The Hausdorff dimension of generalized Sierpinski carpets

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(Received August 27, 2001)

ABSTRACT

We investigate the Hausdorff dimension of generalized Sierpinski carpets by using thermodynamic formalism.

1 Introduction

The Hausdorff dimension of the generalized cookie-cutter set in $\mathbf{R}$ has been obtained in several ways. In [1] and [3], it is obtained by using the pressure P [4] in thermodynamic formalism. As for the generalized Sierpinski carpets, C. McMulle [2] gave the Hausdorff dimension for the special Sierpinski carpet, which is generated by functions $\{f_i(x) = n(x - \frac{i}{n})\}_{i=0}^{n-1}$ and $\{g_j(x) = m(x - \frac{j}{m})\}_{j=0}^{m-1}$. However, the way to obtain the dimension of the generalized Sierpinski carpets is not known.

In this paper, we investigate a way to obtain the Hausdorff dimension of generalized Sierpinski carpets in $\mathbf{R}^2$ by using thermodynamic formalism. We treat the affine functions $\{g_j(x)\}$ with different derivatives and obtain the Hausdorff dimension of the Sierpinski carpet by using symbolic dynamics (Theorem 3.1) and also by using thermodynamic formalism (Theorem 4.1). To consider the case that the functions $\{f_i(x)\}$ and $\{g_j(x)\}$ are not affine, we give a conjecture to obtain the Hausdorff dimension of the generalized Sierpinski carpet by using thermodynamic formalism.

2 Preliminaries

The generalized cookie-cutter set can be considered as follows:

Let $X = [0, 1]$ and take points $x_0, x_1, \cdots, x_{2r-1}$ in X satisfying

$$0 = x_0 < x_1 < \cdots < x_{2r-1} = 1.$$

Put intervals

$$X_i := [x_{2i}, x_{2i+1}] \ \ (i = 0, \dots, r-1) \quad \text{and } D = \bigcup_{i=0}^{r-1} X_i.$$

Define a continuous function $f : D \to X$ with $|f'(x)| > 1$ for any $x \in D$. Then *a generalized cookie-cutter set* E associated with f is the set

$$E := \bigcap_{k=1}^{\infty} f^{-k}(X). \tag{2.1}$$

Let F_i $(i = 0, \dots, r-1)$ be the branches of the inverse of f, that is,

$$F_i(x) := f^{-1}(x) \cap X_i \quad \text{for } i = 0, \dots, r-1.$$

So F_i maps X bijectively onto X_i.

Let $\phi : D \to \mathbf{R}$ be a Lipschitz function and for $k \in \mathbf{N}$ define

$$S_k\phi(x) = \sum_{j=0}^{k-1} \phi(f^j x)$$

for $x \in D$. It is known [1] that the limit

$$P(\phi) = \lim_{k\to\infty} \frac{1}{k} \log \sum_{x \in \mathrm{Fix} f^k} \exp S_k\phi(x)$$

exists, where $\mathrm{Fix}f^k$ denotes the set of fixed points of f^k. By taking $\phi(x) = -s\log|f'(x)|$ for $x \in D$, the Hausdorff dimension $\dim_H E$ of the cookie-cutter set E is known as follows.

Theorem A. [1] *Let E be the cookie-cutter set defined by* (2.1) *and let s be the unique real number satisfying*

$$P(-s\log|f'|) = 0.$$

Then $\dim_H E = s$.

We will extend this result to the subset of $\mathbf{R}^2$ such as the generalized Sierpinski carpet. If we can choose the suitable function $\phi(x)$ on $\mathbf{R}^2$ corresponding to $\phi(x) = -s\log(x)$ on $\mathbf{R}$, the Hausdorff dimension will be obtained by using the relation $P(\phi) = 0$.

In [2], given $n, m \in \mathbf{N}$ with $n \geq m$ and a subset R of the set $\{(i,j) \in \mathbf{N}\times\mathbf{N} : 0 \leq i < n, 0 \leq j < m\}$, a general Sierpinski carpet E is defined as

$$E = \left\{ \left(\sum_{k=1}^{\infty} \frac{x_k}{n^k}, \sum_{k=1}^{\infty} \frac{y_k}{m^k} \right) : (x_k, y_k) \in R \text{ for all } k \right\} \tag{2.2}$$

and the following theorem is given.

Theorem B. [2] *The Hausdorff dimension of E defined by* (2.2) *is given by*

$$\dim_H E = \log_m \left(\sum_{j=0}^{m-1} t_j^{(\log_n m)} \right)$$

where t_j is the number of i such that $(i,j) \in R$.

We shall generalize the theorem B above to the generalized sierpinski carpet in $\mathbf{R}^2$.

3 Symbolic Dynamical System

Let $I = (0, 1]$ and take points $x_0, x_1, \cdots, x_n$ and $y_0, y_1, \cdots, y_m$ in $\overline{I}$ satisfying

$$0 = x_0 < x_1 < \cdots < x_n = 1$$

$$0 = y_0 < y_1 < \cdots < y_m = 1,$$

where $\overline{I}$ is the closure of the set I. Put intervals

$$I_i := (x_{i-1}, x_i] \ \ (i = 1, \ldots, n) \quad \text{and} \quad J_j := (y_{j-1}, y_j] \ \ (j = 1, \ldots, m).$$

Let $f_i : I_i \to I$ $(i = 1, \ldots, n)$ and $g_j : J_j \to I$ $(j = 1, \ldots, m)$ be continuous, monotone functions satisfying $\overline{f_i(I_i)} = \overline{I}$ and $\overline{g_j(J_j)} = \overline{I}$. Define $f : I \to I$ and $g : I \to I$ as

$$f(x) := f_i(x) \quad \text{for } x \in I_i \quad \text{and} \quad g(y) := g_j(y) \quad \text{for } y \in J_j.$$

Take a subset

$$R := \{(u_1, v_1), \ldots, (u_r, v_r)\} \tag{3.1}$$

of the set $\{(i, j) : 1 \leq i \leq n, 1 \leq j \leq m\}$. Define a map $F : \bigcup_{p=1}^{r} I_{u_p} \times J_{v_p} \to I \times I$ by

$$F(x, y) := (f_{u_p}(x), g_{v_p}(y))$$

for $(x, y) \in I_{u_p} \times J_{v_p}$ with $p = 1, \ldots, r$. Put

$$E := \bigcap_{k=1}^{\infty} \overline{F^{-k}(I \times I)}. \tag{3.2}$$

Then E is *a generalized Sierpinski carpet.* we shall consider the Hausdorff dimension $\dim_H E$ of the generalized Sierpinski carpet E.

In this paper we shall consider the following functions $f_i : I_i \to I$ and $g_j : J_j \to I$ defined by

$$f_i(x) := a_i \times (x - x_{i-1}) \quad \text{and} \quad g_j(y) := b_j \times (y - y_{j-1})$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, where $a_i := \frac{1}{x_i - x_{i-1}}$ and $b_j := \frac{1}{y_j - y_{j-1}}$,. Moreover, we suppose that there exists $a > 1$ such that

1. $a_{u_p} = a$ for any $p = 1, \ldots, r$
2. $b_{v_p} \leq a$ for any $p = 1, \ldots, r$.

In order to obtain the Hausdorff dimension of E, we shall consider the symbolic dynamics. Let

$$\begin{aligned} S_r &:= \prod^{\infty} \{1, \ldots, r\} = \{\mathbf{z} = z_1 z_2 \ldots | z_j \in \{1, \ldots, r\} \text{ for } j \in \mathbf{N}\}, \\ S_r^k &:= \prod^{k} \{1, \ldots, r\} = \{\mathbf{i} = i_1 \ldots i_k | i_j \in \{1, \ldots, r\} \text{ for } j \in \{1, \ldots, k\}\} \end{aligned}$$

for any $k \in \mathbf{N}$ and

$$S_r^* := \bigcup_{k=1}^{\infty} S_r^k.$$

For any $\mathbf{i} = i_1 \dots i_k \in S_r^k$, let

$$\tilde{C}(\mathbf{i}) := \{\mathbf{z} \in S_r | z_j = i_j \text{ for } 1 \leq j \leq k\}$$

be a cylinder set of S_r. Let θ_p be the inverse map of $f_{u_p} \times g_{v_p}$ such that

$$\theta_p : I \times I \longrightarrow I_{u_p} \times J_{v_p} \qquad \text{by} \quad \theta_p(x, y) := f_{u_p}^{-1}(x) \times g_{v_p}^{-1}(y) \qquad (p = 1, \dots r).$$

Consider maps $\tau : S_r \to I \times I$, $\tau_j : S_r \to I (j = 1, 2)$ defined by

$$\begin{aligned} \tau(\mathbf{z}) &:= \lim_{k \to \infty} \theta_{z_1} \circ \theta_{z_2} \circ \dots \circ \theta_{z_k}(1, 1) \\ (\tau_1(\mathbf{z}), \tau_2(\mathbf{z})) &:= \tau(\mathbf{z}) \end{aligned}$$

for $\mathbf{z} = z_1 z_2 \dots \in S_r$. Then $\tau(S_r) = E$. A covering of E has correspondence with a covering of S_r. We shall consider a covering of S_r with cylinder sets. For $\mathbf{i} = i_1 i_2 \dots i_k \in S_r^k$, the image $\tau(\tilde{C}(\mathbf{i}))$ in $I \times I$ of a cylinder set $\tilde{C}(\mathbf{i})$ is a rectangle with width $|(f^k)'(x)|^{-1} = a^{-k}$ and height $|(g^k)'(y)|^{-1}$, where $(x, y) \in \tau(\tilde{C}(\mathbf{i}))$.

To consider the Hausdorff dimension of E, we shall consider squares in $I \times I$.

For $k \in \mathbf{N}$, $y \in [0, 1]$ and $j \in \{1, \dots, m\}$, put

$$\begin{aligned} \bar{k}_{y,j} &:= \#\{i \in \{0, \dots, k-1\} \mid g^i(y) \in J_j\}, \\ l_{k,y} &:= \left[\log_a |(g^k)'(y)|\right] = \left[\sum_{j=1}^{m} \bar{k}_{y,j} \log_a |b_j|\right] \qquad \text{and} \\ \bar{l}_{k,y,j} &:= \#\{i \in \{0, \dots, l_{k,y} - 1\} \mid g^i(y) \in J_j\}, \end{aligned}$$

where $[\cdot]$ is the Gauss symbol. Then the following is easily obtained.

Lemma 1. (1) $k = \sum_{j=1}^{m} \bar{k}_{y,j}, \quad l_{k,y} = \sum_{j=1}^{m} \bar{l}_{k,y,j}.$

(2) $a^{-l_{k,y}} < a|(g^k)'(y)|^{-1} \leq a^{-l_{k,y}+1}$.

(3) *For* $k \in \mathbf{N}$, $\mathbf{i} \in S_r^k$ *and* $y, y' \in \tau_2(\tilde{C}(\mathbf{i}))$, *we have*

$$l_{k,y} = l_{k,y'} \quad \textit{and} \quad (g^k)'(y) = (g^k)'(y').$$

For $\mathbf{i} \in S_r^k$ with $k \in \mathbf{N}$, by putting $y \in \tau_2(\tilde{C}(\mathbf{i}))$ and $l = l_{k,y}$, we shall define

$$\begin{aligned} I_{\mathbf{i}} &:= f_{i_1}^{-1} \circ f_{i_2}^{-1} \circ \dots \circ f_{i_l}^{-1}(I), \\ J_{\mathbf{i}} &:= g_{i_1}^{-1} \circ g_{i_2}^{-1} \circ \dots \circ g_{i_k}^{-1}(I) \qquad \text{and} \\ A(\mathbf{i}) &:= I_{\mathbf{i}} \times J_{\mathbf{i}}. \end{aligned}$$

Then for any $x \in \tau_1(\tilde{C}(\mathbf{i}))$ and $y \in \tau_2(\tilde{C}(\mathbf{i}))$, we have

$$|I_{\mathbf{i}}| = |(f^l)'(x)|^{-1} \quad \text{and } |J_{\mathbf{i}}| = |(g^k)'(y)|^{-1} \tag{3.3}$$

and $A(\mathbf{i})$ is a nearly square by lemma 1 (2). For $\mathbf{i}' \in S_r^k$ with $i'_j = i_j (j = 1, \dots, l)$ and $v_{i'_j} = v_{i_j} (j = l+1, \dots, k)$, the set $\tau(\tilde{C}(\mathbf{i}'))$ is also a subset of $A(\mathbf{i})$. So put

$$c_j := \#\{p \in \{1, \dots, r\} | v_p = j\}$$

for $j \in \{1, \dots, r\}$. Then the number of $\tau(\tilde{C}(\mathbf{i}'))$ in $A(\mathbf{i})$ is $c_{v_{i_{l+1}}} c_{v_{i_{l+2}}} \cdots c_{v_{i_k}}$.

Consider a probability measure μ on $I \times I$ such that $\mu(I_{u_p} \times J_{v_p}) = \mu(I_{u_{p'}} \times J_{v_{p'}})$ if $v_p = v_{p'}$. For $j \in \{1, \dots, m\}$, put $\mu_j := \mu(I_{u_p} \times J_{v_p})$ if $j = v_p$ and $\mu_j := 0$ if $j \neq v_p$ for any $p \in \{1, \dots, r\}$. Then $\sum_{j=1}^m c_j \mu_j = 1$ holds.

For $\mathbf{i} = \overbrace{jj \cdots j}^{k} \in S_r^k$ and $A(\mathbf{i}) = I_{\mathbf{i}} \times J_{\mathbf{i}}$, we have $|I_{\mathbf{i}}| \simeq b_j^{-k}$ and $|J_{\mathbf{i}}| = b_j^{-k}$. If we suppose

$$\mu(A(\mathbf{i})) \simeq (b_j^{-k})^s$$

with some $s \geq 0$, then we have

$$(b_j^{-k})^s \simeq c_j^{k-l} \mu_j^k$$

since the number of rectangles $\tau(\tilde{C}(\mathbf{i}'))$ in $A(\mathbf{z})$ is c_j^{k-l} . Hence

$$\mu_j \simeq b_j^{-s} c_j^{\log_a b_j - 1}.$$

So we can guess that the Hausdorff dimension s of E satisfies the following equation:

$$\sum_{j=1}^m b_j^{-s} c_{y_j}^{\log_a b_j} = 1.$$

In fact we can prove the following theorem.

Theorem 3.1. *Let E be the generalized Sierpinski carpet defined by* (3.2) *satisfying*

$$c_j = c_i \quad \textit{for any } i,\ j \in \{1, \dots, m\}.$$

Then the Hausdorff dimension of E is given by the solution s of the following equation:

$$\sum_{j=1}^m b_j^{-s} c_j^{\log_a b_j} = 1. \tag{3.4}$$

Before proving theorem 3.1, we shall give some lemmas. Let $\mathcal{C}$ be a covering of E, which is a subset of $\{A(\mathbf{i}) \mid \mathbf{i} \in S_r^k, k \in \mathbf{N}\}$. For $k \in \mathbf{N}$ and $\mathbf{i} \in S_r^k$ with $y \in \tau_2(\tilde{C}(\mathbf{i}))$, put

$$\lambda(\mathbf{i}) := |(g^k)'(y)| = \prod_{j=1}^k b_{v_{i_j}}. \tag{3.5}$$

Then we have the following lemmas.

Lemma 2. *For $k \in \mathbf{N}$ and $\mathbf{i} \in S_r^k$, consider $A(\mathbf{i}) = I_{\mathbf{i}} \times J_{\mathbf{i}}$, $x \in \tau_1(\tilde{C}(\mathbf{i}))$ and $y \in \tau_2(\tilde{C}(\mathbf{i}))$. Then by putting $l = l_{k,y}$, we have*

$$\begin{aligned} |I_{\mathbf{i}}| &= |(f^l)'(x)|^{-1} = a^{-l} \le a\lambda(\mathbf{i})^{-1} \\ |J_{\mathbf{i}}| &= |(g^k)'(y)|^{-1} = \lambda(\mathbf{i})^{-1} \\ \lambda(\mathbf{i})^{-1} &< |A(\mathbf{i})| \le (1+a)\lambda(\mathbf{i})^{-1} \end{aligned}$$

Proof. By using equations (3.3), (3.5) and Lemma 1 (2), we get the conclusion. □

Lemma 3. *For $s \ge 0$, the following* (1) *and* (2) *are equivalent.*

(1) *For any $\varepsilon > 0$ there exists a covering $\mathcal{C}$ of E satisfying*

   (a) *$\mathcal{C}$ is a subset of $\{A(\mathbf{i}) \mid \mathbf{i} \in S_r^k, k \in \mathbf{N}\}$ and*

   (b) $\sum_{A(\mathbf{i}) \in \mathcal{C}} \lambda(\mathbf{i})^{-s} < \varepsilon$.

(2) $\mathcal{H}^s(E) = 0$.

Proof. (1) $\Rightarrow$ (2): For any $\delta > 0$, take ε with $0 < \varepsilon < (\delta/(a+1))^s$. Then there exists a covering $\mathcal{C}$ of E such that $\sum \lambda(\mathbf{i})^{-s} < \varepsilon$ and so $A(\mathbf{i}) \in \mathcal{C}$ implies $\lambda(\mathbf{i})^{-s} < \varepsilon$. If $A(\mathbf{i}) = I_{\mathbf{i}} \times J_{\mathbf{i}}$ with $\mathbf{i} \in S_r^k$ is contained in $\mathcal{C}$, then

$$\lambda(\mathbf{i})^{-s} < \varepsilon.$$

So as for the diameter $|A(\mathbf{i})|$ of $A(\mathbf{i})$, we have by lemma 2

$$|A(\mathbf{i})| \le (a+1)\lambda(\mathbf{i})^{-1} \le (a+1)\varepsilon^{1/s} < \delta.$$

Hence $\mathcal{C}$ is a δ-cover and

$$\inf\{\sum |U_i|^s : \{U_i\} \text{ is } \delta-\text{cover of } E\} \le \sum_{A(\mathbf{i}) \in \mathcal{C}} |A(\mathbf{i})|^s$$

$$\le (a+1)^s \sum_{A(\mathbf{i}) \in \mathcal{C}} \lambda(\mathbf{i})^{-s} < (a+1)^s \varepsilon.$$

If δ goes to zero, then ε goes to zero and so

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \inf\{\sum |U_i|^s : \{U_i\} \text{ is } \delta-\text{cover of } E\} = 0.$$

(2) $\Rightarrow$ (1): For any $\varepsilon > 0$, there exists $\delta > 0$ and a $\delta-$ covering $\{U_j : j \in \Lambda\}$ of E such that

$$\sum_{j \in \Lambda} |U_j|^s < \frac{\varepsilon}{4}.$$

For each $j \in \Lambda$, there exists $A(\mathbf{i}_{j_1})$, $A(\mathbf{i}_{j_2})$, $A(\mathbf{i}_{j_3})$, $A(\mathbf{i}_{j_4})$ with $\mathbf{i}_{j_k} \in S_r^*$ $(k = 1, \ldots, 4)$ such that

$$U_j \subset A(\mathbf{i}_{j_1}) \cup A(\mathbf{i}_{j_2}) \cup A(\mathbf{i}_{j_3}) \cup A(\mathbf{i}_{j_4})$$

and

$$|A(\mathbf{i}_{j_k})| < 2|U_j| \quad \text{for } k = 1, \ldots, 4.$$

Then $\mathcal{C} = \{A(\mathbf{i}_{j_k}) \mid j \in \Lambda,\ k = 1, \ldots, 4\}$ is a covering of E and

$$\sum_{j\in\Lambda}\sum_{k=1}^{4} \lambda(\mathbf{i}_{j_k})^{-s} \leq \sum_{j\in\Lambda}\sum_{k=1}^{4} |A(\mathbf{i}_{j_k})|^s \leq 4\sum_{j\in\Lambda} |U_j|^s < \varepsilon.$$

□

For $\mathbf{i} \in S_r^k$, put

$$\tilde{A}(\mathbf{i}) := \tau^{-1}(A(\mathbf{i})).$$

Then as for a covering $\tilde{\mathcal{C}}$ of S_r, we have the following similar result.

Lemma 4. *For $s \geq 0$, the following (1) and (2) are equivalent.*

(1) *For any $\varepsilon > 0$ there exists a covering $\tilde{\mathcal{C}}$ of $S_r = \tau^{-1}(E)$ satisfying*

(a) *$\tilde{\mathcal{C}}$ is a subset of $\{\tilde{A}(\mathbf{i}) \mid \mathbf{i} \in S_r^k, k \in \mathbf{N}\}$ and*

(b) $\displaystyle\sum_{\tilde{A}(\mathbf{i})\in\tilde{\mathcal{C}}} \lambda(\mathbf{i})^{-s} < \varepsilon.$

(2) $\mathcal{H}^s(E) = 0.$

For $j \in \{1, \ldots, m\}$, put

$$\nu_j := b_j^{-s} c_j^{\log_a b_j - 1}$$

by using the number s satisfying the equation (3.4). Define a measure $\tilde{\mu}$ on S_r by

$$\tilde{\mu}(\tilde{C}(i_1 \ldots i_k)) := \nu_{v_{i_1}} \cdots \nu_{v_{i_k}}.$$

Then $\tilde{\mu}$ is a probability measure on S_r.

Since the following holds for $\mathbf{i} \in S_r^k$

$$\tilde{A}(\mathbf{i}) = \tau^{-1}(A(\mathbf{i})) = \bigcup_{w_{l+1}\ldots w_k} \tilde{C}(i_1 \ldots i_l w_{l+1} \ldots w_k),$$

where the union is taken over all $\mathbf{w} = w_{l+1} \ldots w_k \in S_r^{k-l}$ such that $v_{w_q} = v_{i_q}$ for all $q \in \{l+1, \ldots, k\}$, we have

$$\begin{aligned}
\tilde{\mu}(\tilde{A}(\mathbf{i})) &= \tilde{\mu}(\tilde{C}(i_1 \ldots i_k)) \prod_{q=l+1}^{k} c_{v_{i_q}} = \frac{\prod_{j=1}^{k} b_{v_{i_j}}^{-s} c_{v_{i_j}}^{\log_a b_{v_{i_j}}}}{\prod_{j=1}^{l} c_{v_{i_j}}} \\
&= \frac{\prod_{j=1}^{k} b_{v_{i_j}}^{-s} \prod_{j=1}^{m} c_j^{\bar{k}_{y,j} \log_a b_j}}{\prod_{j=1}^{m} c_j^{\bar{l}_{k,y,j}}} = \left(\prod_{j=1}^{k} b_{v_{i_j}}\right)^{-s} \prod_{j=1}^{m} c_j^{\bar{k}_{y,j} \log_a b_j - \bar{l}_{k,y,j}},
\end{aligned}$$

where $y \in \tau_2(\tilde{C}(\mathbf{i}))$.

For $\mathbf{i} \in S_r^k$, put

$$h(\mathbf{i}) := \left[\prod_{j=1}^{m} c_j^{\bar{k}_{y,j}\log_a b_j - \bar{l}_{k,y,j}}\right]^{1/k}.$$

Then

$$\tilde{\mu}(\tilde{A}(\mathbf{i})) = h(\mathbf{i})^k \lambda(\mathbf{i})^{-s}. \tag{3.6}$$

For $\mathbf{z} \in S_r$, define the functions

$$\tilde{h}_k(\mathbf{z}) := h(z_1 \dots z_k)$$

with $z_1 \dots z_k \in S_r^k$.

Then we have the following

Lemma 5. *If $c_j = c_i$ holds for any $i,\ j \in \{1, \dots, m\}$, then*

$$\lim_{k\to\infty} \tilde{h}_k(\mathbf{z}) = 1$$

for all $\mathbf{z} \in S_r$.

Proof. If $c_j = c$ holds for any $j \in \{1, \dots, m\}$, then for any $\mathbf{z} \in S_r$ with $y = \tau_2(\mathbf{z})$, $\tilde{h}_k(\mathbf{z}) = \left\{\prod_{j=1}^m c^{\bar{k}_{y,j}\log_a b_j - \bar{l}_{k,y,j}}\right\}^{1/k} = \left\{c^{\sum_{j=1}^m(\bar{k}_{y,j}\log_a b_j - \bar{l}_{k,y,j})}\right\}^{1/k} = c^{\frac{1}{k}(\log_a |(g^k)'(y)| - l_{k,y})}$ goes to 1 as $k \to \infty$.

□

By using Lemmas 4 and 5, we shall prove the following

Lemma 6. *If $c_j = c_i$ holds for any $i,\ j \in \{1, \dots, m\}$, then*

$$\dim_H E \le s.$$

Proof. Put

$$\mathcal{I} = \{\mathbf{i} \in S_r^* \mid h(\mathbf{i}) > (\lambda(\mathbf{i}))^{-\frac{\varepsilon}{k}}\}.$$

For $\mathbf{i} \in \mathcal{I}$, we have

$$\tilde{\mu}(\tilde{A}(\mathbf{i})) > (\lambda(\mathbf{i}))^{-\varepsilon - s} \tag{3.7}$$

by the equation (3.6).

For each $k \in \mathbf{N}$, let

$$\mathcal{I}_k = \left\{\mathbf{i} \in \mathcal{I} \mid 2^{k-1} \le \lambda(\mathbf{i}) < 2^k\right\}.$$

Note that any $\mathbf{z} \in S_r$ is covered by $A(\mathbf{i})$ with $\mathbf{i} \in \mathcal{I}_k$ for infinitely many k, since $\lim \tilde{h}_k(\mathbf{z}) = 1 > (\lambda(z_1 \dots z_k))^{-\varepsilon/k}$. Hence $\mathcal{M}_K := \bigcup_{k\ge K} \bigcup_{\mathbf{i}\in\mathcal{I}_k} A(\mathbf{i})$ is a covering of S_r for any large $K \in \mathbf{N}$. If

$A(\mathbf{i}) \cap A(\mathbf{i}') \neq \emptyset$, then either $A(\mathbf{i})$ or $A(\mathbf{i}')$ contains the other set. So let a subfamily $\mathcal{M}'_K$ of $\mathcal{M}_K$ be a covering of S_r consisting of disjoint sets and for $k \geq K$, put

$$\mathcal{I}'_{K,k} = \{\mathbf{i} \in \mathcal{I}_k \mid A(\mathbf{i}) \in \mathcal{M}'_K.\}$$

As for the number $\#(\mathcal{I}'_{K,k})$ of elements of $\mathcal{I}'_{K,k}$, by using (3.7) we have

$$\#(\mathcal{I}'_{K,k}) \times 2^{-k(s+\varepsilon)} \leq \sum_{\mathbf{i}\in\mathcal{I}'_{K,k}} \lambda(\mathbf{i})^{-(s+\varepsilon)} \leq \sum_{\mathbf{i}\in\mathcal{I}'_{K,k}} \tilde{\mu}(\tilde{A}(\mathbf{i})) \leq 1,$$

and hence $\#(\mathcal{I}'_{K,k}) \leq 2^{k(s+\varepsilon)}$.

By taking K large enough that $(2a+2)^{s+2\varepsilon} \sum_{n=K}^{\infty} 2^{-n\varepsilon} < \varepsilon$, we have by using lemma 2

$$\begin{aligned}\sum_{A(\mathbf{i})\in\mathcal{M}'_K} |A(\mathbf{i})|^{s+2\varepsilon} &= \sum_{k=K}^{\infty} \sum_{\mathbf{i}\in\mathcal{I}'_{K,k}} |A(\mathbf{i})|^{s+2\varepsilon} \leq \sum_{k=K}^{\infty} \#(\mathcal{I}'_k) \times (a+1)^{s+2\varepsilon} \lambda(\mathbf{i})^{-(s+2\varepsilon)} \\ &\leq (a+1)^{s+2\varepsilon} \sum_{n=K}^{\infty} 2^{k(s+\varepsilon)} 2^{-(k-1)(s+2\varepsilon)} = (2a+2)^{s+2\varepsilon} \sum_{n=K}^{\infty} 2^{-n\varepsilon} \leq \varepsilon.\end{aligned}$$

By using Lemma 4, we have $\dim_H E \leq s$. □

Lemma 7. *If $c_j = c_i$ holds for any $i,\ j \in \{1, \ldots, m\}$, then*

$$\dim_H E \geq s.$$

Proof. Taking β with $\beta < s$, we shall show that there exits $\varepsilon > 0$ such that $\sum_{\tilde{A}(\mathbf{i})\in\tilde{\mathcal{C}}} \lambda(\mathbf{i})^{-\beta} > \varepsilon$ for any covering $\tilde{\mathcal{C}}$ of S_r consisting of a subset of $\{A(\mathbf{i}) \mid \mathbf{i} \in S_r^k, k \in \mathbf{N}\}$. For $K \in \mathbf{N}$, put

$$\tilde{E}_K := \{\mathbf{z} \in S_r : \tilde{h}_k(\mathbf{z}) < |(g^k)'(\tau_2(\mathbf{z}))|^{\frac{s-\beta}{k}} \text{ for any } k \geq K\}.$$

By the relation $\beta < s$, we have $|(g^k)'(\tau_2(\mathbf{z}))|^{\frac{s-\beta}{k}} \geq 1$ and $\lim_{k\to\infty} \tilde{h}_k(\mathbf{z}) = 1$ by lemma 5. So we can take $K \in \mathbf{N}$ such that $\tilde{\mu}(\tilde{E}_K) > 0$ and put $\varepsilon = \min\{\tilde{\mu}(\tilde{E}_K), b_{\max}^{-\beta K}\}$, where $b_{\max} = \max\{b_{v_p} \mid p \in \{1, \ldots, r\}\}$. Let $\tilde{\mathcal{C}}$ be any covering of S_r consisting of a subset of $\{\tilde{A}(\mathbf{i}) \mid \mathbf{i} \in S_r^k, k \in \mathbf{N}\}$ and let $D_k = \{\tilde{A}(\mathbf{i}) \in \tilde{\mathcal{C}} \mid \mathbf{i} \in S_r^k\}$ for any $k \in \mathbf{N}$. If there exists $k(< K)$ such that $\#(D_k) \neq 0$, then

$$\sum_{\tilde{A}(\mathbf{i})\in\tilde{\mathcal{C}}} \lambda(\mathbf{i})^{-\beta} \geq \sum_{\tilde{A}(\mathbf{i})\in D_k} \lambda(\mathbf{i})^{-\beta} \geq b_{\max}^{-\beta k} > b_{\max}^{-\beta K} \geq \varepsilon.$$

For $\mathbf{z} \in \tilde{A}(\mathbf{i}) \cap \tilde{E}_K$ with $\mathbf{i} \in S_r^k$ $(k > K)$, we have $z_j = i_j$ for $j = 1, \ldots, l_{k,\tau_2(\mathbf{z})}$ and $v_{z_j} = v_{i_j}$ for $j = l_{k,\tau_2(\mathbf{z})} + 1, \ldots, k$. Hence $h(\mathbf{i}) = \tilde{h}_k(\mathbf{z})$ and

$$\tilde{\mu}(\tilde{A}(\mathbf{i})) = h(\mathbf{i})^k \lambda(\mathbf{i})^{-s} = \tilde{h}_k(\mathbf{z})^k \lambda(\mathbf{i})^{-s} < \lambda(\mathbf{i})^{k\frac{s-\beta}{k}-s} = \lambda(\mathbf{i})^{-\beta}.$$

If $\tilde{A}(\mathbf{i}) \in \tilde{\mathcal{C}}$ implies $\mathbf{i} \in S_r^k$ with $k \geq K$, then

$$\sum_{\tilde{A}(\mathbf{i})\in\tilde{\mathcal{C}}} \lambda(\mathbf{i})^{-\beta} = \sum_{k\geq K} \sum_{\tilde{A}(\mathbf{i})\in D_k} \lambda(\mathbf{i})^{-\beta} \geq \sum_{k\geq K} \sum_{\tilde{A}(\mathbf{i})\in D_k} \tilde{\mu}(\tilde{A}(\mathbf{i})) = \sum_{\tilde{A}(\mathbf{i})\in\tilde{\mathcal{C}}} \tilde{\mu}(\tilde{A}(\mathbf{i})) \geq \tilde{\mu}(\tilde{E}_K) \geq \varepsilon.$$

Therefore $\mathcal{H}^s(E) > 0$ by lemma 4 and we have $\dim_H E \geq s$. □

proof of Theorem 3.1. By using the number s satisfying the equation (3.4), we have proved $\dim_H E \le s$ in lemma 6 and $\dim_H E \ge s$ in lemma 7. So the number s satisfying the equation (3.4) is the Hausdorff dimension of E. $\square$

4 Thermodynamic formalism

Let I, I_i, J_j, a, c_j, $R = \{(u_1, v_1), \ldots, (u_r, v_r)\}$, f and F be defined at Section 3. For $y \in J_j$, put

$$\tilde{c}_y = c_j.$$

As an extension of Theorem A to the set in $\mathbf{R}^2$, we shall consider the function ϕ $I \times I \to \mathbf{R}$ corresponding to $-s \log |f'(x)|$ on I, defined by

$$\phi(x, y) = -s \log |g'(y)| + (\log_a |g'(y)| - 1) \log \tilde{c}_y, \tag{4.1}$$

for $(x, y) \in I \times I$.

We shall consider the pressure function $P(\phi)$ by using the map $F : \bigcup_{p=1}^{r} I_{u_p} \times J_{v_p} \to (0, 1] \times (0, 1]$ as follows:

$$P(\phi) = \lim_{k \to \infty} \frac{1}{k} \log \sum_{(x,y) \in \mathrm{Fix} F^k} \exp \sum_{i=0}^{k-1} \phi\left(F^i(x, y)\right),$$

where F^i is defined on $F^{-i}(I \times I)$ and $\mathrm{Fix} F^k = \{(x, y) \in F^{-k}(I \times I) : (x, y) = F^k(x, y)\}$. The existence of the above limit is shown in a similar way to ([1], Th 5.1).

Then we have the following

Theorem 4.1. *Let E be the generalized Sierpinski carpet defined by* (3.2) *satisfying* (i) *or* (ii)*:*

(i) $b_j = b_i$ *for any* i, $j \in \{1, \ldots, m\}$.

(ii) $c_j = c_i$ *for any* i, $j \in \{1, \ldots, m\}$.

Let s be the unique real number satisfying

$$P\left(-s \log |g'(y)| + (\log_a |g'(y)| - 1) \log \tilde{c}_y\right) = 0.$$

Then

$$\dim_H E = s.$$

Proof. Put $\phi(x, y) = -s \log |g'(y)| + (\log_a |g'(y)| - 1) \log \tilde{c}_y$ for $(x, y) \in I \times I$. We have

$$\begin{aligned} \sum_{(x,y) \in \mathrm{Fix} F^k} \exp \sum_{i=0}^{k-1} \phi\left(F^i(x, y)\right) &= \sum_{(x,y) \in \mathrm{Fix} F^k} \exp \sum_{i=0}^{k-1} \{(-s \log |g'(g^i y)| + (\log_a |g'(g^i y)| - 1) \log \tilde{c}_{g^i y}\} \\ &= \sum_{(x,y) \in \mathrm{Fix} F^k} \prod_{i=0}^{k-1} |g'(g^i y)|^{-s} \tilde{c}_{g^i y}^{\log_a |g'(g^i y)| - 1}. \end{aligned}$$

For $y \in \mathrm{Fix}g^k$, we have $\#\{(x',y') \in \mathrm{Fix}F^k : y' = y\} = \tilde{c}_y \tilde{c}_{gy} \cdots \tilde{c}_{g^{k-1}y}$ and

$$\sum_{(x,y)\in \mathrm{Fix}F^k} \prod_{i=0}^{k-1} |g'(g^i y)|^{-s} \tilde{c}_{g^i y}^{\log_a |g'(g^i y)|-1} = \sum_{y\in \mathrm{Fix}g^k} \prod_{i=0}^{k-1} |g'(g^i y)|^{-s} \tilde{c}_{g^i y}^{\log_a |g'(g^i y)|}. \tag{4.2}$$

Put $\gamma_j = b_j^{-s} c_j^{\log_a b_j}$ for $j \in \{1,\dots,m\}$. Since the right-hand side of (4.2) consists of all the combinations of k elements of $\{\gamma_j\}_{j=1}^m$, we have

$$\sum_{y\in \mathrm{Fix}g^k} \prod_{i=0}^{k-1} |g'(g^i y)|^{-s} \tilde{c}_{g^i y}^{\log_a |g'(g^i y)|} = (\gamma_1 + \cdots + \gamma_m)^k.$$

So we have

$$P(\phi) = \lim_{k\to\infty} \frac{1}{k} \log(\sum_{j=1}^m \gamma_j)^k = \log \sum_{j=1}^m b_j^{-s} c_j^{\log_a b_j}.$$

By the assumption $P(\phi) = 0$, we have

$$\sum_{j=1}^m b_j^{-s} c_j^{\log_a b_j} = 1.$$

In case of (i), the above equation implies that

$$-s \log b + \sum_{j=1}^m c_j^{\log_a b} = 0.$$

So by Theorem B, s is the Hausdorff dimension of E.

In case of (ii), by using theorem 3.1, we get the conclusion.

□

In this paper, we have supposed that the functions $\{f_i\}$ and $\{g_j\}$ are affine functions. We want to get the Hausdorff dimension in more general cases by using thermodynamic formalism. So we give the following conjecture.

5 Conjecture

Let intervals

$$I_i := (x_{i-1}, x_i] \ \ (i = 1,\dots,n) \quad \text{and} \quad J_j := (y_{j-1}, y_j] \ \ (j = 1,\dots,m)$$

and a subset

$$R := \{(u_1,v_1),\dots,(u_r,v_r)\}$$

of the set $\{(i,j) : 1 \le i \le n, 1 \le j \le m\}$ be those defined in Section 3.

Consider the maps $f \in C^2(I)$ and $g \in C^2(I)$ satisfying

(C-1) $f|_{I_i^\circ}$ and $g|_{J_j^\circ}$ are one-to-one maps onto $(0,1)$ respectively for $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

(C-2) $1 < \inf\{|g'(y)| y \in I\} \leq \sup\{|g'(y)| : y \in I\} \leq \inf\{|f'(x)| : x \in I\}$,

where J° denotes the interior of J.

For $p = 1, \ldots, r$, put $\alpha_p = |I_{u_p}|^{-1}$, $\beta_p = |J_{v_p}|^{-1}$, $D_1 = \cup_{p=1}^r I_{u_p}$ and $D_2 = \cup_{p=1}^r J_{v_p}$.

For $t > 0$ and $y \in D_2$, define $w_y(t)$ as the solution w of the following equation:

$$\sum_{p: y \in J_{v_p}} \alpha_p^{-(t+\log_{\beta_p} w)} = 1,$$

where the summation is taken over p such that $(x, y) \in J_{v_p}$ with some $x \in D_1$ and for $y \in I \setminus D_2$ let $w_y(t) = 0$.

For each $x \in I$, define the function β_x on I by

$$\beta_x(y) = \begin{cases} \beta_p & \text{if } (x,y) \in I_{i_p} \times J_{j_p} \text{ with some } p \in \{1, \ldots, r\} \\ 0 & \text{otherwise.} \end{cases}$$

Since for each $x \in I$, $w_y(t)$ is an decreasing function on $(0, \infty)$, there exists some t satisfying the following:

$$\int_0^1 w_y(t) \beta_x(y) dy = 1 \tag{5.1}$$

and let $w_{x,y}$ be $w_y(t)$ satisfying (5.1). For f and g satisfying (C-1) and (C-2), define the function

$$\phi(x,y) = \left(1 - \frac{\log|f'(x)|}{\log|g'(y)|}\right) \log w_{x,y} - s \log|f'(x)|.$$

Then for s satisfying $P(\phi) = 0$, we suppose that $\dim_H E = s$ holds.

Remark. We shall apply the above conjecture to the case of Theorem 4.1. Then $\alpha_p = a$, $\beta_x(y) = |g'(y)|$, $w_y(t) = |g'(y)|^{-t+\log_a \tilde{c}_y}$ and $w_{x,y} = |g'(y)|^{-s+\log_a \tilde{c}_y - 1}$ with s satisfying the equation (3.4). So the function ϕ defined above is the same as that defined by (4.1) as shown in the following:

$$\begin{aligned} \phi(x,y) &= \left(1 - \log_a |g'(y)|\right) \log |g'(y)|^{-s+\log_a \tilde{c}_y - 1} - s \log a \\ &= \left(\log_a |g'(y)| - 1\right) \log \tilde{c}_j - s \log |g'(y)|. \end{aligned}$$

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