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BOUNDARY BEHAVIOR OF DOUBLE LAYER POTENTIALS IN A DOMAIN WITH FRACTAL BOUNDARY

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ABSTRACT. For a bounded domain D with fractal boundary we consider a Besov space on ∂D, with respect to a measure corresponding to the fractal dimension of ∂D. We define double layer potentials of functions in the Besov space and discuss the existence of nontangential limits of the double layer potentials, with an exceptional set, and estimate the size of the exceptional set by using a Hausdorff measure depending on the order of the Besov space.

1. Introduction

Let D be a bounded smooth domain in $\mathbf{R}^d$ $(d \geq 2)$. The double layer potential Φg of $g \in L^p(\partial D)$ is defined by

$$\Phi g(x) = -\int_{\partial D} \langle \nabla_y N(x-y), n_y \rangle g(y) d\sigma(y), \tag{1.1}$$

where $N(x-y)$ is the Newton kernel if $d \geq 3$ and the logarithmic kernel if $d = 2$, n_y is the unit outer normal to ∂D and σ is the surface measure on ∂D. The function Φf is harmonic in $\mathbf{R}^d \setminus \partial D$ and has a nontangential limit at σ-almost every boundary point.

If D is a domain with fractal boundary, then n_y and σ can not be considered and (1.1) is not defined. But in [W1] and [W3] we introduced the double layer potentials on such a domain.

More precisely, let D be a bounded domain in $\mathbf{R}^d$ such that ∂D is a β-set $(d-1 \leq \beta < d)$, i.e., there exist a positive Radon measure μ on ∂D and positive real numbers β, r_0, b_1, b_2 such that

$$b_1 r^\beta \leq \mu(B(z,r) \cap \partial D) \leq b_2 r^\beta \tag{1.2}$$

for all $z \in \partial D$ and all $r \leq r_0$, where $B(z,r)$ stands for the open ball with center z and radius r in $\mathbf{R}^d$.

Such a measure is called a β-measure. Fix a β-measure μ on ∂D. Since D is a bounded, we choose $R \geq 1$ satisfying $\overline{D} \subset B(0, R/2)$.

By the same method as in [JW] we constructed in [W3] an extension operator $\mathcal{E}$ from $L^p(\mu)$ to $L^p(\mathbf{R}^d)$ such that $\mathcal{E}(f) = f$ on ∂D, supp $\mathcal{E}(f) \subset B(0, 2R)$ and $\mathcal{E}(f)$ is a C^∞-function in $\mathbf{R}^d \setminus \partial D$. Using the extension operator $\mathcal{E}$, we define the double layer potential Φf of $f \in L^p(\mu)$ by

$$\Phi f(x) = \int_{\mathbf{R}^d \setminus \overline{D}} \langle \nabla_y \mathcal{E}(f)(y), \nabla_y N(x-y) \rangle dy \tag{1.3}$$

for $x \in D$ and

$$\Phi f(x) = -\int_{D} \langle \nabla_y \mathcal{E}(f)(y), \nabla_y N(x-y) \rangle dy \tag{1.4}$$

for $x \in \mathbf{R}^d \setminus \overline{D}$, where

$$N(x-y) = \begin{cases} \frac{1}{\omega_d (d-2)|x-y|^{d-2}} & \text{if } d \geq 3 \\ -\frac{3R}{2\pi} \log \frac{|x-y|}{3R} & \text{if } d = 2 \end{cases}$$

and ω_d stands for the surface area of the unit ball in $\mathbf{R}^d$.

But the integrals of the right-hand side in (1.3) and (1.4) don't necessarily converge for any $f \in L^p(\mu)$. So we consider the Besov space of functions on ∂D. Let $0 < \alpha < 1$. We define,

$$\Lambda_\alpha^p(\partial D) = \{f \in L^p(\mu); \iint \frac{|f(x)-f(y)|^p}{|x-y|^{\beta+p\alpha}} d\mu(x) d\mu(y) < \infty\}$$

with norm

$$\|f\|_{p,\alpha} = \left(\int |f|^p d\mu\right)^{1/p} + \left(\iint \frac{|f(x)-f(y)|^p}{|x-y|^{\beta+p\alpha}} d\mu(x) d\mu(y)\right)^{1/p}.$$

If $\beta - (d-1) < \alpha < 1$ and $f \in \Lambda_\alpha^p(\partial D)$, then we saw in [W3] that Φf is well-defined and harmonic in $\mathbf{R}^d \setminus \partial D$.

Furthermore let $z \in \partial D$ and τ be a positive real number. We define nontangential approach regions in D and in $\mathbf{R}^d \setminus \overline{D}$ as follows:

$$\Gamma_\tau(z) = \{x \in D : |x-z| < (1+\tau)\delta(x)\} \tag{1.5}$$

and

$$\Gamma_\tau^e(z) = \{x \in \mathbf{R}^d \setminus \overline{D} : |x-z| < (1+\tau)\delta(x)\}, \tag{1.6}$$

where $\delta(x)$ stands for the distance of x from ∂D.

Let $\tau > 0$ and $r_1 > 0$. We consider the following conditions:

$$B(z,r) \cap \Gamma_\tau(z) \neq \emptyset, \tag{1.7}$$

for all $z \in \partial D$ and all $r \leq r_1$;

$$B(z,r) \cap \Gamma_\tau^e(z) \neq \emptyset \tag{1.8}$$

for all $z \in \partial D$ and all $r \leq r_1$.

In [W3] we proved the following theorem.

Theorem A. *Assume that D is a bounded domain in $\mathbf{R}^d$ $(d \geq 2)$ such that the boundary of D is a β-set $(d-1 \leq \beta < d)$. Furthermore assume that there are $\tau > 0$ and $r_1 > 0$ such that D (resp. $\mathbf{R}^d \setminus \overline{D}$) satisfies (1.7) (resp. (1.8)). If $p > 1$, $1 > \alpha > \beta - (d-1)$ and $f \in \Lambda_\alpha^p(\partial D)$, then*

$$\lim_{x \to z, x \in \Gamma_\tau(z)} \Phi f(x) = K_1 f(z)$$

$$\left(\textit{resp.} \lim_{x \to z, x \in \Gamma_\tau^e(z)} \Phi f(x) = K_2 f(z)\right)$$

at μ-almost every boundary point z, where K_1 and K_2 are operators defined by

$$K_1 f(z) = \begin{cases} \int_{\mathbf{R}^d \setminus \overline{D}} \langle \nabla \mathcal{E}(f)(y), \nabla_y N(z-y) \rangle dy & \textit{if it is well-defined} \\ 0 & \textit{otherwise} \end{cases} \tag{1.9}$$

and

$$K_2 f(z) = \begin{cases} -\int_D \langle \nabla \mathcal{E}(f)(y), \nabla_y N(z-y)\rangle dy & \textit{if it is well-defined} \\ 0 & \textit{otherwise.} \end{cases} \tag{1.10}$$

In this paper we shall study more precise boundary behavior of the double layer potentials. To do so, we define

$$N_\tau g(z) = \sup\{|g(x)|; x \in \Gamma_\tau(z) \cap B(z, r_1)\}$$

for a function g on D and $z \in \partial D$, and

$$N_\tau^e g(z) = \sup\{|g(x)|; x \in \Gamma_\tau^e(z) \cap B(z, r_1)\}$$

for a function g on $\mathbf{R}^d \setminus \overline{D}$ and $z \in \partial D$, where r_1 is the positive real number satisfying (1.7) and (1.8), respectively. In §2 we shall prove the following theorem.

Theorem 1. *Let D be a bounded domain in $\mathbf{R}^d$ such that ∂D is a β-set $(d-1 \le \beta < d)$. Furthermore assume that there are $\tau > 0$ and $r_1 > 0$ such that D (resp. $\mathbf{R}^d \setminus \overline{D}$) satisfies* (1.7) *(resp.* (1.8)*). Let $p > 1$, $\beta - (d-1) < \alpha < 1$ and $\beta - p\alpha > 0$. If $\lambda > \beta - p\alpha$ and ν is a positive Borel measure on ∂D satisfying*

$$\nu(B(z,r)) \le cr^\lambda \quad \textit{for every } z \in \partial D \textit{ and for every } r \le 3R,$$

then there exists a constant c such that, for every $f \in \Lambda_\alpha^p(\partial D)$,

$$\int (N_\tau(\Phi f))^p d\nu \le c\|f\|_{p,\alpha}^p \quad \textit{and} \quad \int (K_1 f)^p d\nu \le c\|f\|_{p,\alpha}^p \tag{1.11}$$

$$\left(\textit{resp.} \quad \int (N_\tau^e(\Phi f))^p d\nu \le c\|f\|_{p,\alpha}^p \textit{ and } \int (K_2 f)^p d\nu \le c\|f\|_{p,\alpha}^p\right). \tag{1.12}$$

Using Theorem 1, we shall prove the following theorem in §4.

Theorem 2. *Suppose D (resp. $\mathbf{R}^d \setminus \overline{D}$) satisfies the same assumptions as in Theorem 1.*

(i) *If $\beta - p\alpha > 0$, then, for every $\lambda > \beta - p\alpha$ and every $f \in \Lambda_\alpha^p(\partial D)$, there exists a subset E of ∂D such that $\mathcal{H}^\lambda(E) = 0$ and, for every $z \in \partial D \setminus E$, the integral of the right-hand side of* (1.9) *(resp.* (1.10)*) is well-defined and finite, and*

$$\lim_{x \to z, x \in \Gamma_\tau(z)} \Phi f(x) = K_1 f(z) \tag{1.13}$$

$$\left(\textit{resp.} \quad \lim_{x \to z, x \in \Gamma_\tau^e(z)} \Phi f(x) = K_2 f(z)\right),$$

where $\mathcal{H}^\lambda$ stands for the λ-dimensional Hausdorff measure.

(ii) *If $\beta - p\alpha < 0$, then, for every $f \in \Lambda_\alpha^p(\partial D)$ and for every $z \in \partial D$, the integral of the right-hand side of* (1.9) *(resp.* (1.10)*) is well-defined and finite, and*

$$\lim_{x \to z, x \in D} \Phi f(x) = K_1 f(z)$$

$$\left(\textit{resp.} \quad \lim_{x \to z, x \in \mathbf{R}^d \setminus \overline{D}} \Phi f(x) = K_2 f(z)\right).$$

2. Some lemmas and proof of Theorem 1

Hereafter we assume that D is a bounded domain in $\mathbf{R}^d$ such that the boundary is a β-set satisfying $d-1 \leq \beta < d$. Fix a β-measure μ on ∂D and a positive number R satisfying $\overline{D} \subset B(0, R/2)$ $(R \geq 1)$. We may assume that

$$b_3 r^\beta \leq \mu(B(z,r) \cap \partial D) \leq b_4 r^\beta$$

for all $z \in \partial D$ and all $r \leq 4R$.

Using the Whitney decomposition of $\mathbf{R}^d \setminus \partial D$, we contructed an extension operator $\mathcal{E}$ with the following properties in [W3].

Lemma B. *Let $p > 1$ and $\beta - (d-1) < \alpha < 1$, $\lambda \in \mathbf{R}$ and $f \in \Lambda^p_\alpha(\partial D)$. If $p(\alpha - 1) + d - \beta + p\lambda > 0$ and $p\lambda + d - \beta > 0$, then*

$$\left(\int |\nabla \mathcal{E}(f)(y)|^p \delta(y)^{\lambda p} dy \right)^{1/p} \leq c\|f\|_{p,\alpha},$$

where c is a constant independent of f.

In [W1] we gave the following lemmas.

Lemma C. *Let λ, k be positive numbers satisfying $d - \beta - \lambda > 0$ and $d - \lambda - k > 0$. Then*

$$\int_{B(z,r)} \delta(y)^{-\lambda} |y - z|^{-k} dy \leq c r^{d-\lambda-k}$$

for every $z \in \partial D$ and $0 < r \leq 3R$.

Lemma D. *Under the same assumptions as in Lemma C, the function*

$$x \mapsto \int_{B(0,2R)} \delta(y)^{-\lambda} |y - x|^{-k} dy$$

is bounded on $\mathbf{R}^d$.

It is easy to see that the following lemma holds.

Lemma 2.1. *Let $0 < \lambda \leq \beta$ and ν be a Borel measure on ∂D satisfying*

$$\nu(B(z,r)) \leq c r^\lambda \quad \text{for } z \in \partial D \text{ and for all } r \leq 3R.$$

(i) *If $\eta + \lambda > 0$, then*

$$\int_{|x-z| \leq r} |x - z|^\eta d\nu(x) \leq c r^{\eta+\lambda}$$

for all $r \leq 3R$ and $z \in \partial D$.

(ii) *If $\eta + \lambda < 0$, then*

$$\int_{|x-z| \geq r} |x - z|^\eta d\nu(x) \leq c r^{\eta+\lambda}$$

for all $r \leq 3R$ and $z \in \partial D$.

Here c is a constant independent of z and r.

We are now ready to prove Theorem 1.

Proof of Theorem 1. Set $q = p/(p-1)$. Since the support of $\mathcal{E}(f)$ is included in $B(0,2R)$, we have

$$|K_1 f(z)| \le c_1 \int_{B(0,2R)\setminus\overline{D}} |\nabla_y \mathcal{E}(f)(y)||z-y|^{1-d}dy \tag{2.1}$$

for every $z \in \partial D$.

Let us estimate $N_\tau(\Phi f)$. To do so, let $z \in \partial D$ and $x \in \Gamma_\tau(z) \cap B(z, r_1)$, and set

$$A = \{y \in B(0,2R) \setminus \overline{D}; |y-z| \le 2|x-z|\}, \quad B = \{y \in B(0,2R) \setminus \overline{D}; |y-z| > 2|x-z|\}.$$

If $y \in A$, then

$$|x-y| \ge \delta(x) \ge \frac{|x-z|}{1+\tau} \ge \frac{|y-z|}{2(1+\tau)}.$$

If $y \in B$, then

$$|x-y| \ge |y-z| - |z-x| \ge \frac{|y-z|}{2}.$$

Therefore we have

$$\begin{aligned}|\Phi f(x)| &\le c_1 \int_{\mathbf{R}^d\setminus\overline{D}} |\nabla \mathcal{E}(f)(y)||x-y|^{1-d}dy \\ &\le c_2 \int_{B(0,2R)\setminus\overline{D}} |\nabla \mathcal{E}(f)(y)||z-y|^{1-d}dy,\end{aligned}$$

whence

$$N_\tau(\Phi f)(z) \le c_2 \int_{B(0,2R)\setminus\overline{D}} |\nabla \mathcal{E}(f)(y)||z-y|^{1-d}dy. \tag{2.2}$$

Set

$$I(z) = \int_{\mathbf{R}^d\setminus\overline{D}} |\nabla \mathcal{E}(f)(y)||z-y|^{1-d}dy.$$

From (2.1) and (2.2) we deduce that it suffices to show

$$\int I(z)^p d\nu(z) \le c_3 \|f\|_{p,\alpha}^p. \tag{2.3}$$

To do so, let g be a function in $L^q(\nu)$ such that $\|g\|_{L^q(\nu)} \le 1$, where $q = p/(p-1)$. Since $\lambda > \beta - p\alpha > 0$, we choose $\epsilon > 0$ such that $\lambda - 2p\epsilon > \beta - p\alpha$ and $\alpha - \epsilon > \beta - d + 1$. We write

$$\begin{aligned}J &\equiv \left|\int I(z)g(z)d\nu(z)\right| \\ &\le \left(\int\int_{B(0,2R)\setminus\overline{D}} |\mathcal{E}(f)(y)|^p \delta(y)^{p-p\alpha-d+\beta+p\epsilon}|y-z|^{-\lambda+p\epsilon}dyd\nu(z)\right)^{1/p} \\ &\times \left(\int\int_{B(0,2R)\setminus\overline{D}} \delta(y)^{-q(1-\alpha-(d-\beta)/p+\epsilon)}|y-z|^{q(1-d+\lambda/p-\epsilon)}dyd\nu(z)\right)^{1/q}.\end{aligned}$$

Note that $-q(1-\alpha-(d-\beta)/p+\epsilon) + q(1-d+\lambda/p-\epsilon) + d = q(\lambda/p - 2\epsilon + \alpha - \beta/p) > 0$ and $d - \beta - q(1-\alpha-(d-\beta)/p+\epsilon) = q(\alpha - 1 + d - \beta - \epsilon) > 0$. Using Lemma C, we can show that the last integral is dominated by constant times of $\|g\|_{L^q(\nu)}$. From this we deduce

$$\begin{aligned}J &\le c_4\|g\|_{L^q(\nu)} \\ &\times \left(\int_{B(0,2R)\setminus\overline{D}} |\nabla\mathcal{E}(f)(y)|^p\delta(y)^{p-p\alpha-d+\beta+p\epsilon}dy \int |y-z|^{-\lambda+p\epsilon}d\nu(z)\right)^{1/p}.\end{aligned}$$

On account of Lemma 2.1 we have

$$J \le c_5 \|g\|_{L^q(\nu)} \left(\int_{B(0,2R)\setminus\overline{D}} |\mathcal{E}(f)(y)|^p \delta(y)^{p-p\alpha-d+\beta+p\epsilon} dy \right)^{1/p},$$

whence, by Lemma B,

$$\left(\int I(z)^p d\nu(z) \right)^{1/p} \le c_6 \left(\int_{B(0,2R)\setminus\overline{D}} |\mathcal{E}(f)(y)|^p \delta(y)^{p-p\alpha-d+\beta+p\epsilon} dy \right)^{1/p}$$
$$\le c_7 \|f\|_{p,\alpha}.$$

Thus we see that (1.11) holds. Similarly we can show (1.12). □

3. Approximation by Lipschitz functions

We next approximate $f \in \Lambda^p_\alpha(\partial D)$ by Lipschitz functions on ∂D. To do so we use the C^∞-function in $\mathbf{R}^d$ such that

$$0 \le \phi \le 1, \qquad \text{supp}\ \phi \subset B(0,1), \qquad \phi = 1 \text{ on } \overline{B(0,1/2)}.$$

For each $t > 0$ and each $x \in \partial D$, define

$$g_t(x) = \int \phi(\frac{x-y}{t}) d\mu(y)$$

and for $f \in \Lambda^p_\alpha(\partial D)$

$$H_t f(x) = \int \frac{\phi((x-y)/t)}{g_t(x)} f(y) d\mu(y).$$

We note that $g_t(x) \ge ct^\beta$ for every $x \in \partial D$ and $H_t 1 = 1$ on ∂D.

Lemma 3.1. *Let $f \in \Lambda^p_\alpha(\partial D)$. Then $H_t f$ is a Lipschitz function on ∂D and*

$$\|H_t f - f\|_{p,\alpha} \to 0 \quad \text{as } t \to 0.$$

Proof. First we show that $H_t f$ is a Lipschitz function. Since

$$|g_t(z) - g_t(x)| \le \sup_w |\nabla \phi(w)| \frac{1}{t} |z-x| \mu(\partial D), \tag{3.1}$$

we have

$$\left| \frac{\phi((x-y)/t)}{g_t(x)} - \frac{\phi((z-y)/t)}{g_t(z)} \right|$$
$$\le \left| \frac{\phi((x-y)/t) - \phi((z-y)/t)}{g_t(x)} + \phi(\frac{z-y}{t}) \frac{g_t(z) - g_t(x)}{g_t(z) g_t(x)} \right|$$
$$\le c_1 \frac{1}{t^{\beta+1}} \sup_w |\nabla\phi(w)| |x-z| + c_2 \frac{\mu(\partial D)}{t^{2\beta+1}} \sup_w |\nabla\phi(w)| |z-x|.$$

Hence we see that $H_t f$ is a Lipschitz fuction on ∂D.

We next define

$$J(x) := \int \frac{\phi((x-y)/t)}{g_t(x)} f(y) d\mu(y) - f(x)$$
$$= \int \frac{\phi((x-y)/t)}{g_t(x)} (f(y) - f(x)) d\mu(y).$$

Noting that $|x-y|>t$ implies $\phi((x-y)/t)=0$, we have

$$\begin{aligned}|J(x)| &\le c_3\frac{t^{\beta/p+\alpha}}{g_t(x)}\left(\int \frac{|f(y)-f(x)|^p}{|y-x|^{\beta+p\alpha}}d\mu(y)\right)^{1/p}\left(\int \phi(\frac{x-y}{t})^q d\mu(y)\right)^{1/q}\\ &\le c_4 t^\alpha\left(\int \frac{|f(y)-f(x)|^p}{|y-x|^{\beta+p\alpha}}d\mu(y)\right)^{1/p},\end{aligned}$$

whence

$$\left(\int |J(x)|^p d\mu(x)\right)^{1/p}\le c_4 t^\alpha \|f\|_{p,\alpha}.$$

Therefore we see that $\|H_tf-f\|_p\to 0$ as $t\to 0$.

Next, to see that $\|H_tf-f\|_{p,\alpha}\to 0$, we assume that $|z-x|\le t$ and write

$$\begin{aligned}&|H_tf(x)-f(x)-H_tf(z)+f(z)| \\ &\le \left|\int\left(\frac{\phi((x-y)/t)}{g_t(x)}-\frac{\phi((z-y)/t)}{g_t(z)}\right)(f(y)-f(x))\,d\mu(y)\right| \\ &+\left|\int\frac{\phi((z-y)/t)}{g_t(z)}(f(x)-f(z))d\mu(y)\right| \\ &\equiv I_1+I_2.\end{aligned}\tag{3.2}$$

If $|z-y|\le t$, then $|x-y|\le 2t$. Using the mean-value theorem, we have

$$|\phi(\frac{x-y}{t})-\phi(\frac{z-y}{t})|\le\frac{|x-z|}{t}\sup_w|\nabla\phi(w)|\chi_{B(x,2t)}(y)\tag{3.3}$$

and

$$|g_t(x)-g_t(z)|\le c_5t^{\beta-1}|x-z|.$$

Hence

$$\begin{aligned}I_1&\le c_6\frac{|x-z|}{t^{\beta+1}}\int_{|x-y|\le 2t}|f(y)-f(x)|d\mu(y)\\ &\le c_7t^{-\beta/q+\alpha-1}|x-z|\int_{|x-y|\le 2t}\frac{|f(y)-f(x)|}{|y-x|^{\beta/p+\alpha}}d\mu(y)\\ &\le c_8t^{\alpha-1}|x-z|\left(\int_{|x-y|\le 2t}\frac{|f(y)-f(x)|^p}{|y-x|^{\beta+p\alpha}}d\mu(y)\right)^{1/p}.\end{aligned}$$

Using Lemma 2.1 for $\nu=\mu$, we obtain

$$\begin{aligned}&\int d\mu(x)\int_{|x-z|\le t}\frac{I_1^p}{|x-z|^{\beta+\alpha p}}d\mu(z)\\ &\le c_9t^{\alpha-1}\int d\mu(x)\int_{|x-y|\le 2t}\frac{|f(y)-f(x)|^p}{|y-x|^{\beta+p\alpha}}d\mu(y)\int_{|x-z|\le t}|x-z|^{\beta+p(1-\alpha)}d\mu(z)\\ &\le c_{10}\int d\mu(x)\int_{|x-y|\le 2t}\frac{|f(y)-f(x)|^p}{|y-x|^{\beta+p\alpha}}d\mu(y).\end{aligned}\tag{3.4}$$

Furthermore we have

$$\begin{aligned}&\int d\mu(x)\int_{|x-z|\le t}\frac{I_2^p}{|x-z|^{\beta+\delta p}}d\mu(z)\\ &\le c_{11}\frac{1}{t^{\beta p}}\int d\mu(x)\int_{|x-z|\le t}\frac{|f(x)-f(z)|^p}{|x-z|^{\beta+p\alpha}}d\mu(z)\left(\int\phi(\frac{z-y}{t})d\mu(y)\right)^p\\ &\le c_{12}\int d\mu(x)\int_{|x-z|\le t}\frac{|f(x)-f(z)|^p}{|x-z|^{\beta+p\alpha}}d\mu(z).\end{aligned}\tag{3.5}$$

The inequalities (3.2), (3.4) and (3.5) show that

$$\iint_{|x-z|\le t} \frac{|H_t f(x) - f(x) - H_t f(z) - f(z)|^p}{|x-z|^{\beta+p\alpha}} d\mu(x)d\mu(z) \to 0$$

as $t \to 0$.

We next assume that $|x-z| > t$. Set

$$J_1 = \int \frac{\phi((x-y)/t)}{g_t(x)}(f(y) - f(x))d\mu(y).$$

Then

$$\begin{aligned} J_1 &\le c_{13}\frac{t^{\beta/p+\alpha}}{t^\beta}\int_{|x-y|<t}\frac{|f(y)-f(x)|}{|y-x|^{\beta/p+\alpha}}d\mu(y) \\ &\le c_{14}t^\alpha\left(\int_{|x-y|\le t}\frac{|f(y)-f(x)|^p}{|y-x|^{\beta+p\alpha}}d\mu(y)\right)^{1/p}, \end{aligned}$$

whence

$$\begin{aligned} &\int d\mu(x)\int_{|x-z|>t}\frac{J_1^p}{|x-z|^{\beta+\alpha p}}d\mu(z) \\ &\le c_{15}t^{p\alpha}\int d\mu(x)\int_{|x-y|\le t}\frac{|f(y)-f(x)|^p}{|y-x|^{\beta+p\alpha}}d\mu(y)\int_{|x-z|>t}\frac{d\mu(z)}{|x-z|^{\beta+p\alpha}}. \end{aligned}$$

Therefore we have, by Lemma 2.1,

$$\int\int_{|x-z|>t}\frac{J_1^p}{|x-z|^{\beta+\alpha p}}d\mu(z)d\mu(x) \le c_{16}\int d\mu(x)\int_{|x-y|\le t}\frac{|f(y)-f(x)|^p}{|y-x|^{\beta+p\alpha}}d\mu(y).$$

Similarly we set

$$J_2 := \int \frac{\phi((z-y)/t)}{g_t(z)}(f(y) - f(z))d\mu(y)$$

and also have

$$\int\int_{|x-z|>t}\frac{J_2^p}{|x-z|^{\beta+\alpha p}}d\mu(z)d\mu(x) \le c_{17}\int d\mu(z)\int_{|z-y|\le t}\frac{|f(y)-f(z)|^p}{|y-z|^{\beta+p\alpha}}d\mu(y),$$

whence

$$\iint_{|x-z|>t} \frac{|H_t f(x) - f(x) - H_t f(z) - f(z)|^p}{|x-z|^{\beta+p\alpha}} d\mu(x)d\mu(z) \to 0$$

as $t \to 0$. Thus we have the conclusion. □

4. Proof of Theorem 2

In this section we shall prove Theorem 2.

Proof of Theorem 2 (i) Let $f \in \Lambda_\alpha^p(\partial D)$ and denote by G the set of all $z \in \partial D$ such that the integral of right-hand side of (1.9) is not well-defined or $K_1 f(z) = \pm\infty$ or the indicated limit does not exist or $\lim_{x\to z, x\in\Gamma_\tau(z)} \Phi f(x)(x) \ne K_1 f(z)$. We assume that $\mathcal{H}^\lambda(G) > 0$ and get a contradiction.

We shall first show that G is a Borel measurable set. To see this, let r be a positive real number satisfying $r \le r_1$ and set

$$M_{\tau,r}(\Phi f)(z) = \sup\{\Phi f(x); x \in \Gamma_\tau(z) \cap B(z,r)\}.$$

To see that $M_{\tau,r}(\Phi f)$ is lower semicontinuous on ∂D, let $b \in \mathbf{R}$ and $M_{\tau,r}(\Phi f)(z_0) > b$. Then there is $x_0 \in \Gamma_\tau(z_0) \cap B(z_0, r)$ such that $\Phi f(x_0) > b$. Choose $\epsilon > 0$ satisfying

$$|x_0 - z_0| + \epsilon < (1+\tau)\delta(x_0) \quad \text{and } |x_0 - z_0| + \epsilon < r.$$

Then, $x_0 \in \Gamma_\tau(z) \cap B(z, r)$ for each $z \in \partial D \cap B(z_o, \epsilon)$ and hence $M_{\tau,r}(\Phi f)(z) > b$. Therefore $M_{\tau,r}(\Phi f)$ is lower semicontinuous on ∂D.

Similarly the function

$$z \mapsto \inf\{\Phi f(x); x \in \Gamma_\tau(z) \cap B(z,r)\}$$

is upper semicontinuous on ∂D. So we see that the set

$$\{z \in \partial D; \limsup_{x\to z, x\in\Gamma_\tau(z)} \Phi f(x) \neq \liminf_{x\to z, x\in\Gamma_\tau(z)} \Phi f(x)\}$$

is Borel measurable. Since the function $z \mapsto K_1 f(z)$ is Borel measurable, the set G is Borel measurable.

Since we assumed that $\mathcal{H}^\lambda(G) > 0$, there is a compact subset F of G such that $\mathcal{H}^\lambda(F) > 0$. Using Frostman's lemma, we can find a Borel measure ν supported on F such that $\nu(F) > 0$, and $\nu(B(a,r)) \le r^\lambda$ for every $a \in \mathbf{R}^d$ and $r > 0$ (cf. [C, Theorem 1 in Chapter 2]).

We claim that

$$\lim_{x\to z, x\in\Gamma_\tau(z)} \Phi f(x) = K_1 f(z)$$

for all $z \in \partial D$ with the exception of a set E with $\nu(E) = 0$. To see this, denote by E_0 the set of all $z \in \partial D$ such that the integral of the right-hand side of (1.9) is not well-defined or $K_1 f(z) = \pm\infty$. By (2.3) we see that $\nu(E_0) = 0$.

For each $\epsilon > 0$ we put

$$E_\epsilon = \{z \in \partial D \setminus E_0; \limsup_{x\to z, x\in\Gamma_\tau(z)} |\Phi f(x) - K_1 f(z)| > \epsilon\}.$$

We show that $\nu(E_\epsilon) > 0$. To do so, we choose a sequence $\{f_n\}$ of Lipschits functions such that $\|f_n - f\|_{p,\alpha} \to 0$. This choice is possible by Lemma 3.1. In [W2, Lemma 3.3 and Theorem 1] we saw that if g is a Lipschitz function on ∂D, then

$$\lim_{x\to z, x\in D} \Phi g(x) = K_1 g(z) \quad \text{for every } z \in \partial D. \tag{4.1}$$

Let $z \in E_\epsilon$. Then, by (4.1),

$$\begin{aligned}\epsilon &< \limsup_{x\to z, x\in\Gamma_\tau(z)} |\Phi f(x) - K_1 f(z)| \\ &\le \limsup_{x\to z, x\in\Gamma_\tau(z)} |\Phi f(x) - \Phi f_n(x)| + |K_1 f_n(z) - K_1 f(z)|,\end{aligned}$$

whence

$$1 \le \frac{c_1}{\epsilon^p}\left(N_\tau(f-f_n)(z)^p + K_1(f-f_n)(z)^p\right) \quad \text{on } E_\epsilon.$$

This and Theorem 1 yield

$$\begin{aligned}\nu(E_\epsilon) &\le \frac{c_1}{\epsilon^p}\left(\int N_\tau(f-f_n)(z)^p d\nu(z) + \int K_1(f-f_n)(z)^p d\nu(z)\right) \\ &\le \frac{c_2}{\epsilon^p}\|f - f_n\|_{p,\alpha}^p.\end{aligned}$$

As $n \to \infty$, we see that $\nu(E_\epsilon) = 0$ and the claim is true. This yields the desired contradiction.

(ii) Let $f \in \Lambda^p_\alpha(\partial D)$. Since $p\alpha - \beta > 0$, we choose $\epsilon > 0$ satisfying $p\alpha - \beta - 2p\epsilon > 0$ and $\alpha - \epsilon > \beta - (d-1)$. Let $x \in D$ and $z \in \partial D$. Then

$$\begin{aligned} I &\equiv \int_{\mathbf{R}^d \setminus \overline{D}} |\langle \nabla \mathcal{E}(f)(y), \nabla_y N(x-y) - \nabla_y N(z-y)\rangle| dy \\ &\leq c_3 |x-z|^\epsilon \int_{\mathbf{R}^d \setminus \overline{D}} |\nabla \mathcal{E}(f)(y)||x-y|^{1-d-\epsilon} \\ &+ c_3 |x-z|^\epsilon \int_{\mathbf{R}^d \setminus \overline{D}} |\nabla \mathcal{E}(f)(y)||z-y|^{1-d-\epsilon} \equiv I_1 + I_2. \end{aligned}$$

By Hölder's inequality we have

$$\begin{aligned} I_1 &\leq c_3 |x-z|^\epsilon \left(\int_{\mathbf{R}^d \setminus \overline{D}} |\nabla \mathcal{E}(f)(y)|^p \delta(y)^{p-p\alpha-d+\beta+p\epsilon} dy \right)^{1/p} \\ &\times \left(\int_{B(0,2R)} \delta(y)^{-q(1-\alpha-(d-\beta)/p+\epsilon)} |x-y|^{q(1-d-\epsilon)} dy \right)^{1/q}. \end{aligned}$$

Noting that $-q(1-\alpha-(d-\beta)/p+\epsilon) + q(1-d-\epsilon) + d = q(\alpha - 2\epsilon - \beta/p) > 0$ and using Lemmas D, B, we obtain

$$I_1 \leq c_4 |x-z|^\epsilon \left(\int_{\mathbf{R}^d \setminus \overline{D}} |\nabla \mathcal{E}(f)(y)|^p \delta(y)^{p-p\alpha-d+\beta+p\epsilon} dy \right)^{1/p} \leq c_5 |x-z|^\epsilon \|f\|_{p,\alpha}.$$

Similarly we have the same estimate for I_2 and hence

$$I \leq c_6 |x-z|^\epsilon \|f\|_{p,\alpha}. \tag{4.2}$$

Since $\Phi f(z)$ is finite, we see, by (4.2), that the integral of the right-hand side of (1.9) is well-defined at each $z \in \partial D$. We also see that $I \to 0$ as $x \to z$ in D.

Since we can prove similarly the assertions related to the set $\mathbf{R}^d \setminus \overline{D}$ and $K_2 f$, we have the conclusion. □

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