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LIOUVILLE TYPE THEOREM FOR SOLUTIONS OF INFRA-EXPONENTIAL GROWTH OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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Abstract

We discuss existence of global solutions of infra-exponential growth to a linear partial differential equation with constant coefficients whose total symbol $P(\xi)$ has the origin as its only real zero. We show that such a solution necessarily reduces to an entire infra-exponential function if and only if the complex zeros of $P(\zeta)$ is absent in a strip $|\mathrm{Im}\,\zeta| < \delta$, $|\mathrm{Re}\,\zeta| > 1/\delta$. This is a generalization of Schwartz's theorem, which in turn generalizes the classical Liouville theorem in the theory of functions.

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1. Introduction

In the theory of functions a well known theorem of Liouville asserts that a bounded entire function reduces to a constant. A more generalized form asserts that an entire function of tempered growth (i.e. of polynomial growth) is a true polynomial. Schwartz[S] explained this theorem employing the space $\mathcal{S}'$ of tempered distributions and suggested the following generalization:

Theorem 1.1. *Let $P(D)$ be a linear partial differential operator with constant coefficients. Assume that its total symbol $P(\xi)$ has the origin as its only real zero. Then every classical solution of $P(D)u = 0$, defined on the whole space $\boldsymbol{R}^n$ and has tempered growth, reduces to a polynomial. If the solution is bounded, then it reduces to a constant.*

To illustrate our fundamental argument, we sketch the proof: u can be considered as a global section of $\mathcal{S}'$. Hence we can apply the Fourier transform and we obtain

$$P(\xi)\widehat{u}(\xi) = 0.$$

Therefore the support of $\widehat{u}$ is contained in the real zeros of $P(\xi)$ which by assumption comprises the single point $\{0\}$. By the structure theorem of distributions supported by one point, we see that

$$\widehat{u}(\xi) = Q(D_\xi)\delta(\xi)$$

with a polynomial Q. After the inverse Fourier transform we conclude that $u(x) = Q(-x)$ is a polynomial.

The additional assertion follows from the fact that no polynomials other than constants are bounded on the whole real space.

We need not limit the solutions to classical ones and treat tempered distribution solutions directly. But usually we limit so in order to give a classical fragrance to our theorems of such type.

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Schwartz's argument can be generalized without modification to the spaces of non-quasianalytic type ultradistributions or the classical solutions of the corresponding growth. On the other hand, we encounter a more complicated situation when we consider the space of quasianalytic type ultradistributions.

In this article, as a prototype of such study we shall discuss the extreme case of quasianalytic ultradistributions, namely, hyperfunctions, or the infra-exponential growth condition corresponding to it. The case of solutions with growth corresponding to general ultradistributions will be treated in our forthcoming paper. Some of our final results might be already known since their appearance is very classical. But we believe our proof is new even in that case.

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2. Case of operators with no zeros in a strip

As is easily seen by means of the exponential solutions employed in Fundamental Principle, an operator $P(D)$ whose total symbol has real zeros other than the origin possesses bounded non-constant global solutions. Therefore we assume hereafter that the total symbol of our operator has only the origin as its real zero. In spite of this assumption, the matter is not so simple as in the case of $\mathcal{S}'$. This is because we now have to consider the support at infinity of $\widehat{u}(\xi)$. In this section we shall show the following

Theorem 2.1. *Let $P(D)$ be a linear partial differential operator with constant coefficients. If there exists $\delta > 0$ such that $P(\zeta)$ has no complex zeros in the strip*

$$|\mathrm{Im}\,\zeta| < \delta,$$

then every classical solution of $P(D)u = 0$ of infra-exponential growth is trivial. If $P(\zeta)$ has only real zero at the origin and no complex zeros on

$$|\mathrm{Im}\,\zeta| < \delta, \qquad |\mathrm{Re}\,\zeta| > \delta^{-1},$$

then every classical solution of $P(D)u = 0$ of infra-exponential growth is in fact an infra-exponential entire function.

Proof. Infra-exponential growth means the following: for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|u(x)| \le C_\varepsilon e^{\varepsilon |x|}.$$

Such u can naturally be considered as a Fourier hyperfunction (see [Kw], or [Kn2], Chapter 8). Then applying the Fourier transform we obtain $P(\xi)\widehat{u}(\xi) = 0$. At finite points where $P(\xi) \neq 0$, we can multiply the real analytic function $1/P(\xi)$ to both sides of the above equation. Thus we conclude that $\widehat{u}(\xi) = 0$ outside the origin. Also at a complex neighborhood of the sphere at infinity, in view of the Seidenberg-Tarski theorem (see e.g. [H], Appendix), we have in this region an estimate of the form

$$|P(\zeta)| \ge C|\zeta|^{-\mu}, \tag{2.1}$$

where $\mu > 0$ is a rational number. Therefore $1/P(\xi)$ serves as a multiplier to the Fourier hyperfunctions in this region, and we conclude that $\widehat{u}$ is also zero there. By now the support of $\widehat{u}$ reduces to the origin, and by the well-known structure theorem for such hyperfunctions, it is of the form $J(D_\xi)\delta(\xi)$ with an infra-exponential entire function $J(z)$. Thus we conclude that $u(x) = J(-x)$. If P has no real zero even at the origin, we conclude by the same discussion that $u \equiv 0$. □

We can sharpen the first assertion of the above theorem to solutions of exponential growth with exponential type restricted by the distance of the complex zeros of $P(\zeta)$ from the real axis:

Proposition 2.2. *Assume that $P(\zeta)$ has no complex zeros in the strip*

$$|\mathrm{Im}\,\zeta| \le \delta.$$

Then every classical solution of growth $O(e^{\delta|x|})$ is identically equal to 0.

Instead of Fourier hyperfunctions, we now employ the space of Fourier ultrahyperfunctions introduced by Park-Morimoto[PM] and Sargos-Morimoto[SM]: A solution u of indicated growth accepts the Fourier transform which is interpreted as an element of the space $\mathcal{Q}(\boldsymbol{D}^n+iB_\delta;\{0\})$ of Fourier ultrahyperfunctions. Recall first the general notation of Fourier ultrahyperfunctions (we change the notation of [SM] a little in order to fit better with Sato's original notation for Fourier hyperfunctions): For two convex sets $K, L\subset \boldsymbol{R}^n$, we define the space of test functions

$$\mathcal{P}_*(\boldsymbol{D}^n+iK;L)=\bigcup_{\varepsilon>0}\{\varphi(z)\in\mathcal{O}(\boldsymbol{R}^n+iK_\varepsilon);\ |\varphi(z)e^{\varepsilon|z|+H_L(\operatorname{Re}z)}|<\infty\}.$$

This is naturally endowed with the structure of DFS(dual Fréchet-Schwartz)-type topological linear space. Its dual, denoted by $\mathcal{Q}(\boldsymbol{D}^n+iK;L)$, is an FS(Fréchet-Schwartz)-type space and is called the space of Fourier ultrahyperfunctions of growth $e^{H_L(x)}$ and defined on the strip $\boldsymbol{R}^n+iK$. $H_L(x)$ denotes the supporting function of L:

$$H_L(x)=\sup_{\xi\in L}\langle x,\xi\rangle.$$

The Fourier transform brings the space $\mathcal{P}_*(\boldsymbol{D}^n+iK;L)$ isomorphically to $\mathcal{P}_*(\boldsymbol{D}^n+iL;-K)$, hence $\mathcal{Q}(\boldsymbol{D}^n+iL;K)$ to $\mathcal{Q}(\boldsymbol{D}^n-iK;L)$. What we cited in the above is the special case where $K=B_\delta$, $L=\{0\}$, since we have obviously $u(x)\in\mathcal{Q}(\boldsymbol{D}^n;B_\delta)$, where B_δ denotes the closed δ-ball centered at the origin.

These understood, from $P(D)u=0$ via the Fourier transform we have $P(\zeta)\widehat{u}(\zeta)=0$. Since by the assumption $P(\zeta)$ has no zeros on $\boldsymbol{R}^n+iB_\delta$ and bounded from below as (2.1), we conclude that $\widehat{u}(\zeta)=0$ as an element of $\mathcal{Q}(\boldsymbol{D}^n+iB_\delta;\{0\})$, hence $u=0$.

The assumption on the complex zeros at infinity is satisfied by any hypoelliptic operator. Hence our theorem applies e.g. to the heat equation. Thus its classical solutions of infra-exponential growth are necessarily entire infra-exponential. Notice that for the heat equation, any classical solution of growth $O(e^{\varepsilon|x|^2})$ for $\forall\varepsilon>0$ becomes entire holomorphic. In fact, by the Tihonov-Täcklind uniqueness theorem applied for any fixed $s\in\boldsymbol{R}$ such a solution agrees on $t>s$ with the one given by

$$\int E(t-s,x-y)u(s,y)dy,\qquad \text{where } E(t,x)=\frac{1}{(4\pi t)^{n/2}}e^{-x^2/4t}.j$$

(The analytic continuation with respect to the space variables can also be shown by the Cauchy-Kowalevsky-Zerner theorem. The analytic continuation with respect to the time variable is a property of analytic semigroup.) On the other hand, the heat equation obviously has entire solutions of order greater than 1. (Such can be given e.g. by the integral of the form

$$\int_{\boldsymbol{R}^n}e^{\xi\cdot x+|\xi|^2t-|\xi|^3}d\xi.)$$

Thus our theorem asserts that for entire solutions of the heat equation, the infra-exponential growth order posed on the real axis necessarily extends to the imaginary direction, too. This can be considered as a kind of Phragmén-Lindelöf principle.

3. Case of operators with zeros approaching the real axis

The argument of the preceding section leaves the possibility that if $P(\zeta)$ has zeros approaching faster the real points at infinity, there may be a solution of infra-exponential growth of which the Fourier transform has support not only at the origin but also at the corresponding real zeros at infinity. In this section we shall prove that this really takes place.

Theorem 3.1. *Let $P(D)$ be a linear partial differential operator with constant coefficients. Assume that for any $\delta>0$ the region*

$$|\operatorname{Im}\zeta|<\delta,\qquad |\operatorname{Re}\zeta|>\delta^{-1}$$

contains complex zeros of $P(\zeta)$. Then $P(D)u = 0$ always admits a C^∞ solution of which all derivatives are of infra-exponential growth, but which is not entire infra-exponential.

Proof. We shall first show that there exists a non-trivial Fourier hyperfunction $f(\xi)$ supported by the sphere at infinity S_∞^{n-1} and satisfying $P(\xi)f(\xi) = 0$. Unfortunately we cannot present concrete examples of such $f(\xi)$. But it can be shown in an abstract way by observing that the kernel of the following multiplication mapping is non-trivial:

$$P(\xi)\cdot \ : \ \mathcal{Q}[S_\infty^{n-1}] \to \mathcal{Q}[S_\infty^{n-1}]. \tag{3.1}$$

Here $\mathcal{Q}[S_\infty^{n-1}]$ denotes the space of Fourier hyperfunctions with support in S_∞^{n-1}. This is an FS-type space in good duality with the DFS-type space $\mathcal{P}_*(S_\infty^{n-1})$ of real analytic functions of exponential decay defined in a complex neighborhood of the real set S_∞^{n-1}. More concretely, in this case, an element $\varphi(\zeta)$ of $\mathcal{P}_*(S_\infty^{n-1})$ is defined in a complex domain of the form

$$U^\delta := \{\zeta = \xi + i\eta;\ |\xi| > \delta^{-1},\ |\eta| < \delta\}$$

and there satisfies the estimate $|\varphi(\zeta)| \leq Ce^{-\delta|\zeta|}$ for some $\delta > 0$. Let us denote by Φ^δ the corresponding Banach space defined with the natural supremum norm

$$\|\varphi\| := \sup_{\zeta \in U^\delta} |\varphi(\zeta)|e^{\delta|\zeta|}.$$

The DFS-structure of $\mathcal{P}_*(S_\infty^{n-1})$ is defined by the natural inductive limit of these Φ^δ with respect to $\delta > 0$. Hence by Grothendieck's theorem a bounded set of this space is contained in some of the Banach spaces Φ^δ. Moreover, a bounded closed set of $\mathcal{P}_*(S_\infty^{n-1})$ becomes compact and contained in some Φ^δ.

Now consider the mapping dual to (3.1)

$$P(\xi)\cdot \ : \ \mathcal{P}_*(S_\infty^{n-1}) \to \mathcal{P}_*(S_\infty^{n-1}). \tag{3.2}$$

The non-triviality of (3.1) is equivalent to the non-denseness of the image $\mathcal{I}$ of this mapping. Let $\overline{\mathcal{I}}$ denote the sequential closure of $\mathcal{I}$, that is, the space of elements $\varphi \in \mathcal{P}_*(S_\infty^{n-1})$ which can be obtained as the limit of a sequence from $\mathcal{I}$. A converging sequence is bounded, hence converges in some Φ^δ by its supremum norm. Thus any element of $\overline{\mathcal{I}}$ vanishes on the complex zero set of $P(\zeta)$ in the region U^δ for some $\delta > 0$, which is non-void by the assumption.

Choose any bounded closed, hence compact, subset K of $\mathcal{P}_*(S_\infty^{n-1})$. As remarked before, it is compact and contained in some Φ^{δ_0}, hence in Φ^δ for $\forall\delta < \delta_0$. For such δ, the topology of the Banach space Φ^δ and the original topology of $\mathcal{P}_*(S_\infty^{n-1})$ agree on K. Thus $\overline{\mathcal{I}} \cap K$ is closed. Therefore by Krein-Shmulyan's theorem $\overline{\mathcal{I}}$ is the true closure of $\mathcal{I}$. This implies that $\overline{\mathcal{I}}$ cannot be the total space.

Now we have found an abstract element $f(\xi) \in \mathcal{Q}[S_\infty^{n-1}]$ which satisfies $P(\xi)f(\xi) = 0$ as Fourier hyperfunctions. The inverse Fourier transform u of f gives a solution in Fourier hyperfunctions of the equation $P(D)u = 0$.

But we can say almost nothing about the nature of the inverse Fourier transform of f. We therefore replace this element by one with better property. Let

$$f(\xi) = \sum_{j=1}^{N} F_j(\xi + i\Gamma_j 0)$$

be the boundary value representation of the Fourier hyperfunction by infra-exponential defining functions $F_j(\zeta), j = 1, \ldots, N$ on the respective wedges. By the discussion of [Kn1], we can find a positive function $\chi(t)$ of $t \geq 0$ monotone increasing to ∞ and a sequence of constants C_k such that

$$|F_j(\xi + i\eta)| \leq C_k e^{|\xi|/\chi(|\xi|)} \qquad \text{on } 1/k \leq |\eta| \leq \delta, \qquad j = 1, \ldots, N.$$

Then we can find an infra-exponential entire function $J(\xi + i\eta)$ with no zeros in $|\eta| < \delta$ for some $\delta > 0$ and satisfying there

$$|J(\xi + i\eta)| \geq e^{|\xi|/\chi(|\xi|)+|\xi|^{1/2}} \tag{3.3}$$

Thus $G_j(\zeta) = F_j(\zeta)/J(\zeta)$, $j = 1, \dots, N$ are of $O(|\zeta|^{-m})$ for $m = 1, 2, \dots$ uniformly on $1/k \le |\mathrm{Im}\,\zeta| \le \delta$ for $k = 1, 2, \dots$. Now the new Fourier hyperfunction $g(\xi) = f(\xi)/J(\xi)$, or more precisely the one defined by $G_j(\zeta)$, $j = 1, \dots, N$, again has support in S^{n-1}_∞ and satisfies $P(\xi)g(\xi) = 0$.

The inverse Fourier transform v of g is a Fourier hyperfunction satisfying $P(D)v = 0$. Moreover, by the above estimate for the defining functions of g, v can be directly calculated by the integral

$$v(x) = \sum_{j=1}^{N} (2\pi)^{-n} \int_{\boldsymbol{R}^n} G_j(\xi + i\varepsilon\eta_j) e^{ix(\xi + i\varepsilon\eta_j)} d\xi, \tag{3.4}$$

where $\eta_j \in \Gamma_j$, $j = 1, \dots, N$ are unit vectors. This gives a continuous function of growth $e^{\varepsilon|x|}$. Since the integral is independent of the choice of $\varepsilon > 0$, $v(x)$ is actually of infra-exponential growth. The same holds for all finite order derivatives of v. This proves our assertion. □

Remark In the above proof we gave a solution of infra-exponential growth in C^∞ regularity. But we can always present such a solution in entire analytic functions. To show this, we replace the entire infra-exponential function $J(\zeta)$, employed to make g rapidly decreasing, by the entire function $e^{\varepsilon\zeta^2}$. It is obvious that $g(\xi) = f(\xi)e^{-\varepsilon\xi^2}$ has entire function v as its inverse Fourier image, as is seen from formula (3.4). It is also obvious that v satisfies $P(D)v = 0$. But we have to check that $v \not\equiv 0$, or equivalently $g \not\equiv 0$ as a Fourier hyperfunction. This can be proved by the Phragmén-Lindelöf principle as in Palamodov[P]. Here for our purpose it suffices to see that this holds at least for some $\varepsilon > 0$, which can be shown more easily: Set $g_\varepsilon(\xi) = f(\xi)e^{-\varepsilon\xi^2}$ and assume that these are equal to zero as Fourier hyperfunctions for all $\varepsilon > 0$. But when $\varepsilon \to 0$ we have $g_\varepsilon \to f$ as Fourier hyperfunctions. In fact, take a test function $\varphi \in \mathcal{P}_*$. Then we have

$$\langle g_\varepsilon, \varphi \rangle = \langle f, \varphi e^{-\varepsilon\xi^2} \rangle. \tag{3.5}$$

Here if φ belongs to the Banach space Φ^δ by the notation used in the proof of Theorem 3.1, then as $\varepsilon \to 0$ we have

$$\sup_{\zeta \in U^{\delta/2}} |\varphi(\zeta)(1 - e^{-\varepsilon\zeta^2})| e^{(\delta/2)|\zeta|} \to 0,$$

hence $\varphi(\xi)e^{-\varepsilon\xi^2} \to \varphi$ in $\Phi^{\delta/2}$, hence in $\mathcal{P}_*$. Thus (3.5) tends to $\langle f, \varphi \rangle$. Therefore $g_\varepsilon \to f$ (weakly, hence strongly, because $\mathcal{Q}(\boldsymbol{D}^n)$ is an FS-space). Thus, given that $f \ne 0$, some of g_ε should be non-trivial.

Notice, however, that the growth order of such entire solutions u to the imaginary direction is very high in general.

One might think that the discussion of Theorem 3.1 together with use of modified Fourier hyperfunctions would give similar counter-examples for general non-elliptic operators including hypoelliptic ones, with respect to the zeros at infinity of their symbols. But the inverse Fourier image of a modified Fourier hyperfunction calculated by the modified version of formula (3.4) need not be of infra-exponential growth on the real axis.

References

[H] Hörmander L., *Linear Partial Differential Operators*, Springer, 1963.

[Kn1] Kaneko A., *On the structure of Fourier hyperfunctions*, Proc. Japan Acad. **48** (1972), 651-653.

[Kn2] ———, *Introduction to Hyperfunctions* (in Japanese), Univ. of Tokyo Press, 1980-1982 (English translation from Kluwer, 1988).

[Kw] Kawai T., *On the theory of Fourier hyperfunctions and its applications to partial differential equations with constant coefficients*, J. Fac. Sci. Univ. Tokyo Sec.1A **17** (1970), 467-517.

[P] Palamodov V. P., *From hyperfunctions to analytic functionals*, Soviet Math. Dokl. **18** (1977), 975-979.

[PM] Park Y.S. and Morimoto M., *Fourier ultra-hyperfunctions in the Euclidean n-space*, J. Fac. Sci. Univ. Tokyo Sect.1A **20** (1973), 121-127.

[SM] Sargos P. and Morimoto M., *Transformation des fonctionnelles analytiques à porteurs non compacts*, Tokyo J. Math. **4** (1981), 457-492.

[S] Schwartz L., *Théorie des Distributions* (New Edition), Hermann, Paris, 1966.