) \subset \mathcal{B}(X,r) \subset C(X,r). \tag{2.1}$$

The following lemma will be shown by the same method as in the proof of the Vitali covering lemma.

LEMMA 2.1. *Suppose that $\{C(X_j,r_j)\}_{j=1}^m$ are parabolic cylinders. Then we can choose a subsequence $\{C(X_{j_k},r_{j_k})\}_{k=1}^l$ of $\{C(x_j,r_j)\}_{j=1}^m$ such that they are mutually disjoint and*

$$\cup_{j=1}^m C(X_j,r_j) \subset \cup_{k=1}^l C(X_{j_k}, 3r_{j_k}).$$

The following lemma is an easy consequence of the property (1.1).

LEMMA 2.2. *Let $Z \in S_D$ and $0 < r \leq 3R$. Then*

$$b_3 r^{\beta+2} \leq \mu_T(\mathcal{B}(Z,r)\cap S_D) \leq b_4 r^{\beta+2}, \tag{2.2}$$

where b_3 and b_4 are constants independent of Z and r.

LEMMA 2.3. *Let $\lambda > 0$ and $Z \in S_D$. Further, let b_4 be the positive real number in (2.2).*
(i) *If $\beta+2 < \lambda$ and $3R \geq a > 0$, then*

$$\int_{S_D\cap\{a\leq\rho(X,Z)\}} \rho(X,Z)^{-\lambda} d\mu_T(X) \leq b_4 \frac{\lambda}{\lambda-\beta-2} a^{\beta+2-\lambda}.$$

(ii) *If* $\beta+2>\lambda>0$ *and* $0<b\le 3R$, *then*

$$\int_{S_D\cap\{\rho(X,Z)\le b\}}\rho(X,Z)^{-\lambda}d\mu_T(X)\le b_4\frac{\lambda}{\beta+2-\lambda}b^{\beta+2-\lambda}.$$

PROOF. (i) By (2.2) we have

$$\begin{aligned}&\int_{S_D\cap\{a<\rho(X,Z)\}}\rho(X,Z)^{-\lambda}d\mu_T(X)\\&=\int_0^{a^{-\lambda}}\mu_T(\{X\in S_D;\rho(X,Z)^{-\lambda}>t\})dt=\int_0^{a^{-\lambda}}\mu_T(\mathcal{B}(Z,t^{-1/\lambda}))\cap S_D)dt\\&\le b_4\int_0^{a^{-\lambda}}t^{-(\beta+2)/\lambda}dt=\frac{b_4\lambda}{\lambda-\beta-2}a^{\beta+2-\lambda},\end{aligned}$$

which shows (i).

(ii) This is shown by the same method as (i).

□

LEMMA 2.4. *Let* $X\in(D\cap B(0,R))\times(0,T)$ *and* $0<r\le R$. *Then*

$$|\mathcal{B}(X,r)\cap((D\cap B(0,R))\times[0,T])|\le s_2r^{d+2}. \tag{2.3}$$

Furthermore, if D *satisfies the condition (b), then*

$$s_1r^{d+1}\le|\mathcal{B}(X,r)\cap((D\cap B(0,R))\times[0,T])|. \tag{2.4}$$

Here s_1, s_2 *are constants independent of* X *and* r.

PROOF. It is clear that (2.3) holds. If D satisfies the condition (b), then, by [W4, Lemma 2.2], there exists a constant c such that

$$cr^d\le|B(x,r)\cap D|$$

for each $x\in D$ and each positive real number $r\le R$. This and (2.2) lead to (2.4).

□

We fix positive real numbers s_1 and s_2 satisfying (2.4) and (2.3), respectively.

Using Lemma 2.4, we can easily show the following lemma by the same method as in the proof of Lemma 2.3.

LEMMA 2.5. *Let* $\lambda>0$ *and* $X\in(D\cap B(0,R))\times(0,T)$.

(i) *If* $\lambda>d+2$ *and* $R\ge a>0$, *then*

$$\int_{(D\cap B(0,R))\times(0,T)\cap\{\rho(X,Y)\ge a\}}\rho(X,Y)^{-\lambda}dY\le s_2\frac{\lambda}{\lambda-d-2}a^{d+2-\lambda}.$$

(ii) *If* $d+2>\lambda$ *and* $0<b\le R$, *then*

$$\int_{(D\cap B(0,R))\times(0,T)\cap\{\rho(X,Y)\le b\}}\rho(X,Y)^{-\lambda}dY\le s_2\frac{\lambda}{d+2-\lambda}b^{d+2-\lambda}.$$

We see that the following lemma holds as in the proofs of [W1, Lemma 2.1 and W2, Lemma 2.1].

LEMMA A. *Suppose that D is a domain such that ∂D is a compact β-set $(d-1 \leq \beta < d)$ and satisfies the condition* (b). *Further assume that $\partial D \subset B(0, R/2)$. Let $0 < \epsilon < R$, $0 < r \leq 3R$ and $z \in \partial D$. Then there exist positive numbers c_1, c_2 such that*

$$c_1 r^\beta \epsilon^{d-\beta} \leq \int_{\{\delta(y)<\epsilon\}\cap B(z,r)} dy \leq c_2 r^\beta \epsilon^{d-\beta},$$

where c_1 and c_2 are independent of r, ϵ and z.

The following lemma is an easy consequence of Lemma A.

LEMMA 2.6. *Suppose that D satisfies the condition (b). Let $0 < \epsilon < R$, $0 < r \leq 3R$ and $Z \in S_D$. Then there exist positive numbers s_3, s_4 such that*

$$s_3 r^{\beta+2} \epsilon^{d-\beta} \leq \int_{\{\delta(Y)<\epsilon\}\cap B(Z,r)\cap(D\times(0,T))} dY \leq s_4 r^{\beta+2} \epsilon^{d-\beta}, \tag{2.5}$$

where s_3 and s_4 are independent of r, ϵ and Z.

3. Estimates of parabolic atoms

Fix a C^∞-function ϕ on $\mathbf{R}^d$ such that

$$\phi = 1 \text{ on } \overline{B(0,1/2)}, \quad 0 \leq \phi \leq 1, \quad \text{supp } \phi \subset B(0,1), \quad \phi(x) = \phi(-x)$$

and a C^∞-function ψ on $\mathbf{R}$ such that

$$\psi = 1 \text{ on } [-\frac{1}{2}, \frac{1}{2}], \quad 0 \leq \psi \leq 1, \quad \text{supp } \psi \subset (-1,1), \quad \psi(t) = \psi(-t).$$

Define, for $X = (x,t) \in \mathbf{R}^{d+1}$ and $r > 0$,

$$g_{X,r}(Y) = \phi(\frac{y-x}{r})\psi(\frac{s-t}{r^2})$$

for $Y = (y,s)$. Note that $g_{X,r} \in C^\infty(\mathbf{R}^{d+1})$, $g_{X,r} = 1$ on $\overline{C(X, r/2)}$ and supp $g_{X,r} \subset C(X,r)$. Furthermore, $|\nabla_y g_{X,r}| \leq c/r$ and $|\frac{\partial}{\partial s} g_{X,r}| \leq \frac{c}{r^2}$, where c is a constant independent of x, r.

We have the following lemma.

LEMMA 3.1. *Let X_0, $Y_0 \in S_D$, $0 < 4r < \rho(X_0, Y_0)$ and a, $b \in \mathbf{R}$. Assume that $p - p\alpha - d + \beta > 0$. Then*

$$\iint \frac{|a(g_{X_0,r}(X) - g_{X_0,r}(Y)) + b(g_{Y_0,r}(X) - g_{Y_0,r}(Y))|^p}{\rho(X,Y)^{\beta+2+p\alpha}} d\mu_T(X) d\mu_T(Y) \leq c(|a|^p + |b|^p) r^{\beta+2-p\alpha}, \tag{3.1}$$

$$\int_{D\times(0,T)} \int_{S_D} \frac{|a(g_{X_0,r}(X) - g_{X_0,r}(Y)) + b(g_{Y_0,r}(X) - g_{Y_0,r}(Y))|^p}{\rho(X,Y)^{d+2+p\alpha}} d\mu_T(Y) dX \leq c(|a|^p + |b|^p) r^{\beta+2-p\alpha} \tag{3.2}$$

and

$$\int_{D\times(0,T)}\int_{D\times(0,T)} \frac{|a(g_{X_0,r}(X)-g_{X_0,r}(Y))+b(g_{Y_0,r}(X)-g_{Y_0,r}(Y))|^p}{\rho(X,Y)^{d+2+p\alpha+d-\beta}}dXdY \tag{3.3}$$
$$\le c(|a|^p+|b|^p)r^{\beta+2-p\alpha}$$

PROOF. Set $X_0=(x_0,t_0)$ and $Y_0=(y_0,s_0)$. To show (3.1) we write

$$\iint \frac{|a(g_{X_0,r}(X)-g_{X_0,r}(Y))+b(g_{Y_0,r}(X)-g_{Y_0,r}(Y))|^p}{\rho(X,Y)^{\beta+2+p\alpha}}d\mu_T(X)d\mu_T(Y)$$
$$\le 2^p\iint \frac{|a(g_{X_0,r}(X)-g_{X_0,r}(Y))|^p}{\rho(X,Y)^{\beta+2+p\alpha}}d\mu_T(X)d\mu_T(Y)$$
$$+2^p\iint \frac{|b(g_{Y_0,r}(X)-g_{Y_0,r}(Y)|^p}{\rho(X,Y)^{\beta+2+p\alpha}}d\mu_T(X)d\mu_T(Y)\equiv I_1+I_2.$$

Then

$$\iint_{\rho(X,Y)<3r} \frac{|a(g_{X_0,r}(X)-g_{X_0,r}(Y))|^p}{\rho(X,Y)^{\beta+2+p\alpha}}d\mu_T(X)d\mu_T(Y)$$
$$\le c_1|a|^p\iint_{\rho(X,Y)<3r} \frac{|g_{X_0,r}(X)-g_{X_0,r}(X')|^p}{\rho(X,Y)^{\beta+2+p\alpha}}d\mu_T(X)d\mu_T(Y)$$
$$+c_1|a|^p\iint_{\rho(X,Y)<3r} \frac{|g_{X_0,r}(X')-g_{X_0,r}(Y)|^p}{\rho(X,Y)^{\beta+2+p\alpha}}d\mu_T(X)d\mu_T(Y)$$
$$\equiv I_{11}+I_{12},$$

where $X=(x,t)$, $Y=(y,s)$ and $X'=(x,s)$. Since

$$|g_{X_0,r}(X)-g_{X_0,r}(X')| \tag{3.4}$$
$$\le \sup_{s\in[0,T]}|\frac{\partial}{\partial s}g_{X_0,r}(x,s)||s-t|\chi_{\{|x-x_0|<r\}}(x)\chi_{\{|s-t_0|<4^2r^2\}}(s)$$
$$\le c_2\frac{|s-t|}{r^2}\chi_{\{|x-x_0|<r\}}(x)\chi_{\{|s-t_0|<4r^2\}}(s)$$

and

$$|g_{X_0,r}(X')-g_{X_0,r}(Y)| \tag{3.5}$$
$$\sup_{x\in\partial D}|\nabla_x g_{X_0,r}(x,s)||x-y|\chi_{\{|x_0-y|<4r\}}(y)\chi_{\{|t_0-s|<r^2\}}(s)$$
$$\le c_3\frac{|x-y|}{r}\chi_{\{|x_0-y|<4r\}}(y)\chi_{\{|t_0-s|<r^2\}}(s),$$

we have, by (2.2) and Lemma 2.3,

$$I_{11}\le c_4\frac{|a|^p}{r^{2p}}\int_{C(X_0,4r)\cap S_D}d\mu_T(X)\int_{\rho(X,Y)<3r}\rho(X,Y)^{-\beta-2-p\alpha+2p}d\mu_T(Y)$$
$$\le c_5|a|^pr^{-2p}(3r)^{2p-p\alpha}(4r)^\beta 4r^2=c_6|a|^pr^{\beta+2-p\alpha}$$

and

$$I_{12}\le c_7|a|^p\frac{1}{r^p}\int_{C(X_0,4r)\cap S_D}d\mu_T(Y)\int_{\rho(X,Y)<3r}\rho(X,Y)^{-\beta-2-p\alpha+p}d\mu_T(X)$$
$$\le c_8|a|^pr^{-p}r^{p-p\alpha}r^\beta r^2=c_9|a|^pr^{\beta+2-p\alpha}.$$

Using Lemma 2.3 again, we also have

$$\begin{aligned}
&\iint_{\rho(X,Y)\geq 3r} \frac{|a(g_{X_0,r}(X)-g_{X_0,r}(Y))|^p}{\rho(X,Y)^{\beta+2+p\alpha}} d\mu_T(X)d\mu_T(Y) \\
&\leq 2^p \iint_{\rho(X,Y)\geq 3r} |a|^p \frac{|g_{X_0,r}(X)|^p}{\rho(X,Y)^{\beta+2+p\alpha}} d\mu_T(X)d\mu_T(Y) \\
&+ 2^p \iint_{\rho(X,Y)\geq 3r} |a|^p \frac{|g_{X_0,r}(Y)|^p}{\rho(X,Y)^{\beta+2+p\alpha}} d\mu_T(X)d\mu_T(Y) \\
&\leq c_{10}|a|^p \int_{C(X_0,r)} d\mu_T(X) \int_{\rho(X,Y)\geq 3r} \rho(X,Y)^{-\beta-2-p\alpha} d\mu_T(Y) \\
&+ c_{10}|a|^p \int_{C(X_0,r)} d\mu_T(Y) \int_{\rho(X,Y)\geq 3r} \rho(X,Y)^{-\beta-2-p\alpha} d\mu_T(X) \\
&\leq c_{11} r^{\beta+2-p\alpha}.
\end{aligned}$$

From these we deduce

$$I_1 \leq c_{12}|a|^p r^{\beta+2-p\alpha}.$$

Similarly we also have

$$I_2 \leq c_{13}|b|^p r^{\beta+2-p\alpha}.$$

Thus we have (3.1).

We next prove (3.2). To do so, we write

$$\begin{aligned}
&\int_{D\times(0,T)} \int_{S_D\cap\{\rho(X,Y)<3r\}} \frac{|a(g_{X_0,r}(X)-g_{X_0,r}(Y))|^p}{\rho(X,Y)^{d+2+p\alpha}} d\mu_T(Y)dX \\
&\leq 2^p|a|^p \int_{D\times(0,T)} \int_{S_D\cap\{\rho(X,Y)<3r\}} \frac{|(g_{X_0,r}(X)-g_{X_0,r}(X')|^p}{\rho(X,Y)^{d+2+p\alpha}} d\mu_T(Y)dX \\
&+ 2^p|a|^p \int_{D\times(0,T)} \int_{S_D\cap\{\rho(X,Y)<3r\}} \frac{|g_{X_0,r}(X')-g_{X_0,r}(Y)|^p}{\rho(X,Y)^{d+2+p\alpha}} d\mu_T(Y)dX \\
&\equiv I_{31} + I_{32}.
\end{aligned}$$

Noting that $p - p\alpha - d + \beta > 0$ and using (3.4) and Lemma 2.3, we have

$$\begin{aligned}
I_{31} &\leq c_{14}|a|^p r^{-2p} \int_{C(X_0,4r)} dX \int_{\{\rho(X,Y)<3r\}\cap S_D} \rho(X,Y)^{-d-2-p\alpha+2p} d\mu_T(Y) \\
&\leq c_{15}|a|^p r^{\beta+2-p\alpha}.
\end{aligned}$$

Similarly we have, by (3.5) and Lemma 2.5,

$$\begin{aligned}
I_{32} &\leq c_{16}|a|^p r^{-p} \int_{C(X_0,4r)\cap S_D} d\mu_T(Y) \int_{\rho(X,Y)<3r} \rho(X,Y)^{-d-2-p\alpha+p} dX \\
&\leq c_{17}|a|^p r^{\beta+2-p\alpha}.
\end{aligned}$$

We also have, by Lemmas 2.3 and 2.5,

$$\begin{aligned}
&\int_{D\times(0,T)}\int_{S_D\cap\{\rho(X,Y)\geq 3r\}}\frac{|a(g_{X_0,r}(X)-g_{X_0,r}(Y))|^p}{\rho(X,Y)^{d+2+p\alpha}}d\mu_T(Y)dX\\
&\leq \int_{D\times(0,T)}\int_{S_D\cap\{\rho(X,Y)\geq 3r\}}\frac{|a|^p|g_{X_0,r}(X)|^p}{\rho(X,Y)^{d+2+p\alpha}}d\mu_T(Y)dX\\
&+\int_{D\times(0,T)}\int_{S_D\cap\{\rho(X,Y)\geq 3r\}}\frac{|a|^p|g_{X_0,r}(Y)|^p}{\rho(X,Y)^{d+2+p\alpha}}d\mu_T(Y)dX\\
&\leq c_{18}|a|^p\int_{C(X_0,r)}dX\int_{\{\rho(X,Y)\geq 3r\}\cap S_D}\frac{1}{\rho(X,Y)^{d+2+p\alpha}}d\mu_T(Y)\\
&+c_{18}|a|^p\int_{C(X_0,r)\cap S_D}d\mu_T(Y)\int_{\rho(X,Y)\geq 3r}\frac{1}{\rho(X,Y)^{d+2+p\alpha}}dX\\
&\leq c_{19}|a|^p r^{\beta+2-p\alpha}.
\end{aligned}$$

Thus we have (3.2). Similarly we can also show (3.3) by Lemma 2.5. □

LEMMA 3.2. *Suppose that D satisfies the condition (b). Let $0<r\leq R$, $0<a\leq R$ and $Z\in S_D$. Then there is a positive real number c_0 such that*

$$|\mathcal{B}(Z,a)\cap F_{r,c_0}|\geq s_5 a^{\beta+2}r^{d-\beta}\quad \textit{and}\ |\mathcal{B}(Z,a)\cap F_{r,c_0}|\leq s_6 a^{\beta+2}r^{d-\beta}. \tag{3.6}$$

PROOF. With the aid of Lemma 2.6 we have, for $c>0$,

$$\begin{aligned}
&|\mathcal{B}(Z,a)\cap F_{r,c}|\\
&=|\mathcal{B}(Z,a)\cap\{Y\in\Omega_D;\delta(Y)<r\}|-|\mathcal{B}(Z,a)\cap\{Y\in\Omega_D;\delta(Y)<cr\}|\\
&\geq s_3a^{\beta+2}r^{d-\beta}-s_4a^{\beta+2}(cr)^{d-\beta}=(s_3-c^{d-\beta}s_4)a^{\beta+2}r^{d-\beta},
\end{aligned}$$

where s_3, s_4 are constants in (2.5). If we set

$$c_0=\frac{1}{2}\left(\frac{s_3}{s_4}\right)^{1/(d-\beta)}<1,$$

then

$$|\mathcal{B}(Z,a)\cap F_{r,c_0}|\geq c_1a^{\beta+2}r^{d-\beta},$$

which is the first inequality.

We next show the second inequality. By (2.5) we have

$$|\mathcal{B}(Z,a)\cap F_{r,c_0}|\leq s_4a^{\beta+2}r^{d-\beta}-s_3c_0^{d-\beta}a^{\beta+2}r^{d-\beta}=(s_4-s_3c_0^{d-\beta})a^{\beta+2}r^{d-\beta}.$$

Since $s_4-s_3c_0^{d-\beta}>0$, we have the conclusion. □

Set

$$t_1=2\max\{1,\frac{2}{3}\left(\frac{b_4}{b_3}\right)^{1/(\beta+2)},\ \frac{2}{3}\left(\frac{s_6}{s_5}\right)^{1/(\beta+2)}\}, \tag{3.7}$$

where b_3, b_4 and s_5, s_6 are constants in (2.2) and (3.6), respectively. Then we have

LEMMA 3.3. *Suppose that D satisfies the condition (b). Let $R/(4t_1) \geq r > 0$, $a, b \in \mathbf{R}$, $Z_1, Z_2 \in S_D$ and $\rho(Z_1, Z_2) \geq 4rt_1$ Then*

(i)

$$\int_{F_{r,c_0}} \int_{F_{r,c_0}} \frac{|a\chi_{C(Z_1,r)}(X) - b\chi_{C(Z_2,r)}(Y)|^p}{\rho(X,Y)^{d+2+p\alpha+d-\beta}} dX dY \geq cr^{\beta-p\alpha}. \tag{3.8}$$

(ii)

$$\int_{F_{r,c_0}} \int_{S_D} \frac{|a\chi_{C(Z_1,r)}(X) - b\chi_{C(Z_2,r)}(Y)|^p}{\rho(X,Y)^{d+2+p\alpha}} d\mu_T(Y) dX \geq cr^{\beta-p\alpha}. \tag{3.9}$$

(iii)

$$\int_{S_D} \int_{S_D} \frac{|a\chi_{C(Z_1,r)}(X) - b\chi_{C(Z_2,r)}(Y)|^p}{\rho(X,Y)^{\beta+2+p\alpha}} d\mu_T(X) d\mu_T(Y) \geq cr^{\beta-p\alpha}. \tag{3.10}$$

Here c is a constant independent of Z_1, Z_2 and r.

PROOF. (i) Note that

$$\begin{aligned}
&\int_{F_{r,c_0}} dY \int_{F_{r,c_0}} \frac{|a\chi_{C(Z_1,r)}(X) - b\chi_{C(Z_2,r)}(Y)|^p}{\rho(X,Y)^{d+p\alpha+d-\beta}} dX dY \\
&\geq |a|^p \int_{F_{r,c_0}} \chi_{C(Z_1,r)}(X) dX \int_{F_{r,c_0} \cap \{\rho(Z_2,Y) > \sqrt{2}r\}} \frac{1}{\rho(X,Y)^{d+2+p\alpha+d-\beta}} dY \\
&+ |b|^p \int_{F_{r,c_0}} \chi_{C(Z_2,r)}(Y) dY \int_{F_{r,c_0} \cap \{\rho(Z_1,X) > \sqrt{2}r\}} \frac{1}{\rho(X,Y)^{d+2+p\alpha+d-\beta}} dx \\
&\equiv I_{11} + I_{12}.
\end{aligned}$$

Since $X \in C(Z_1, r)$ and $2r < \rho(Z_1, Y) < 3rt_1$ imply $\rho(Z_2, Y) > rt_1 > \sqrt{2}r$ and $\rho(X, Y) < (\sqrt{2}/2 + 1)\rho(Z_1, Y)$, we have, by Lemma 2.6,

$$\begin{aligned}
I_{11} &\geq |a|^p \int_{F_{r,c_0}} \chi_{C(Z_1,r)}(X) dX \\
&\times \int_{F_{r,c_0} \cap \{2r < \rho(Z_1,Y) < 3rt_1\}} (\frac{\sqrt{2}}{2} + 1)^{-d-2-p\alpha-d+\beta} \frac{1}{\rho(Z_1,Y)^{d+2+p\alpha+d-\beta}} dY \\
&\geq c_1 |a|^p (3rt_1)^{-d-2-p\alpha-d+\beta} |C(Z_1,r) \cap F_{r,c_0}| |F_{r,c_0} \cap (\mathcal{B}(Z_1, 3rt_1) \setminus \mathcal{B}(Z_1, 2r))|.
\end{aligned}$$

Lemma 3.2 yields

$$|C(Z_1,r) \cap F_{r,c_0}| \geq |\mathcal{B}(Z_1,r) \cap F_{r,c_0}| \geq s_5 r^{\beta+2} r^{d-\beta}.$$

Noting that t_1 is defined by (3.7), we have, by Lemma 3.2,

$$\begin{aligned}
&|\mathcal{B}(Z_1, 3rt_1) \cap F_{r,c_0}| - |\mathcal{B}(Z_1, 2r) \cap F_{r,c_0}| \\
&\geq s_5 (3rt_1)^{\beta+2} r^{d-\beta} - s_6 (2r)^{\beta+2} r^{d-\beta} \geq (3^{\beta+2} t_1^{\beta+2} s_5 - s_6 2^{\beta+2}) r^{d+2}.
\end{aligned}$$

Therefore we have

$$I_{11} \geq c_2 |a|^p r^{\beta+2-p\alpha}.$$

The same estimate is obtained for I_{12}. Thus we see that (3.8) holds.

(iii) Similarly we write

$$\int_{S_D}\int_{S_D} \frac{|a\chi_{C(Z_1,r)}(X) - b\chi_{C(Z_2,r)}(Y)|^p}{\rho(X,Y)^{\beta+2+p\alpha}} d\mu_T(X) d\mu_T(Y)$$

$$\geq |a|^p \int_{S_D} \chi_{C(Z_1,r)}(X) d\mu_T(X) \int_{S_D\cap\{\rho(Z_2,X)>\sqrt{2}r\}} \rho(X,Y)^{-\beta-2-p\alpha} d\mu_T(Y)$$

$$+ |b|^p \int_{S_D} \chi_{C(Z_2,r)}(Y) d\mu_T(Y) \int_{S_D\cap\{\rho(Z,X)>\sqrt{2}r\}} \rho(X,Y)^{-\beta-2-p\alpha} d\mu_T(X)$$

$$\equiv I_{21} + I_{22}.$$

Using Lemma 2.2, we have

$$I_{21} \geq c_3|a|^p \int_{S_D} \chi_{C(Z_1,r)}(X) d\mu_T(X)$$

$$\times \int_{S_D\cap\{2r<\rho(Z_1,Y)<3rt_1\}} (\frac{\sqrt{2}}{2}+1)^{-\beta-2-p\alpha} \rho(Z_1,Y)^{-\beta-2-p\alpha} d\mu_T(Y)$$

$$\geq c_4|a|^p r^{\beta+2}(3t_1r)^{-\beta-2-p\alpha} \mu_T(\mathcal{B}(Z,3rt_1)\setminus\mathcal{B}(Z_1,2r)).$$

Since

$$\mu_T(\mathcal{B}(Z_1,3rt_1)\setminus\mathcal{B}(Z_1,2r)) \geq b_3(3rt_1)^{\beta+2} - b_4(2r)^{\beta+2}$$

$$\geq (3^{\beta+2}t_1^{\beta+2}b_3 - b_4 2^{\beta+2})r^{\beta+2},$$

we have

$$I_{21} \geq c_5|a|^p r^{\beta+2-p\alpha}.$$

We also have the same estimate for I_{22}. Thus we have (3.10).

Similarly we can also show (3.9). □

4. Proofs of Theorem 1 and Theorem 2

In this section we shall prove Theorem 1 and Theorem 2.

PROOF of THEOREM 1. We first prove the second inequality of (1.4). Suppose that f is non-negative and continuous on $\overline{D}\times[0,T]$ with respect to the metric ρ. Since f is uniformly continuous on $\overline{D}\times[0,T]$ with respect to ρ, there is, for each $\epsilon>0$, a positive real number $\delta>0$ such that $\rho(X,Y)<\delta$ implies $|f(X)-f(Y)|<\epsilon$. We consider any positive real number t satisfying $t<\delta/15$. Since

$$S_D \cup \overline{F_{r,c_0}} \subset \cup_{Z\in S_D} C(Z,2r)$$

and $S_D\cup\overline{F_{r,c_0}}$ is compact, there is a subfamily of $\{C(Z,2r)\}_{Z\in S_D}$ which covers $S_D\cup\overline{F_{r,c_0}}$ and consists of finitely many cylinders. Using Lemma 2.1, we can find, $Z_1, Z_2,\cdots,Z_m\in S_D$ such that $\{C(Z_j,2r)\}_{j=1}^m$ is a subfamily of $\{C(Z,2r)\}_{Z\in S_D}$ and $\{C(Z_j,2r)\}_{j=1}^m$ are mutually disjoint and

$$S_D\cup\overline{F_{r,c_0}} \subset \cup_{j=1}^m C(Z_j,6r).$$

Noting that f is also continuous with respect the Euclidian topology, we have

$$\int_{S_D} f(Z)d\mu_T(Z) \leq \sum_{j=1}^m \int_{C(Z_j,6r)\cap S_D} f(Z)d\mu_T(Z)$$

$$\leq c_1\sum_{j=1}^m \max\{f(Z);\, z\in\overline{C(Z_j,6r)}\cap S_D\}(6r)^{\beta+2}$$

$$\leq c_2 r^{\beta-d}\sum_{j=1}^m (\min\{f(Y);\, Y\in\overline{C(Z_j,r)}\cap\overline{F_{r,c_0}}\}+\epsilon)r^{d+2}.$$

Since, by Lemma 3.2,

$$|C(Z_j,2r)\cap F_{r,c_0}|\geq c_3r^{d+2}$$

and $\{C(Z_j,2r)\}_{j=1}^m$ are mutually disjoint, we have

$$\int_{S_D} f(Z)d\mu_T(Z)\leq c_4r^{\beta-d}\int_{F_{r,c_0}}(f(Y)+\epsilon)dY. \tag{4.1}$$

On the other hand we have, by Lemma 2.6,

$$\int_{F_{r,c_0}} dY\leq |C(Z_0,R)\cap F_{r,c_0}|\leq c_5r^{d-\beta}R^{\beta+2},$$

where Z_0 is a fixed point on S_D. This and (4.1) yield

$$\int_{S_D} f(Z)d\mu_T(Z)\leq c_6(r^{\beta-d}\int_{F_{r,c_0}} f(Y)dY+\epsilon).$$

Thus we have the second inequality of (1.4).

We next prove the first inequality of (1.4). Using the above covering, we have, by (2.2),

$$\begin{aligned}
r^{\beta-d}\int_{F_{r,c_0}} f(Y) &\leq r^{\beta-d}\sum_{j=1}^m\int_{C(Z_j,6r)\cap F_{r,c_0}} f(Y)dY\\
&\leq c_7r^{\beta-d}\sum_{j=1}^m \max\{f(Y);\, Y\in\overline{C(Z_j,6r)}\cap\overline{F_{r,c_0}}\}(6r)^{d+2}\\
&\leq c_8\sum_{j=1}^m(\min\{f(Z);\, Z\in\overline{C(Z_j,r)}\cap S_D\}+\epsilon)r^{\beta+2}\\
&\leq c_9\int_{S_D}(f(Z)+\epsilon)d\mu_T(Z)=c_9\Big(\int_{S_D} f(Z)d\mu_T(Z)+\epsilon\mu_T(S_D)\Big).
\end{aligned}$$

This leads the first inequality of (1.4). □

We next prove Theorem 2.

PROOF of THEOREM 2. We first prove (1.5). Choose $\eta>0$ satisfying $(d-\beta)/p+\alpha<\eta<\lambda$ and $\epsilon>0$. Since f is λ-Hölder continuous on $\overline{D}\times[0,T]$ with the parabolic meric ρ, we can find $t_0>0$ such that

$$\rho(X,Y)<t_0 \text{ implies } \frac{|f(X)-f(Y)|}{\rho(X,Y)^\eta}<\epsilon$$

and $t_0\leq R$.

Consider any positive real number r satisfying $r<t_0/(80t_1)$, where t_1 is the positive real number defined by (3.7). Since the set $S_D\cup\overline{F_{r,c_0}}$ is compact, we cover

$$S_D\cup\overline{F_{r,c_0}}\subset\cup_{k=1}^m C(Y_k,2r),$$

where $Y_k\in S_D$. Using Lemma 2.1, we can find a subfamily $\{C(Z_j,2r)\}$ of $\{C(Y_k,2r)\}$ such that $\{C(Z_j,2r)\}$ are mutually disjoint and

$$S_D\cup\overline{F_{r,c_0}}\subset\cup_j C(Z_j,6r).$$

Using the family, we define functions $\{v_{i,j}\}$ on $\mathbf{R}^{d+1} \times \mathbf{R}^{d+1}$ as follows. If $\rho(Z_i, Z_j) \leq 24rt_1$, then $v_{i,j}(X,Y) \equiv 0$. If $\rho(Z_i, Z_j) > 24rt_1$, then we define

$$v_{i,j}(X,Y) = f(Z_i)(g_{Z_i,12r}(X) - g_{Z_i,12r}(Y)) + f(Z_j)(g_{Z_j,12r}(X) - g_{Z_j,12r}(Y)).$$

Let $(X,Y) \in \left(C(Z_i,6r) \cap (\overline{D} \times [0,T])\right) \times \left(C(Z_j,6r) \cap (\overline{D} \times [0,T])\right)$. If $\rho(Z_i, Z_j) > 24rt_1$, we have

$$\begin{aligned}
&|v_{i,j}(X,Y) - (f(X) - f(Y))| \\
&\leq |f(Z_i)g_{Z_i,12r}(X) - f(X)| + |f(Z_j)g_{Z_j,12r}(Y) - f(Y)| \\
&= |f(Z_i) - f(X)| + |f(Z_j) - f(Y)| \\
&\leq \epsilon\rho(Z_i,X)^\eta + \epsilon\rho(Z_j,Y)^\eta \leq 2\epsilon(6\sqrt{2}r)^\eta < 2\epsilon\rho(X,Y)^\eta.
\end{aligned}$$

If $\rho(Z_i, Z_j) \leq 24rt_1$, then

$$|v_{i,j}(X,Y) - (f(X) - f(Y))| = |f(X) - f(Y)| < \epsilon\rho(X,Y)^\eta.$$

We also define functions $\{w_{ij}\}$ on $\mathbf{R}^{d+1} \times \mathbf{R}^{d+1}$ as follows. If $\rho(Z_i, Z_j) > 4rt_1$, we define

$$w_{ij}(X,Y) = f(Z_i)\chi_{C(Z_i,2r)}(X) - f(Z_j)\chi_{C(Z_j,2r)}(Y).$$

Then we can also estimate

$$|w_{ij}(X,Y) - (f(X) - f(Y))| < c_1\epsilon\rho(X,Y)^\eta$$

for each pair $(X,Y) \in (C(Z_i,r) \cap (\overline{D} \times [0,T])) \times (C(Z_j,r) \cap (\overline{D} \times [0,T]))$.

We note that each $X \in \overline{D} \times [0,T]$ belongs to at most N many numbers of $\{C(Z_i,6r)\}$, where N is a constant depending only on d. Hence

$$\begin{aligned}
I_1 &\equiv \iint \frac{|f(X) - f(Y)|^p}{\rho(X,Y)^{\beta+2+p\alpha}} d\mu_T(X) d\mu_T(Y) \\
&\leq \sum_{i,j} \int_{C(Z_i,6r)\cap S_D} \int_{C(Z_j,6r)\cap S_D} \frac{|f(X) - f(Y)|^p}{\rho(X,Y)^{\beta+2+p\alpha}} d\mu_T(X) d\mu_T(Y) \\
&\leq c_2 \sum_{i,j} \iint \frac{|v_{i,j}(X,Y)|^p}{\rho(X,Y)^{\beta+2+p\alpha}} d\mu_T(X) d\mu_T(Y) \\
&+ c_2\epsilon^p \iint \rho(X,Y)^{-\beta-2-p\alpha+p\eta} d\mu_T(X) d\mu_T(Y) \\
&\leq c_2 \sum_{i,j} \iint \frac{|v_{i,j}(X,Y)|^p}{\rho(X,Y)^{\beta+2+p\alpha}} d\mu_T(X) d\mu_T(Y) + c_3\epsilon^p.
\end{aligned}$$

Noting that $v_{ij}(X,Y) = 0$ if $\rho(Z_i, Z_j) \leq 24rt_1$ and $\{C(Z_i,2r)\}$ are mutually disjoint, we have

$$\begin{aligned}
&\sum_{i,j} \iint \frac{|v_{i,j}(X,Y)|^p}{\rho(X,Y)^{\beta+2+p\alpha}} d\mu_T(X) d\mu_T(Y) \\
&\leq c_4 \sum_i |f(Z_i)|^p \int_{S_D\cap C(Z_i,6r)} d\mu_T(X) \int_{\rho(X,Y)>12rt_1} \rho(X,Y)^{-\beta-2-p\alpha} d\mu_T(Y) \\
&+ c_4 \sum_j |f(Z_j)|^p \int_{S_D\cap C(Z_j,6r)} d\mu_T(Y) \int_{\rho(X,Y)>12rt_1} \rho(X,Y)^{-\beta-2-p\alpha} d\mu_T(X) \\
&\leq c_5 r^{\beta+2-p\alpha} \sum_i |f(Z_i)|^p,
\end{aligned}$$

whence

$$I_1 \le c_6 \sum_i{}' \left(|f(Z_i)|^p\right) r^{\beta+2-p\alpha} + c_6 \epsilon^p. \tag{4.2}$$

Let $C(Z_i, r)$ be one of parabolic cylinders $\{C(Z_j, r)\}$ and denote by Z_{j_i} one of the points of the centers of $\{C(Z_j, r)\}$ such that $\rho(Z_i, Z_{j_i}) > 4rt_1$ and $\rho(Z_i, Z_j) \ge \rho(Z_i, Z_{j_i})$ for each center Z_j satisfying $\rho(Z_i, Z_j) > 4rt_1$. Since $\{C(Z_j, 6r)\}$ is a covering of $\overline{F_{r,c_0}}$, we see that $\rho(Z_i, Z_{j_i}) \le t_2 r$, where t_2 is a constant independent of r and i. Therefore, if $X \in C(Z_i, r)$ and $Y \in C(Z_{j_i}, r)$, then $\rho(X, Y) \le t_3 r$, where t_3 is a constant independent of r and i. Consequently we have, by Lemma 3.2,

$$\begin{aligned}
&\int_{F_{r,c_0}} \int_{F_{r,c_0}} \frac{|f(X) - f(Y)|^p}{\rho(X,Y)^{d+2+p\alpha+d-\beta}} dX dY \\
&\ge \sum_{i,j} \int_{F_{r,c_0} \cap C(Z_i,r)} \int_{F_{r,c_0} \cap C(Z_j,r)} \frac{|f(X) - f(Y)|^p}{\rho(X,Y)^{d+2+p\alpha+d-\beta}} dX dY \\
&\ge c_7 \sum_{i,j} \int_{F_{r,c_0} \cap C(Z_i,r)} \int_{F_{r,c_0} \cap C(Z_j,r)} \frac{|w_{ij}(X,Y)|^p}{\rho(X,Y)^{d+2+p\alpha+d-\beta}} dX dY \\
&- c_8 \epsilon^p \int_{C(Z_0,R)\cap D} \int_{C(Z_0,R)\cap D} \rho(X,Y)^{-d-2-p\alpha-d+\beta+\eta} dX dY \\
&\ge c_9 \sum_i \int_{F_{r,c_0} \cap C(Z_i,r)} |f(Z_i)|^p dX \int_{F_{r,c_0} \cap C(Z_{j_i},r)} \rho(X,Y)^{-d-2-p\alpha-d+\beta} dY \\
&+ c_9 \sum_j \int_{F_{r,c_0} \cap C(Z_j,r)} |f(Z_j)|^p dY \int_{F_{r,c_0} \cap C(Z_{i_j},r)} \rho(X,Y)^{-d-2-p\alpha-d+\beta} dX - c_9 \epsilon^p \\
&\ge c_{10} \sum_i \left(|f(Z_i)|^p\right) r^{\beta+2-p\alpha} - c_{11} \epsilon^p,
\end{aligned} \tag{4.3}$$

where Z_0 is some point of S_D. Combining (4.2) with (4.3), we have the second inequality of (1.5).

We next show the first inequality of (1.5). Since

$$\begin{aligned}
I_2 &\equiv \int_{F_{r,c_0}} \int_{F_{r,c_0}} \frac{|f(X) - f(Y)|^p}{\rho(X,Y)^{d+2+p\alpha+d-\beta}} dX dY \\
&\le c_{12} \sum_{i,j} \int_{F_{r,c_0} \cap C(Z_i,6r)} \int_{F_{r,c_0} \cap C(Z_j,6r)} \frac{|v_{i,j}(X,Y)|^p}{\rho(X,Y)^{d+2+p\alpha+d-\beta}} dX dY \\
&+ c_{12} \epsilon^p \int_{C(Z_0,R)} \int_{C(Z_0,R)} \rho(X,Y)^{-d-2-p\alpha+d+\beta+p\eta} dX dY,
\end{aligned}$$

we have, by the above methods,

$$I_2 \le c_{13} \sum_i \left(|f(Z_i)|^p|^p\right) r^{\beta+2-p\alpha} + c_{13} \epsilon^p.$$

On the other hand, we also have

$$
\begin{aligned}
&\int_{S_D}\int_{S_D} \frac{|f(X)-f(Y)|^p}{\rho(X,Y)^{\beta+2+p\alpha}} d\mu_T(X) d\mu_T(Y) \\
&\geq \sum_{i,j} \int_{S_D\cap C(Z_i,r)} \int_{S_D\cap C(Z_j,r)} \frac{|f(X)-f(Y)|^p}{\rho(X,Y)^{\beta+2+p\alpha}} d\mu_T(X) d\mu_T(Y) \\
&\geq c_{14} \sum_{i,j} \int_{S_D\cap C(Z_i,r)} \int_{S_D\cap C(Z_j,r)} \frac{w_{i,j}(X,Y)}{\rho(X,Y)^{\beta+2+p\alpha}} d\mu_T(X) d\mu_T(Y) \\
&- c_{15} \int_{S_D}\int_{S_D} \rho(X,Y)^{-\beta-2-p\alpha+p\eta} d\mu_T(X) d\mu_T(Y) \\
&\geq c_{16} \sum_i \int_{S_D\cap C(Z_i,r)} |f(Z_i)|^p d\mu_T(X) \int_{S_D\cap C(Z_{J_i},r)} \rho(X,Y)^{-\beta-2-p\alpha} d\mu_T(Y) \\
&+ c_{16} \sum_j \int_{S_D\cap C(Z_j,r)} |f(Z_j)|^p d\mu_T(Y) \int_{S_D\cap C(Z_{i_j},r)} \rho(X,Y)^{-\beta-2-p\alpha} d\mu_T(X) - c_{17}\epsilon^p \\
&\geq c_{18} \sum_i (|f(Z_i)|^p) r^{\beta+2-p\alpha} - c_{17}\epsilon^p.
\end{aligned}
$$

Thus we also have the first inequality of (1.5).

Similarly we can prove the inequality (1.6).

□

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