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Large deviation around the origin for sums of nonnegative i.i.d. random variables

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Abstract

Let $X_1, X_2, \dots$ be nonnegative independent random variables with a common distribution attracted to the stable law G_α, and put $S_n = X_1+X_2+\cdots+X_n$. That is for some monotone increasing function $\sigma(n)$, $P[S_n/\sigma(n) \le x] \to G_\alpha(x)$, for every $x > 0$ as $n \to \infty$. The aim of the present paper is to study the asymptotic behaviour of $P[S_n/\sigma(n) \le x_n]$, where x_n is a positive sequence such that $x_n \to 0$ as $n \to \infty$.

1. Introduction

Let $G_\alpha(x)$ $(0 < \alpha < 1)$ be the distribution function such that

$$(1.1) \qquad \int_0^\infty e^{-sx} dG_\alpha(x) = \exp\{-s^\alpha\}, \qquad \text{for } s > 0.$$

So G_α is a one-sided stable law with index α. Throughout the paper, let $X, X_1, X_2, \dots$ be nonnegative independent random variables with a common distribution which belongs to the domain of attraction of G_α, i.e., for some monotone increasing function $\sigma(x)$ tending to ∞ as $x \to \infty$, it holds that

$$(1.2) \qquad \lim_{n\to\infty} P[\frac{1}{\sigma(n)}(X_1 + X_2 + \cdots + X_n) \le x] = G_\alpha(x), \qquad \text{for every } x > 0.$$

We remark here that $\sigma(x)$ is asymptotically equal to the inverse function of $1/P[X_1 > x]$ up to a multiplicative constant, and that $\sigma(x)$ is a regularly varying function with exponent $1/\alpha$, i.e., $\sigma(x) = x^{1/\alpha}L(x)$ (L: slowly varying function).

Thus if we denote by $F_n(x)$ the distribution function of $(X_1+X_2+\dots X_n)/\sigma(n)$, then (1.2) implies $F_n(x) \to G_\alpha(x)$, for every $x > 0$ as $n \to \infty$.

The aim of the present paper is to study the relationship between the asymptotic behaviour of $F_n(x_n)$ and that of $G_\alpha(x_n)$ when x_n is a positive sequence such that $\lim_{n\to\infty} x_n = 0$. In other words, we shall consider the case where $x \to 0$ as $n \to \infty$ in (1.2) instead of a fixed $x > 0$. To this end, we first recall the following well-known results. As for the tail probability, we have

$$1 - G_\alpha(x) \sim \frac{x^{-\alpha}}{\Gamma(1-\alpha)}, \quad \text{as } x \to \infty. \tag{1.3}$$

On the other hand, G_α decreases exponentially as x goes to 0:

$$-\log G_\alpha(x) \sim D_\alpha x^{-\beta}, \quad \text{as } x \to 0, \tag{1.4}$$

where $\beta = \alpha/(1-\alpha)$, and $D_\alpha = (1-\alpha)\alpha^\beta$. This fact can be shown by applying the result of [2]. In fact, more precise result is known (see [3]) as follows: For $c > 0$,

$$G_\alpha(x) \sim cx^{\beta/2} \exp\{-D_\alpha x^{-\beta}\}, \quad \text{as } x \to 0. \tag{1.5}$$

Now our problem is the following. Let $x = \gamma_n \to 0$ $(n \to \infty)$. Since $G_\alpha(0) = 0$, (1.2) implies

$$\lim_{n\to\infty} P[\frac{1}{\sigma(n)}(X_1 + X_2 + \cdots + X_n) \le \gamma_n] = 0. \tag{1.6}$$

Thus it is of interest to know its logarithmic rate, i.e., the asymptotic behaviour of

$$-\log P[\frac{1}{\sigma(n)}(X_1 + X_2 + \cdots + X_n) \le \gamma_n]. \tag{1.7}$$

Keeping (1.4) in mind, we can generally say that

$$-\log P[\frac{1}{\sigma(n)}(X_1 + X_2 + \cdots + X_n) \le \gamma_n] \sim D_\alpha \gamma_n^{-\beta}, \quad \text{as } \gamma_n \to 0, \tag{1.8}$$

if γ_n goes to 0 slowly enough, but the aim of the present paper is to give the conditions on γ_n $(\to 0)$ which assure (1.8).

2. Main results

Theorem 2.1. *Let $X, X_1, X_2, \ldots$ be nonnegative independent random variables with a common distribution, and assume* (1.2) *holds for some monotone regularly varying function $\sigma(x) = x^{1/\alpha}L(x)$. Put $\tau(t) = \sigma(t)/t$ and $a_n = n/\tau^{-1}(\gamma_n\tau(n))$. If $\gamma_n \to 0$ and $\gamma_n\tau(n) \to \infty$, then $a_n \to \infty$ as $n \to \infty$, and we have*

(i)

$$-\log P[\frac{1}{\sigma(n)}(X_1 + X_2 + \cdots + X_n) \le \gamma_n] \sim D_\alpha a_n, \quad \text{as } n \to \infty, \tag{2.1}$$

where D_α is as in (1.4).

(ii) *Furthermore, a necessary and sufficient condition for $a_n \sim \gamma_n^{-\beta}$ $(\beta = \alpha/(1-\alpha))$ is*

$$\frac{L(n\gamma_n^\beta)}{L(n)} \to 1, \quad \text{as } n \to \infty. \tag{2.2}$$

A sufficient condition for (2.2) *is that* $L(x)$ *is a constant, that is* X *belongs to the normal domain of attraction of the stable law* G_α.

Remark. The above theorem treats the case where $\gamma_n\tau(n) \to \infty$ as $n \to \infty$. So let us here discuss a little on the case where $\lim_{n\to\infty} \gamma_n\tau(n) < \infty$. Notice that the left side of (2.1) can be rewritten as

$$-\log P[\frac{1}{n}(X_1 + X_2 + \cdots + X_n) \le \gamma_n\tau(n)]. \tag{2.3}$$

Thus, in the special case where $\gamma_n\tau(n)$ is a constant, (2.3) can be regarded as a problem of Cramer's large deviation principle. Also, if $\gamma_n\tau(n) \to 0$, then the result was obtained in one of the author's paper ([4]) which studied the relationship between the asymptotic behaviour of (2.3) and that of the distribution function of X around the origin.

3. Proof of Theorem 2.1

For the proof of Theorem 2.1, we use the following theorem and a lemma.

Theorem A ([4]) . *Let* $\varphi(s) \in C^1(0,\infty)$ *be a decreasing strictly convex function such that*

$$\lim_{\varepsilon\to 0+} \varphi'(\varepsilon) = -\infty, \quad \lim_{s\to\infty} \varphi'(s) = 0,$$

and define

$$\varphi^*(x) = \inf_{s>0}\{sx + \varphi(s)\}, \quad x > 0.$$

Suppose $U_n(x)$ *be a sequence of nondecreasing right-continuous functions on* $[0,\infty)$ *with* $U_n(0) = 0$, *and* a_n *be a positive sequence which tends to* ∞ *as* $n \to \infty$. *If*

$$\lim_{n\to\infty} \frac{1}{a_n} \log \int_0^\infty e^{-a_n sx} dU_n(x) = \varphi(s), \quad \textit{for all } s > 0,$$

then

$$\lim_{n\to\infty} \frac{1}{a_n} \log U_n(x) = \varphi^*(x), \quad \textit{for all } x > 0.$$

Lemma 3.1. *Let* $f(x)$ *be a regularly varying function with exponent* $\rho(> 0)$, *and* a_n *and* b_n *be positive sequences tending to* ∞ *as* n *goes to* ∞. *Then* $\lim_{n\to\infty} a_n/b_n = c$ *for* $0 \le c \le \infty$ *is a necessary and sufficient condition for*

$$\lim_{n\to\infty} \frac{f(a_n)}{f(b_n)} = c^\rho.$$

Proof. Since $f(x)$ is regularly varying, it holds that

$$\lim_{x\to\infty}\frac{f(\lambda x)}{f(x)}=\lambda^{\rho},$$

the convergence being uniform for λ on every compact sub-interval of $(0,\infty)$, which proves the assertion. □

From (1.2), we have

$$\xi\log E[e^{(-s/\sigma(\xi))X}]\longrightarrow\varphi_\alpha(s):=-s^\alpha,\quad \text{as } \xi\to\infty. \tag{3.1}$$

Since $\sigma(t)$ is regularly varying with exponent $1/\alpha$ $(0<\alpha<1)$, so is $\tau(t)$ with exponent $1/\alpha-1$. From the assumption, $\gamma_n\tau(n)\to\infty$ $(n\to\infty)$, and thus we have

$$\lim_{n\to\infty}\frac{n}{a_n}=\lim_{n\to\infty}\tau^{-1}(\gamma_n\tau(n))=\infty. \tag{3.2}$$

By applying Lemma 3.1, we have

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{n}{\tau^{-1}(\gamma_n\tau(n))}=\infty, \tag{3.3}$$

since $\lim_{n\to\infty}\tau(n)/(\gamma_n\tau(n))=\infty$. Therefore, we can put $\xi=n/a_n$ in (3.1) and have

$$\frac{n}{a_n}\log E[e^{(-s/\sigma(n/a_n))X}]\longrightarrow\varphi_\alpha(s),\quad \text{as } n\to\infty. \tag{3.4}$$

From the definition of a_n, $\tau(n/a_n)=\gamma_n\tau(n)$, and thus multiplying both sides by n/a_n, we have

$$\sigma(n/a_n)=\gamma_n\sigma(n)/a_n,$$

and therefore, we can rewrite (3.4) as

$$\frac{n}{a_n}\log E[\exp\{-\frac{sa_n}{\gamma_n\sigma(n)}X\}]\longrightarrow\varphi_\alpha(s),\quad \text{as } n\to\infty. \tag{3.5}$$

Thus, replacing X by $S=X_1+X_2+\cdots+X_n$, we obtain

$$\frac{1}{a_n}\log E[\exp\{-\frac{sa_n}{\gamma_n\sigma(n)}S_n\}]\longrightarrow\varphi_\alpha(s),\quad \text{as } n\to\infty. \tag{3.6}$$

Applying Theorem A, we have

$$\frac{1}{a_n}\log P[\frac{S_n}{\gamma_n\sigma(n)}\le x]\longrightarrow\varphi_\alpha^*(x),\quad \text{as } n\to\infty, \tag{3.7}$$

where $\varphi_\alpha^*(x)=-(1-\alpha)\alpha^{\alpha/(1-\alpha)}x^{-\alpha/(1-\alpha)}$.

If we put $x=1$ in (3.7), we have the assertion of Theorem 2.1(i).

Next, we prove Theorem 2.1(ii). Suppose $a_n\sim\gamma_n^{-\beta}$. From the definition of a_n, we have

$$\frac{n}{\tau^{-1}(\gamma_n\tau(n))}\sim\gamma_n^{-\beta}, \tag{3.8}$$

which can be rewritten as

$$\frac{n\gamma_n^\beta}{\tau^{-1}(\gamma_n\tau(n))} \to 1, \qquad \text{as } n \to \infty. \tag{3.9}$$

By Lemma 3.1, (3.9) is equivalent to

$$\frac{\tau(n\gamma_n^\beta)}{\gamma_n\tau(n)} \to 1, \qquad \text{as } n \to \infty. \tag{3.10}$$

Replacing $\tau(x)$ by $\sigma(x)/x$, we get

$$\frac{n\sigma(n\gamma_n^\beta)}{n\gamma_n^{\beta+1}\sigma(n)} \to 1, \qquad \text{as } n \to \infty. \tag{3.11}$$

Furthermore, by putting $\sigma(x) = x^{1/\alpha}L(x)$, we get (2.2), which proves the necessity, and similarly, we can prove the sufficiency.

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