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A NOTE ON THE PROPERTY OF LINEAR CELLULAR AUTOMATA

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Abstract

For a p^r-state linear cellular automaton L, we give a systematic set which has one-to-one, onto correspondence with the set $\{0,1,2,\dots,p^r-1\}$. This set may play an important role in examing the limit set of space-time patterns as a $\mathbb{Z}_{p^r}$-valued upper semi continuous function.

1 Introduction

Cellular automata are discrete dynamical systems with simple construction. We define a cellular automaton as follows.

Put $\mathbb{Z}_n = \{0,1,\dots,n-1\}$ for $n \in \mathbb{N}$. Let p be a prime number, r a natural number and $\mathcal{P}$ the set of all configurations $w : \mathbb{Z} \to \mathbb{Z}_{p^r}$ with compact support. Let a linear transition rule $L : \mathcal{P} \to \mathbb{Z}_{p^r}$ be defined as follows:

$$Lw(x) \equiv \sum_{k\in G} c_k w(x+k) \pmod{p^r} \quad \text{for } w \in \mathcal{P}, \tag{1.1}$$

where the set G is a finite subset of $\mathbb{Z}$ with $\sharp G \geq 2$ and $c_k \in \mathbb{N}$.

Patterns of linear cellular automata were studied by some people. Existence of the limit of a series of space-time patterns is proved [1, 3, 5, 6]. E. Jen showed that a series $\{L^t w(x) \,|\, t \in \mathbb{N}\}$ is aperiodic for some L with $p = 2$ in [2]. In [5], S. Takahashi considered the case where L is p^r-state linear cellular automata with the initial state δ_0 which is 1 at the origin and 0 at others. He examined the limit set with respect to each non-zero state, by using the fact that every state appears in the set $\{L^t\delta_0(-(t-1)r_1 - r_2) \,|\, t = 1,\dots,p^{r+1}\}$. However, when we consider the limit set as a multi-valued function, the set $\{L^t\delta_0(-(t-1)r_1 - r_2) \,|\, t = 1,\dots,p^{r+1}\}$ does not work well. Hence we need another set which includes every state and plays a useful role in examing the limit set. So we give a systematic set which has one-to-one, onto correspondence with the set $\{0,1,2,\dots,p^r-1\}$. This set may play an important role in examing the limit set of space-time patterns as a $\mathbb{Z}_{p^r}$-valued upper semi continuous function [4].

2 The result

We deal with the specified transition rule, which L satisfies some condition. We say that L *satisfies the condition* (A) if there exists $r_1, r_2 \in G$ satisfying

(I) $c_{r_1}/p, c_{r_2}/p \notin \mathbb{N}$,

(II) r_1 is an either maximum or minimum element of the set G,

(III) r_2 is extreme or $r_2 = \sum_{k\in G, r_2\neq k} \beta_k k$ with $0 \leq \beta_k < 1$ such that $\sum_{k\in G, r_2\neq k} \beta_k = 1$ and $p^{r-1}\beta_k \notin \mathbb{N}$,

(IV) if r_1 is maximum [resp. minimum], then for $s \in \{1,2,\dots,p-1\}$, $l \in \{1,2,\dots,|r_1-r_2|-1\}$ there does not exist a path from $-r_1 s(p^r - p^{r-1}) + l$ [resp. $-r_1 s(p^r - p^{r-1}) - l$] to the origin, that is, we have

$$-\sum_{k\in G} \alpha_k k \notin \{-r_1 s(p^r - p^{r-1}) + l \,|\, l \in \{1,2,\dots,|r_1-r_2|-1\}\}$$

$$[\text{resp. } -\sum_{k\in G}\alpha_k k \notin \{-r_1 s(p^r - p^{r-1}) - l \,|\, l \in \{1,2,\ldots,|r_1 - r_2| - 1\}\}],$$

for the set $\{\alpha_k\}_{k\in G} \subset \mathbb{N}\cup\{0\}$ such that $\sum_{k\in G}\alpha_k = s(p^r - p^{r-1})$ with $s \in \{1,2,\ldots,p-1\}$.

Here, an element $k \in G$ is *extreme* if an element k is not expressed as a convex linear combination of other elements of G. We note that a maximum or minimum element of the set G is extreme.

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Figure 1: Space-time pattern of $L_3\delta_0(x) \equiv 3\delta_0(x-1) + 5\delta_0(x+1) \pmod{2^3}$

By using $r_1, r_2 \in G$ which satisfy (I),(II),(III) and (IV), put

$$t(r,j) = j(p^r - p^{r-1}) \tag{2.1}$$

and

$$i(r,j) = -(t(r,j) - p^{r-1})r_1 - p^{r-1}r_2 \tag{2.2}$$

for $j \in \mathbb{N}$. We define $\delta_0 \in \mathcal{P}$ as

$$\delta_0(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0. \end{cases}$$

We shall prove that the set $\{L^{t(r,j)}\delta_0(i(r,j)) | 1 \le j \le p^r\}$ has one-to-one, onto correspondence with the set $\{0,1,2,\ldots,p^r - 1\}$. We shall call the set $\{a_n \,|\, n = 1,\ldots,k\}$ a *k-set*, if the set has one-to-one, onto correspondence with the set $\{0,1,\ldots,k-1\}$.

We need the following lemmas later.

Lemma 1. [5] *Suppose L is defined as (1.1). For $j, l \in \mathbb{N}$, we have*

$$L^{p^{l+r-1}j}\delta_0(x) = \begin{cases} L^{jp^{r-1}}\delta_0(y) & \textit{if there exists } y \textit{ such that } p^l y = x \\ 0 & \textit{otherwise} \end{cases}.$$

Lemma 2. *Let $q \in \{1,2,\ldots,p-1\}$ and $t = jp^{r-1}$ with $j \in \mathbb{N}$. If $v = p^l q$ for l $(0 \le l \le r-2)$, then $t - v = p^l q'$ for some $q' \in \mathbb{N}$ such that $q'/p \notin \mathbb{N}$.*

Proof. We have $t - v = p^l(jp^{r-l-1} - q)$. Put $q' = jp^{r-l-1} - q$. So we obtain $q'/p \notin \mathbb{N}$ by $q \in \{1, 2, \dots, p-1\}$. □

Lemma 3. *Let L be defined as (1.1) and satisfy the condition (A). Suppose $r \in \mathbb{N}$, $r \geq 2$ and $j \in \{1, 2, \dots, p^r\}$. Let $r_1, r_2 \in G$ satisfy (I),(II),(III) and (IV) of the condition (A). Then the following assertions hold:*

(i) $L^{t(r,1)}\delta_0(i(r,1))/p \notin \mathbb{N}$.

(ii) *The set* $\{nL^{t(1,1)}\delta_0(i(1,1)) \bmod p \,|\, 1 \leq n \leq p\}$ *is a p-set.*

(iii) *Suppose* $|r_1 - r_2| \geq 2$. *If* $r_1 > r_2$ [*resp.* $r_1 < r_2$], *then*

$$\begin{aligned} L^{t(r,j)}\delta_0(-t(r,j)r_1 + l) &= 0 \\ [\textit{resp. } L^{t(r,j)}\delta_0(-t(r,j)r_1 - l) &= 0] \end{aligned} \tag{2.3}$$

holds for $l \in \{1, 2, \dots, |r_1 - r_2| - 1\}$ *and*

$$\begin{aligned} L^{t(r,sp^{r-1})}\delta_0(-t(r,sp^{r-1})r_1 + l) &= 0 \\ [\textit{resp. } L^{t(r,sp^{r-1})}\delta_0(-t(r,sp^{r-1})r_1 - l) &= 0] \end{aligned}$$

holds for $l \in \{1, 2, \dots, p^{r-1}|r_1 - r_2| - 1\}$ *and* $s \in \{1, 2, \dots, p-1\}$.

Proof. (i) For $a, b \in \mathbb{N}$ put ${}_{a+b}\mathrm{C}_b = (a+b)!/(a!b!)$. We have

$$L^{t(r,1)}\delta_0(i(r,1)) \equiv {}_{p^r - p^{r-1}}\mathrm{C}_{p^{r-1}} c_{r_1}^{p^r - p^{r-1} - p^{r-1}} c_{r_2}^{p^{r-1}} \pmod{p^r}$$

by (II) and (III) of the condition (A). Since

$${}_{p^r - p^{r-1}}\mathrm{C}_{p^{r-1}} = \frac{(p^r - p^{r-1})(p^r - p^{r-1} - 1)\cdots(p^r - p^{r-1} - p^{r-1} + 1)}{p^{r-1}!},$$

p does not divide ${}_{p^r - p^{r-1}}\mathrm{C}_{p^{r-1}}$ by Lemma 2. Then p does not divide $L^{t(r,1)}\delta_0(i(r,1))$ by (I) of the condition (A).

(ii) We shall show that $nL^{t(1,1)}\delta_0(i(1,1)) \not\equiv 0 \pmod{p}$ holds for all $n \in \{1, 2, \dots, p-1\}$. The proof is by contradiction. Assume that there exists $n_0 \in \{1, 2, \dots, p-1\}$ such that $n_0 L^{t(1,1)}\delta_0(i(1,1)) = s_0 p$ holds with some $s_0 \in \mathbb{N}$. Then we have $L^{t(1,1)}\delta_0(i(1,1)) = s_0 p/n_0$. Since $n_0 \leq p-1$, $s_0/n_0 \in \mathbb{N}$ holds. Therefore p divide $L^{t(1,1)}\delta_0(i(1,1))$, which contradicts assumption.

(iii) We only show the case where $r_1 > r_2$. Since $1 \leq l \leq r_1 - r_2 - 1$, there does not exist the path from the origin to the point $-t(r,j)r_1 + l$ by (IV) of the condition (A). So we have $L^{t(r,j)}\delta_0(-t(r,j)r_1 + l) = 0$. Since $t(r, sp^{r-1}) = p^{r-1}t(r,s)$, we have

$$\begin{aligned} L^{t(r,sp^{r-1})}\delta_0(-t(r,sp^{r-1})r_1) &= L^{t(r,s)}\delta_0(-t(r,s)r_1) \\ L^{t(r,sp^{r-1})}\delta_0(-t(r,sp^{r-1})r_1 + p^{r-1}(r_1 - r_2))) &= L^{t(r,s)}\delta_0(-t(r,s)r_1 + (r_1 - r_2)) \\ L^{t(r,sp^{r-1})}\delta_0(-t(r,sp^{r-1})r_1 + l) &= 0 \end{aligned}$$

for $l \in \{1, 2, \dots, p^{r-1}(r_1 - r_2) - 1\}$ and $s \in \{1, 2, \dots, p-1\}$ by Lemma 1 and the equation (2.3). In case $r_1 < r_2$, we can prove it in the same way as above.

□

Lemma 4. *Suppose $r \geq 2$ and L satisfies the condition (A). Then*

$$L^{t(r,j)}\delta_0(i(r,j)) \equiv L^{t(r,m)}\delta_0(i(r,m)) + sL^{t(r,p^{r-1})}\delta_0(i(r,p^{r-1})) \pmod{p^r}$$

holds for $j = sp^{r-1} + m$ with $s \in \{0, 1, \dots, p-1\}$ and $m \in \{1, 2, \dots, p^{r-1}\}$.

Proof. Let $r_1, r_2 \in G$ satisfy (I),(II),(III) and (IV) of the condition (A). We first consider the case where $r_1 > r_2$. We compute $L^{t(r,j)}\delta_0(i(r,j))$ from the values at time $t(r,j-1)$. By the property of L there exists $B(r,k) \in \mathbb{N}$ for $k \in \{1,2,\ldots,p^{r-1}(r_1-r_2)-1\}$ such that $B(r,k)$ does not depend on j and

$$\begin{aligned} L^{t(r,j)}\delta_0(i(r,j)) &\equiv {}_{p^r-p^{r-1}}\mathrm{C}_{p^{r-1}} c_{r_1}^{p^r-2p^{r-1}} c_{r_2}^{p^{r-1}} \\ &\quad + \sum_{k=1}^{p^{r-1}(r_1-r_2)-1} B(r,k)b(r,j-1,k) \\ &\qquad + c_{r_1}^{p^r-p^{r-1}} L^{t(r,j-1)}\delta_0(i(r,j-1)) \pmod{p^r}, \end{aligned}$$

where $b(r,j,k) = L^{t(r,j)}\delta_0(-t(r,j)r_1+k)$ for $k \in \{1,2,\ldots,p^{r-1}(r_1-r_2)-1\}$. Put

$$d(j) = \sum_{k=1}^{p^{r-1}(r_1-r_2)-1} B(r,k)b(r,j,k),$$

and we rewrite the equation above as follows:

$$\begin{aligned} L^{t(r,j)}\delta_0(i(r,j)) &\equiv L^{t(r,1)}\delta_0(i(r,1)) + d(j-1) + L^{t(r,j-1)}\delta_0(i(r,j-1)) \\ &\equiv jL^{t(r,1)}\delta_0(i(r,1)) + \sum_{l=1}^{j-1} d(l) \pmod{p^r} \end{aligned}$$

from Euler's theorem ($n^{p^r-p^{r-1}} \equiv 1 \pmod{p^r}$).

We have $b(r,sp^{r-1},k) = 0$ for all $k \in \{1,2,\ldots,p^{r-1}(r_1-r_2)-1\}$ and all $s \in \{1,2,\ldots,p-1\}$ by Lemma 3(iii) and $L^{t(r,sp^{r-1})}\delta_0(-t(r,sp^{r-1})r_1) \equiv 1 \pmod{p^r}$. Since $B(r,k)$ does not depend j, we have

$$d(m+sp^{r-1}) \equiv d(m) \pmod{p^r} \tag{2.4}$$

for $m \in \{1,2,\ldots,p^{r-1}\}$ and $s \in \{1,2,\ldots,p-1\}$ by (II) of the condition (A).

For $j = sp^{r-1}+m$ with $m \in \{1,2,\ldots,p^{r-1}\}$ and $s \in \{0,1,\ldots,p-1\}$

$$\begin{aligned} L^{t(r,j)}\delta_0(i(r,j)) &\equiv jL^{t(r,1)}\delta_0(i(r,1)) + \sum_{l=1}^{j-1} d(l) \\ &\equiv sp^{r-1}L^{t(r,1)}\delta_0(i(r,1)) + s\sum_{l=1}^{p^{r-1}-1} d(l) + mL^{t(r,1)}\delta_0(i(r,1)) + \sum_{l=1}^{m-1} d(l) \\ &\equiv L^{t(r,m)}\delta_0(i(r,m)) + sL^{t(r,p^{r-1})}\delta_0(i(r,p^{r-1})) \pmod{p^r} \end{aligned}$$

by (2.4). In case $r_1 < r_2$, putting $b(r,j,k) = L^{t(r,j)}\delta_0(-t(r,j)r_1-k)$, we can prove in the same way. □

Put

$$a(r,j) = L^{t(r,j)}\delta_0(i(r,j)) \tag{2.5}$$

for convenience. We will prove the set $\{a(r,j) \mid 1 \le j \le p^r\}$ is a p^r-set. In order to prove the following lemma, we define a map $\mathcal{L} : \mathcal{P} \to \mathbb{N}$ as follows:

$$\mathcal{L}w(x) = \sum_{k \in G} c_k w(x+k), \tag{2.6}$$

where the set G and c_k are as in the definition of L. We note that there exists $k(r,j) \in \mathbb{N}$ such that

$$\mathcal{L}^{t(r,j)}\delta_0(i(r,j)) = k(r,j)p^r + a(r,j). \tag{2.7}$$

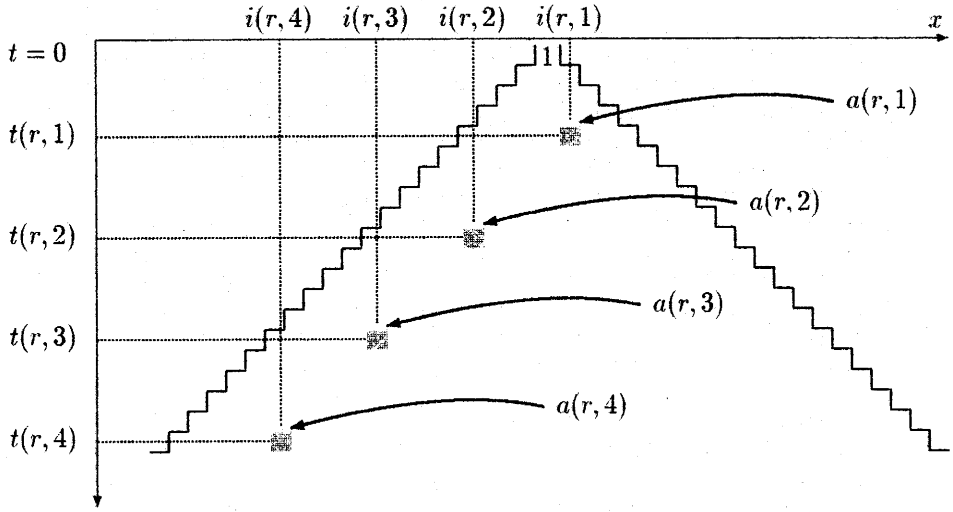

Figure 2: The relation among $t(r,j), i(r,j)$ and $a(r,j)$.

Lemma 5. *Suppose $a(r,j)$ is defined as (2.5) and the set $\{a(r,j) \mid 1 \le j \le p^r\}$ is a p^r-set. Then the following assertions hold:*

(i) $a(r+1,j) \ne a(r+1,l)$ *holds for* $j,l \in \{1,2,\dots,p^r\}$ *with* $j \ne l$.

(ii) *There exists* $k_0 \in \mathbb{N}(0 \le k_0 \le p-1)$ *such that* $a(r,p^{r-1}) = p^{r-1}k_0$.

(iii) $a(r+1,j) \not\equiv a(r+1,l) + ka(r+1,p^r) \pmod{p^{r+1}}$ *holds for any* $k \in \{1,2,\dots,p-1\}$ *and* $j,l \in \{1,2,\dots,p^r\}$ *with* $j \ne l$.

(iv) $a(r+1,j)+k_1a(r+1,p^r) \not\equiv a(r+1,j)+k_2a(r+1,p^r) \pmod{p^{r+1}}$ *holds for any* $k_1,k_2 \in \{1,2,\dots,p-1\}$ *with* $k_1 \ne k_2$ *and* $j \in \{1,2,\dots,p^r\}$.

Proof. We have

$$a(r,j) \equiv \mathcal{L}^{t(r,j)}\delta_0(i(r,j)) \equiv \mathcal{L}^{t(r+1,j)}\delta_0(i(r+1,j)) \pmod{p^r} \tag{2.8}$$

for $j \in \{1,2,\dots,p^r\}$ by Lemma 4, since

$$\mathcal{L}^{pt(r,j)}\delta_0(pi(r,j)) = \mathcal{L}^{t(r+1,j)}\delta_0(i(r+1,j)) \tag{2.9}$$

by Lemma 1 and

$$\mathcal{L}^{t(r,j)}\delta_0(i(r,j)) \equiv \mathcal{L}^{pt(r,j)}\delta_0(pi(r,j)) \pmod{p^r}. \tag{2.10}$$

Therefore by (2.8) there exists $k'(r,j) \in \mathbb{N}$ for $j \in \{1,2,\dots,p^r\}$ such that

$$\mathcal{L}^{t(r+1,j)}\delta_0(i(r+1,j)) = k'(r,j)p^r + a(r,j). \tag{2.11}$$

So we obtain by (2.7) and (2.11)

$$k(r+1,j)p^{r+1} + a(r+1,j) = k'(r,j)p^r + a(r,j) \tag{2.12}$$
$$k(r+1,l)p^{r+1} + a(r+1,l) = k'(r,l)p^r + a(r,l). \tag{2.13}$$

for $j,l \in \{1,2,\dots,p^r\}$.

(i) Assume $a(r+1,j) = a(r+1,l)$ holds for $j,l \in \{1,2,\dots,p^r\}$ with $j \ne l$. By (2.12) and (2.13) $a(r,j)-a(r,l) = (k(r+1,j)-k(r+1,l))p^{r+1} - (k'(r,j)-k'(r,l))p^r$, which contradicts the assumption.

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Figure 3: Example of $L_3\delta_0(x) \equiv 3\delta_0(x-1) + 5\delta_0(x+1) \pmod{2^3}$. The values in the dotted regions are $a(3,1)$, $a(3,2)$, $\ldots$, $a(3,5)$.

(ii) We will prove it by induction on r.

(a) In case $r = 1$, it is clear by definition.

(b) In case $r > 1$, assume that it is true for $r = r'$. Then we have $a(r', p^{r'}) \equiv pa(r', p^{r'-1}) \equiv pp^{r'-1}k_0 \equiv p^{r'}k_0 \pmod{p^{r'}}$ by Lemma 4 and the assumption of induction. So there exists $k_0' \in \mathbb{N}(0 \le k_0' \le p-1)$ such that $a(r'+1, p^{r'}) = p^{r'}k_0'$ by (2.12).

(iii) The proof is by contradiction. Assume that there exists $k_1 \in \mathbb{N}$ such that $a(r+1, j) \equiv a(r+1, l) + k_1 a(r+1, p^r) \pmod{p^{r+1}}$. Then there exists $s_0 \in \mathbb{N}$ such that $a(r+1,j) - a(r+1,l) - k_1 a(r+1,p^r) = s_0 p^{r+1}$. There exists $k_0 \in \{0, 1, \ldots, p-1\}$ such that $a(r+1, p^r) = p^r k_0$ by the assertion (ii). By the equations (2.12) and (2.13), we have

$$\begin{aligned} a(r,j) - a(r,l) &= (k(r+1,j) - k(r+1,l))p^{r+1} - (k'(r,j) - k'(r,l))p^r \\ &\qquad + a(r+1,j) - a(r+1,l) \\ &= (k(r+1,j) - k(r+1,l))p^{r+1} - (k'(r,j) - k'(r,l))p^r \\ &\qquad + s_0 p^{r+1} + k_0 k_1 p^r \\ &= p^r\{(k(r+1,j) - k(r+1,l))p - (k'(r,j) - k'(r,l)) + s_0 p + k_0 k_1\}, \end{aligned}$$

which contradicts $a(r,j) \ne a(r,l)$ by $0 \le a(r,j) - a(r,l) \le p^r - 1$.

(iv) It is clear by the property of modulus.

□

Proposition. *Suppose $r \in \mathbb{N}$. If the set $\{a(r,j) \mid 1 \le j \le p^r\}$ is a p^r-set, then the set $\{a(r+1,j) \mid 1 \le j \le p^{r+1}\}$ is a p^{r+1}-set.*

Proof. We obtain the conclusion by the assertion (i), (iii) and (iv) of Lemma 4 and Lemma 5. □

Theorem. *For a prime number p and $r \in \mathbb{N}$, let L be defined as* (1.1) *and satisfy the condition* (A). *Put* $t(r,j) = j(p^r - p^{r-1})$ *and* $i(r,j) = -(t(r,j) - p^{r-1})r_1 - p^{r-1}r_2$, *where* $r_1, r_2 \in G$ *satisfy* (I),(II),(III) *and* (IV) *of the condition* (A).

Then the set $\{L^{t(r,j)}\delta_0(i(r,j)) \,|\, 1 \le j \le p^r\}$ *is a* p^r*-set.*

Proof. The proof is by induction on r.

(i) In case $r = 1$, from Lemma 3(iii),

$$\begin{aligned} L^{t(1,j)}\delta_0(i(1,j)) &\equiv {}_{t(1,1)}C_p c_{r_1}^{t(1,1)-1} c_{r_2} + c_{r_1}^{t(1,1)} L^{t(1,j-1)}\delta_0(i(1,j-1)) \\ &\equiv L^{t(1,1)}\delta_0(i(1,1)) + L^{t(1,j-1)}\delta_0(i(1,j-1)) \\ &\equiv jL^{t(1,1)}\delta_0(i(1,1)) \qquad \pmod p \end{aligned}$$

for $1 \le j \le p$ by Euler's theorem ($n^{p^r - p^{r-1}} \equiv 1 \pmod{p^r}$ for any $r \in \mathbb{N}$) and $L^{t(1,1)}\delta_0(i(1,1)) \equiv {}_{t(1,1)}C_p c_{r_1}^{t(1,1)-1} c_{r_2} \pmod p$. So the assertion holds for $r = 1$ from Lemma 3(ii).

(ii) In case $r \ge 2$, we get the conclusion by the proposition above and the assumption of induction.

□

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