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Julia sets of $z^2 + c$ and laminations

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Abstract

We introduce an α - invariant equivalence relation on $\{0,1\}^\infty$ with $\alpha \in \{0,1\}^\infty$ and construct a lamination S_s^α using this relation ($s \in \{0,1\}^\infty$). We shall give a condition for α and s that S_s^α corresponds to a Julia set.

1. Introduction

Julia sets play an important role on a complex dynamical system. Concerning these sets there are some researches of representation to study their locally connected property or to analyze the structure of self similarity. For instance, W.P.Thurston introduced "invariant lamination" on a circle. A.Bandt and K.Keller showed the relationship between Thurston's invariant lamination and the symbolic dynamics represented by "itineraries" (infinite sequences of $\{0,1,*\}$), and they got an interesting result involving the correspondence of the dynamics of Julia sets to double-angle motions on a circle [1,2]. Here an invariant lamination is mainly determined by an invariant equivalence relation. So if we define an invariant equivalence relation, we can get the lamination by binding some points on a circle with some chords, where the points belong to the same equivalence class. (Fig.2)

In this paper, we introduce the concept of α-invariance and construct an α-invariant lamination corresponding to a locally connected Julia set without using itineraries. Moreover we give a necessary condition for the lamination to correspond to a Julia set.

In the section 2, we show the existence of a function which makes a correspondece of the binary sequences obtained by the Jordan curve to the point on a Julia set (Theorem 1), and we prove it by using Caratheodory's theorem. Since the correspondence of the binary sequences to the point on a Julia set is not one-to-one, we define an equivalence relation on $\{0,1\}^\infty$ such that if two different binary sequences correspond to the same point on a Julia set, the two sequences belong to the same equivalence class. Theorem 2 shows the property of the equivalence relation.

In the section 3, we shall show a construction of α-invariant lamination S_s^α. The first step is to define an α-invariance without using itineraries such that it satisfies (i) $\sim$ (iii) in Theorem 2. Next we define an α-invariant lamination. The construction of laminations using itineraries in references [1,2] is easier in a non-periodic case than in a periodic case. So in this paper we shall treat the case of $\underline{s}$ being periodic. The examples of a Julia set and of a corresponding α-invariant lamination are shown in Fig.1 and Fig.2 respectively. Moreover we introduce an equivalence relation $\sim_s^\alpha$ such that all points on a lamination connecting by chords belong to the same equivalence class. Theorem 3 shows a one-to-one correspondence of the quotient space $T/_{\sim_s^\alpha}$ ($T = S^1$) to the α-invariant lamination S_s^α. Theorem 4 describes a neccessary condition for $\underline{\alpha}$ and $\underline{s}$ when a lamination S_s^α corresponds to a Julia set.

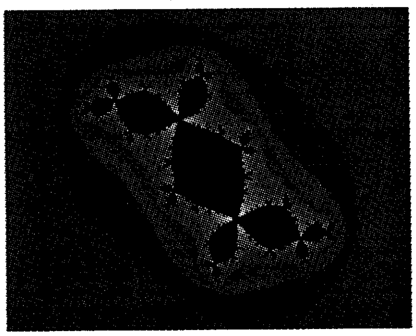


Fig.1 **An example of Julia set**

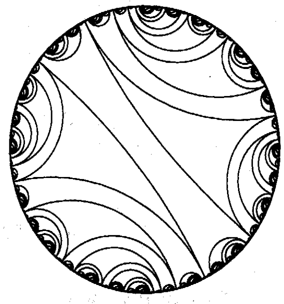


Fig.2 **An example of Lamination**

2. locally connected Julia sets and binary sequences

We recall the definition of Julia sets of $g_c(z) = z^2 + c$. Let $O_c(z) = \{z, g_c(z), g_c^2(z), \cdots\}$ denote the forward orbit of z for $c, z \in C$, and call $K_c = \{z \in C \mid O_c(z) \text{ is bounded}\}$ the filled-in Julia set. The boundery J_c of K_c is said to be the Julia set of g_c. $K_0 = D$ is the unit disk. The set $M = \{c \in C \mid J_c \text{ is connected}\}$ is said to be the Mandelbrot set, and it is known that it is equal to the set $\{c \in C \mid O_c(0) \text{ is bounded}\}$ [3]. If J_c is locally connected, then $c \in M$ and therefore $c \in K_c$ by the definitions of M and K_c. Hearafter we shall consider the case of locally connected Julia sets J_c. Let I denote the closed set $[0, 1]$.

Definition 1.

- $$E_c \stackrel{\text{def}}{=} \left\{ h \in C_c^1[0,1] \;\middle|\; \begin{array}{l} h(0) = h(1) \\ h(I) \text{ is a differentiable Jordan closed curve surrounding } J_c \\ h(t) = -h(t+\frac{1}{2}) \quad (0 \le t \le \frac{1}{2}) \\ g_c(h(I)) \text{ encloses } h(I) \end{array} \right\}.$$

- For $h \in E_c$, define $\theta, r \in C[0,1]$ satisfying $h(t) - c = r(t)e^{i\theta(t)}$ and $-\pi \le \theta(0) < \pi$.

  and define $f_0, f_1 : E_c \to C_c^1[0,1]$ by using θ, r as follows:

  $$\begin{cases} f_0 \cdot h(t) = \quad r(t)^{\frac{1}{2}} \cdot e^{i\frac{\theta(t)}{2}} \\ f_1 \cdot h(t) = -r(t)^{\frac{1}{2}} \cdot e^{i\frac{\theta(t)}{2}} \end{cases}.$$

  Define $S : E_c \to E_c$ by using f_0, f_1 defined above as follows.

  $$Sh(t) = \begin{cases} f_0 \cdot h(2t) & (0 \le t \le \frac{1}{2}) \\ f_1 \cdot h(2t-1) & (\frac{1}{2} \le t \le 1) \end{cases}$$

Remark. (i) For $0 \le t \le \frac{1}{2}$, $(Sh(t))^2 = h(2t) - c$ holds and for $\frac{1}{2} \le t \le 1$, $(Sh(t))^2 = h(2t-1) - c$ holds. (ii) r and θ defined above satisfy $r(0) = r(1)$ and $|\theta(0) - \theta(1)| = 2\pi$ since $c \in K_c$ and $h(I)$ surrounds J_c.

Lemma 1. *$h \in E_c$ implies $Sh \in E_c$.*

Proof. For $h \in E_c$ and $r, \theta \in C[0,1]$ we shall define functions $\hat{r}, \hat{\theta}, \hat{h} \in C(R)$ and an operator $\hat{f} : C(R) \to C(R)$ as follows. For $t \in R$, put $\hat{r}(t) = r(t-[t])$, $\hat{\theta}(t) = \theta(t-[t]) + [t](\theta(1) - \theta(0))$, $\hat{h}(t) - c = \hat{r}(t) \cdot e^{i\hat{\theta}(t)}$, and

$\hat{f}\cdot\hat{h}(t)=\hat{r}(t)^{\frac{1}{2}}\cdot e^{i\frac{\hat{\theta}(t)}{2}}$. Then $Sh(t)=\hat{f}\cdot\hat{h}(2t)$ holds. Since $\hat{f}\cdot\hat{h}(0)=\hat{f}\cdot\hat{h}(2)$ and $\hat{f}\cdot\hat{h}(2t)=-\hat{f}\cdot\hat{h}(2t-1)$ (for $0\le t\le\frac{1}{2}$) hold, it follows that $Sh(0)=Sh(1)$, $Sh(t)=-Sh(t+\frac{1}{2})$ (for $0\le t\le\frac{1}{2}$) and $Sh(I)$ is a differentiable Jordan closed curve.
By simple calculations, we obtain $h(I)=g_c(Sh(I))$ and $g_c(Sh(I))$ surrounds J_c. So $Sh(I)$ surrounds J_c, since J_c and K_c are g_c-invariant.

By the above considerations and $h(I)=g_c(Sh(I))$, g_c maps the interior and exterior of the closed curve $Sh(I)$ into the interior and exterior of the closed curve $h(I)$ respectively. Since $g_c(h(I))$ is the exterior of $h(I)$, $h(I)$ is also the exterior of $Sh(I)$, that is, $g_c(Sh(I))$ is the exterior of $Sh(I)$, which means $g_c(Sh(I))$ surrounds $Sh(I)$. So $Sh\in E_c$ holds. □

The next Lemma 2 will be used in proving Theorem 1.

Lemma 2. *For $0\le t\le\frac{1}{2}\in I$ and $h\in E_c$, $Sg_cSh(t)=g_cSSh(t)=Sh(2t)$ holds. For $\frac{1}{2}\le t\le 1\in I$ and $h\in E_c$, $Sg_cSh(t)=g_cSSh(t)=Sh(2t-1)$ holds.*

Proof. By simple calculations, we obtain the following:

$$g_cSh(t) = \begin{cases} (f_0\cdot h(2t))^2+c & (0\le t\le\frac{1}{2}) \\ (f_1\cdot h(2t-1)^2+c & (\frac{1}{2}\le t\le 1)\end{cases}$$
$$= \begin{cases} r(2t)\cdot e^{i\theta(2t)}+c & (0\le t\le\frac{1}{2}) \\ r(2t-1)\cdot e^{i\theta(2t-1)}+c & (\frac{1}{2}\le t\le 1)\end{cases}$$
$$= \begin{cases} h(2t) & (0\le t\le\frac{1}{2}) \\ h(2t-1) & (\frac{1}{2}\le t\le 1)\end{cases}.$$

So the next follows.

$$Sg_cSh(t) = \begin{cases} Sh(2t) & (0\le t\le\frac{1}{2}) \\ Sh(2t-1) & (\frac{1}{2}\le t\le 1)\end{cases} \qquad \text{(L1)}$$

Since the definition of $Sh(t)$, it follows:

$$S^2h(t) = \begin{cases} f_0\cdot Sh(2t) & (0\le t\le\frac{1}{2}) \\ f_1\cdot Sh(2t-1) & (\frac{1}{2}\le t\le 1)\end{cases}$$
$$= \begin{cases} (r_1(2t)\cdot e^{i\theta_1(2t)})^{\frac{1}{2}} & (0\le t\le\frac{1}{2}) \\ -(r_1(2t-1)\cdot e^{i\theta_1(2t-1)})^{\frac{1}{2}} & (\frac{1}{2}\le t\le 1)\end{cases}$$

where $\theta_1, r_1\in C[0,1]$ satisfy $Sh(t)-c = r_1(t)e^{i\theta_1(t)}$. So the following holds.

$$g_cS^2h(t) = \begin{cases} r_1(2t)\cdot e^{i\theta_1(2t)}+c & (0\le t\le\frac{1}{2}) \\ r_1(2t-1)\cdot e^{i\theta_1(2t-1)}+c & (\frac{1}{2}\le t\le 1)\end{cases}$$
$$= \begin{cases} Sh(2t) & (0\le t\le\frac{1}{2}) \\ Sh(2t-1) & (\frac{1}{2}\le t\le 1)\end{cases} \qquad \text{(L2)}$$

By (L1) and (L2), we obtain the conclusion. □

We recall the next well-known fact. If J_c is locally connected, there is a unique conformal isomorphism $\Phi_c : C \setminus K_c \to C \setminus D$ with $\lim_{z\to\infty} \Phi_c(z)/z = 1$ satisfying $\Phi_c g_c \Phi_c^{-1} = g_0$. Let define field lines $\beta_c = \{z \in C \setminus K_c \mid \arg(\Phi_c(z)) = 2\pi\beta\}$. The next theorems are important in proving the following theorem 1.

Caratheodory's theorem [1] *Each field line β_c has a continuous extension to a unique point z_β of J_c, and each point of J_c is obtained if and only if J_c is locally connected.*

Based on the theorem, we consider the correspondence of locally connected Julia sets to binary sequences. The next theorem is proved by using the above theorem, and we use them to prove Theorem 1.

Theorem A (Keller, Bandt [1]) *J_c $(c \in C)$ is locally connected if and only if the functional equations*

$$\varphi(2\beta) = \varphi(\beta)^2 + c \quad \textit{and} \quad -\varphi(\beta) = \varphi(\beta + \tfrac{1}{2}), \quad \beta \in R \tag{1}$$

have a continuous periodic solution. In this case, $J_c = \varphi(R)$. Moreover, every continuous solution of (1) *with minimal period 1 coincides with either φ_c^+ or φ_c^- where $\varphi_c^+(\beta) = z_{\beta mod 1}$ and $\varphi_c^-(\beta) = \varphi_c^+(-\beta)$ for $\beta \in R$.*

Theorem 1. *Concerning a locally connected Julia set J_c and the operator S^n defined at Definition 1, the following holds.*
(i) *For $t = \Sigma_{i=1}^{\infty} \frac{t_i}{2^i} \in I$ and $h \in E_c$, $S^n h(t) = f_{t_1} \cdot f_{t_2} \cdots f_{t_n} \cdot h(\Sigma_{i=1}^{\infty} \frac{t_{i+n}}{2^i})$ holds.*
(ii) *If J_c is locally connected, there exists $\phi \in \bar{E}_c$ depended on c ($\bar{E}_c$ is the closure of E_c with sup norm) such that $\lim_{n\to\infty} S^n h(t) = \phi(t)$ $(t \in I)$ for any $h \in E_c$ and $\phi(t) \in J_c$.*
Moreover if $c \notin J_c$, there exists r_ϕ, $\theta_\phi \in C[0,1]$ such that $\phi(t) - c = r_\phi(t) e^{i\theta_\phi(t)}$ with $-\pi \leq \theta_\phi(0) \leq \pi$.

Proof. (i) The equation will be shown by induction.
(ii) Since the functions f_0, f_1 make the radius and the argument in half, there exists the limit of $S^n h(t) = f_{t_1} \cdot f_{t_2} \cdots f_{t_n} \cdot h(\Sigma_{i=1}^{\infty} \frac{t_{i+n}}{2^i})$. So put $\phi(t) = \lim_{n\to\infty} S^n h(t)$. Then by induction it holds that $S^n h(t) = -S^n h(t + \frac{1}{2})$ for all n, which implies

$$\phi(t) = -\phi(t + \tfrac{1}{2}). \tag{T1}$$

By Lemma 2, for $0 \leq t \leq \frac{1}{2}$ we have

$$\begin{aligned}\phi(2t) &= \lim_{n\to\infty} S^n h(2t) \\ &= \lim_{n\to\infty} S^n(gSh(t)) \\ &= g \lim_{n\to\infty} S^{n+1} h(t) \\ &= g\phi(t) \\ &= \phi(t)^2 + c.\end{aligned} \tag{T2}$$

By Theorem A, ϕ coincides with φ^+ or φ^-, and by Caratheodory's theorem, $\phi(t)$ is in J_c and ϕ is uniquely determined.

We define $r_{S^n h}(t)$, $\theta_{S^n h}(t)$ satisfying $S^n h(t) - c = r_{S^n h}(t) e^{i\theta_{S^n h}(t)}$. Then there exists $r_\phi(t) = \lim_{n\to\infty} r_{S^n h}(t)$ and $\theta_\phi(t) = \lim_{n\to\infty} \theta_{S^n h}(t)$. Hence $\phi(t) - c = r_\phi(t) e^{i\theta_\phi(t)}$ holds. Since $S^n h(t)$ converges to a point of J_c, $r_\phi(t)$ belongs $C[0,1]$. If $c \notin J_c$ then c is inside of J_c, so $\phi - c \neq 0$ for any $t \in I$, that is $r_\phi(t) \neq 0$ and $\theta_\phi(t) \in C[0,1]$. □

By the equation $\phi(t) = \lim_{n\to\infty} f_{t_1} f_{t_2} \cdots f_{t_n} \cdot h(\Sigma_{i=1}^{\infty} \frac{t_{i+n}}{2^i})$, we can consider a correspondence of points of J_c to binary sequences $t_1 t_2 \cdots t_n$. But the correspondence is not one to one, so we introduce an equivalence relation such that the same points in the Julia set are in the same equivalence class.

Definition 2. Let $c \in C$ be fixed. Using ϕ obtained in Theorem 1, we shall consider the map $\psi : \{0,1\}^\infty \to J_c$ by $\psi(\underline{x}) = \phi(\Sigma_{n=1}^{\infty} \frac{x_n}{2^n})$ with $\underline{x} = x_1x_2\cdots$, and define an equivalence relation $\approx$ on $\{0,1\}^\infty$ as follows.
For $\underline{x} = x_1x_2\cdots$, $\underline{y} = y_1y_2\cdots \in \{0,1\}^\infty$, let

$$\underline{x} \approx \underline{y} \overset{\text{def}}{\Leftrightarrow} \psi(\underline{x}) = \psi(\underline{y}).$$

For $\underline{\alpha} \in \{0,1\}^\infty$, we define the function τ_α as follows:
$\tau_\alpha : \{0,1\}^\infty \to \{0,1\}^\infty$

$$\tau_\alpha(\underline{s}) \overset{\text{def}}{=} \begin{cases} 0\underline{s} & k(\underline{s}) \le k(\underline{\alpha}) \\ 1\underline{s} & k(\underline{s}) > k(\underline{\alpha}) \end{cases} \qquad \text{for} \quad \underline{s} \in \{0,1\}^\infty$$

where $k(\underline{s}) = \Sigma_{n=1}^{\infty} \frac{s_n}{2^n}$ with $\underline{s} = s_1s_2\cdots$.

The next Theorem 2 shows the property of the equivalence relation $\approx$.

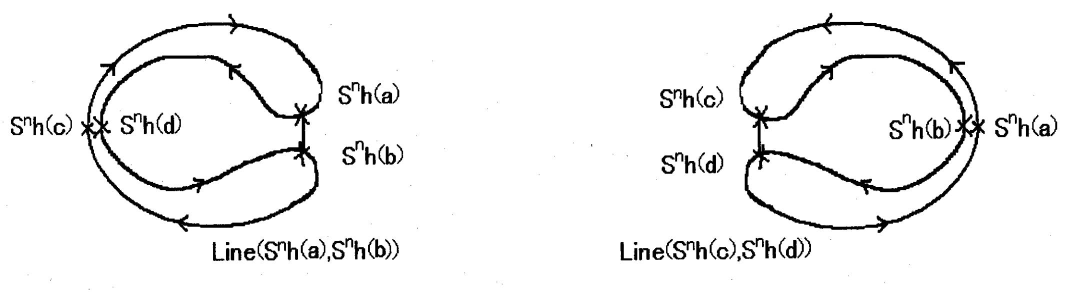
Theorem 2. *The equivalence relation $\approx$, due to a locally connected Julia set J_c with $c \notin J_c$, defined at Definition 2 satisfies the following.*
(i) $\underline{x} \approx \underline{y}$ *implies* $\sigma\underline{x} \approx \sigma\underline{y}$ $(\sigma(x_1x_2\cdots) = x_2x_3\cdots)$.
(ii) $\underline{x} \approx \underline{y}$ *implies* $x_1'\sigma\underline{x} \approx y_1'\sigma\underline{y}$ $(x_1' = 1 - x_1, y_1' = 1 - y_1)$.
(iii) *There exists* $\alpha \in \{0,1\}^\infty$ *such that* $\underline{x} \approx \underline{y}$ *implies* $\tau_\alpha(\underline{x}) \approx \tau_\alpha(\underline{y})$.
(iv) $\underline{x} \approx \underline{u}, \underline{y} \approx \underline{v}, \underline{x} \not\approx \underline{y}$ *implies* $(k(\underline{x}), k(\underline{u})) \cap (k(\underline{y}), k(\underline{v})) = \phi$ *(empty set) or* $(k(\underline{x}), k(\underline{u})) \supset (k(\underline{y}), k(\underline{v}))$ *or* $(k(\underline{x}), k(\underline{u})) \subset (k(\underline{y}), k(\underline{v}))$*, while* $(k(*), k(*))$ *is an interval set.*

Before proving Theorem 2 we show the following lemma.

Lemma 3. *For $a, b, c, d \in I$, suppose that it holds the relations $a < b$, $c < d$, $a < c$, $\phi(a) = \phi(b)$, $\phi(c) = \phi(d)$ and $\phi(a) \neq \phi(c)$. Then the case $a < c < b < d$ doesn't occur.*

Proof. Suppose $a < c < b < d$. By $\phi(a) \neq \phi(c)$, we put $d = |\phi(a) - \phi(c)| > 0$. Then there exists n_0 such that $|S^n h(w) - \phi(w)| < \frac{d}{4}$ $(w = a, b, c, d)$ for $n \ge n_0$. Let D_n be the interior of $S^n h(I)$ and let $Line(S^n h(a), S^n h(b))$ be the line connecting $S^n h(a)$ and $S^n h(b)$. Since for any n, $S^n h(I)$ is a Jordan closed curve surrounding locally connected set J_c, either of the following cases occurs.(See Fig.3)
(the case 1) D_n^c (the complement of D_n) $\cap\ Line(S^n h(a), S^n h(b)) \neq \phi$ for $n \ge n_0$ or
(the case 2) $D_n^c \cap Line(S^n h(c), S^n h(d)) \neq \phi$ for $n \ge n_0$.

In the case 1, if $Line(S^n h(a), \phi(a)) \cap D_n^c \neq \phi$, then there exists $\epsilon > 0$ such that $|S^n h(a) - \phi(a)| > \epsilon$ for any $n \ge n_0$, which contradicts $\lim_{n\to\infty} S^n h(a) = \phi(a)$. In other cases, it will be shown in a similar way. □.



(In the case 1) (In the case 2)

Fig.3 **Some examples of $S^n h(I)$**

Proof of Theorem 2. (i) $\underline{x} \approx \underline{y}$ implies $\phi(\Sigma_{n=1}^{\infty}\frac{x_n}{2^n}) = \phi(\Sigma_{n=1}^{\infty}\frac{y_n}{2^n})$. So $\phi(\Sigma_{n=1}^{\infty}\frac{x_n}{2^n})^2 + c = \phi(\Sigma_{n=1}^{\infty}\frac{y_n}{2^n})^2 + c$ holds. By using (T2), $\phi(2\Sigma_{n=1}^{\infty}\frac{x_n}{2^n}) = \phi(2\Sigma_{n=1}^{\infty}\frac{y_n}{2^n})$ holds, which implies $\sigma\underline{x} \approx \sigma\underline{y}$.
(ii) $\underline{x} \approx \underline{y}$ implies $\phi(\Sigma_{n=1}^{\infty}\frac{x_n}{2^n}) = \phi(\Sigma_{n=1}^{\infty}\frac{y_n}{2^n})$. By using (T1), $-\phi(\Sigma_{n=1}^{\infty}\frac{x_n}{2^n} + \frac{1}{2}) = -\phi(\Sigma_{n=1}^{\infty}\frac{y_n}{2^n} + \frac{1}{2})$ holds, which implies $x_1'\sigma\underline{x} \approx y_1'\sigma\underline{y}$.
(iii) Put $k(\underline{\alpha}) = \sup\{k(\underline{x}) \mid \exists k(\underline{y}) > k(\underline{x})\ ;\ |\theta_\phi(k(\underline{y})) - \theta_\phi(k(\underline{x}))| = 2\pi, \underline{x} \approx \underline{y}\}$ where θ_ϕ is determined in Theorem 1. $\underline{x} \approx \underline{y}$ and $|\theta(k(\underline{y})) - \theta(k(\underline{x}))| = 2\pi$ means that the sets $\{\psi(\underline{r}) \mid k(\underline{x}) \le k(\underline{r}) \le k(\underline{y})\}$ is a closed curve surrounding c. Since ψ is the limit of a simple Jordan curve, the relations $k(\underline{x}) < k(\underline{y})$, $k(\underline{a}) < k(\underline{b})$, $\underline{x} \approx \underline{y}$, $\underline{a} \approx \underline{b}$, and $|\theta(k(\underline{y}))-\theta(k(\underline{x}))| = |\theta(k(\underline{b}))-\theta(k(\underline{a}))| = 2\pi$ imply either $k(\underline{x}) < k(\underline{a}) < k(\underline{b}) < k(\underline{y})$ or $k(\underline{a}) < k(\underline{x}) < k(\underline{y}) < k(\underline{b})$. So if $\underline{x} \approx \underline{y}$ with $k(\underline{x}) \le k(\underline{y}) < k(\underline{\alpha})$ [resp. $k(\underline{\alpha}) < k(\underline{x}) \le k(\underline{y})$] then $\theta(k(\underline{x})) = \theta(k(\underline{y}))$, which implies $\psi(0\underline{x}) = \psi(0\underline{y})$ [resp. $\psi(1\underline{x}) = \psi(1\underline{y})$], that is $\tau_\alpha(\underline{x}) \approx \tau_\alpha(\underline{y})$. If $\underline{x} \approx \underline{y}$ with $k(\underline{x}) \le k(\underline{\alpha}) < k(\underline{y})$, then $\theta(k(\underline{y})) - \theta(k(\underline{x})) = 2\pi$, which implies $\psi(0\underline{x}) = \psi(1\underline{y})$, that is $\tau_\alpha(\underline{x}) \approx \tau_\alpha(\underline{y})$.
(iv) follows from Lemma 3. □

3. The construction of α - invariant lamination

In this section we shall define an α - invariant equivalence relation satisfying (i) $\sim$ (iii) in Theorem 2 and construct laminations by using this relation. We also give the conditions for $\underline{\alpha}$ and $\underline{s}$ that a lamination S_s^α corresponds to a Julia set. If $\underline{s} = \overline{w}$ with $w \in \{0,1\}^n$, we call the sequence $\underline{s}$ to be n-periodic.

Definition 3. Let $\underline{\alpha} = \alpha_1\alpha_2\cdots$ be an element of $\{0,1\}^\infty$.
An equivalence relation $\sim$ on $\{0,1\}^\infty$ is called to be *α-invariant* if it satisfies the following (i) and (ii).
(i) For $\underline{s}$, $\underline{t} \in \{0,1\}^\infty$, $\underline{s} \sim \underline{t}$ implies $\sigma(\underline{s}) \sim \sigma(\underline{t})$ where $\sigma(s_1s_2\cdots) = s_2s_3\cdots$.
(ii) For $\underline{s}$, $\underline{t} \in \{0,1\}^\infty$, $\underline{s} \sim \underline{t}$ implies $\tau_\alpha(\underline{s}) \sim \tau_\alpha(\underline{t})$ and $\tau_\alpha'(\underline{s}) \sim \tau_\alpha'(\underline{t})$,
where $\tau_\alpha(\underline{s}) \stackrel{\text{def}}{=} \begin{cases} 0\underline{s} & \text{if } k(\underline{s}) \le k(\underline{\alpha}) \\ 1\underline{s} & \text{if } k(\underline{s}) > k(\underline{\alpha}) \end{cases}$ and $\tau_\alpha'(\underline{s}) \stackrel{\text{def}}{=} \begin{cases} 1\underline{s} & \text{if } k(\underline{s}) \le k(\underline{\alpha}) \\ 0\underline{s} & \text{if } k(\underline{s}) > k(\underline{\alpha}) \end{cases}$.

Let $\sim_{\bar{0}}$ be the smallest $\bar{0}$ - invariant equivalence relation satisfying $\bar{0} \sim \bar{1}$. Let $T = R/Z$. It is easy to show $\{0,1\}^\infty/\sim_{\bar{0}}\ \simeq\ T$. So let δ be the isomorphism from $\{0,1\}^\infty/\sim_{\bar{0}}$ onto T. For $\underline{a}, \underline{b} \in \{0,1\}^\infty$, let $C_{\underline{a},\underline{b}}$ be a chord connecting $\delta(\underline{a})$ and $\delta(\underline{b})$ on T such that its Poincare metric is minimum.

Definition 4. For $\underline{\alpha} \in \{0,1\}^\infty$, and periodic $\underline{s} \in \{0,1\}^\infty$, the equivalence relation $\sim_s^\alpha$ is defined as the smallest closed α-invariant equivalence relation on $\{0,1\}^\infty/\sim_{\bar{0}}$, satisfying $\underline{s} \sim \sigma(\underline{s})$. Let S_s^α be the closure of the collection of the chords

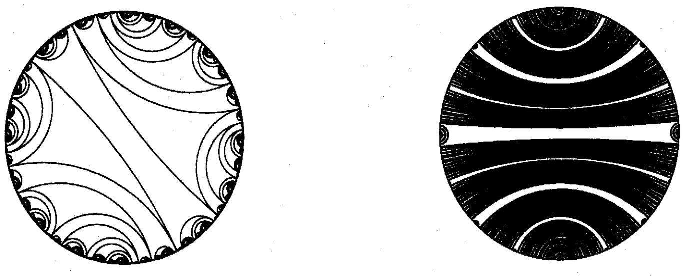
$$\{C_{\lambda_1\lambda_2\cdots\lambda_n(\underline{s}),\lambda_1\lambda_2\cdots\lambda_n(\sigma(\underline{s}))} \mid \lambda_j \in \{\sigma, \tau_\alpha, \tau_\alpha'\}, n \in N \cup \{0\}\},$$

which we call *α-invariant lamination.*

Then the α - invariant lamination S_s^α (some examples are showm in Fig.4) is considered as the quotient space of T where points connected by a chord belong to the same class and we have the following theorem.

Theorem 3. *There is a one-to-one correspondence of elements of the quotient space $T/\sim_s^\alpha$ to those of the α - invariant lamination S_s^α.*

Proof. It is obvious that the element of the quotient space corresponds to that of S_s^α by the construction of S_s^α. □



$\underline{\alpha} = \underline{s} = \overline{001}$ $\underline{\alpha} = \overline{01111000}$, $\underline{s} = \overline{00001111}$

Fig.4 **Some examples of S^{α}_{s}**

If $\underline{s}$ is p-periodic, $k(\underline{s}) = \frac{q}{2^p-1}$ $(q \in N)$ holds. Though the equivalence relation $\approx$ induced from Julia sets satisfies (i) $\sim$ (iv) in Theorem 2, the α - invariant equivalence relation $\sim^{\alpha}_{s}$ satisfies (i) $\sim$ (iii) in Theorem 2 but not necessarily (iv). So for an arbitrary q, there doesn't necessarily exist $\underline{\alpha} \in \{0,1\}^{\infty}$ such that S^{α}_{s} corresponds to a Julia set. Hence we examine the condition for $\underline{\alpha}$ and $\underline{s}$ that S^{α}_{s} corresponds to a Julia set and get the following Theorem 4. First we define the equivalent class of $\underline{s}$ and the minimum of the equivalent class.

Definition 5. *For $\underline{s} \in \{0,1\}^{\infty}$, let $Q\underline{s}$ be the equivalence class of $\underline{s}$ with the equivalence relation $\sim^{\alpha}_{s}$.*

Theorem 4. *Let $\underline{s} \in \{0,1\}^{\infty}$ be p-periodic such that $k(\underline{s}) \leq k(\sigma^j(\underline{s}))$ for any $j \geq 0$, and suppose $Q\underline{s} = \{\underline{s}, \sigma(\underline{s}), \cdots, \sigma^{p-1}(\underline{s})\}$.*
Then $k(\underline{s}) = \frac{q}{2^p-1}$ $(q \in N)$ holds. As for the correspondence of a Julia set to a lamination, we have the following, according to the value of q.
(i) *Suppose there exists $j \geq 1$ and $k_1 \geq 1$ such that $\Sigma_{n=0}^{k_1-1} 2^{nj} = q$ and $p = jk_1 + 1$.*
If the lamination S^{α}_{s} correspondes to a Julia set, then $\underline{\alpha}$ satisfies the following

$$k(\sigma^{j+1}(\underline{s})) \leq k(\underline{\alpha}) < k(\sigma(\underline{s})).$$

(ii) *Suppose there exists $j \geq 1$ and $k_2 \geq 1$ such that $1 + \Sigma_{n=1}^{k_2} 2^{nj+(n-1)} = q$ and $p = (j+1)k_2 + j$.*
If the lamination S^{α}_{s} corresponds to a Julia set, then $\underline{\alpha}$ satisfies the following

$$k(\sigma^{(j+1)k_2}(\underline{s})) \leq k(\underline{\alpha}) < k(\sigma(\underline{s})).$$

Proof. (i) Let $u = \underbrace{0\cdots0}_{j-1}1 \in \{0,1\}^j$ and $w = 0\underbrace{u\cdots u}_{k_1} \in \{0,1\}^{k_1 j+1}$. Then $\underline{s} = \overline{w}$.
The next inequality

$$\begin{aligned} k(\underline{s}) &< k(\sigma^{(k_1-1)j+1}(\underline{s})) < k(\sigma^{(k_1-2)j+1}(\underline{s})) < \cdots < k(\sigma^{j+1}(\underline{s})) < k(\sigma^1(\underline{s})) \\ &< k(\sigma^{(k_1-1)j+2}(\underline{s})) < k(\sigma^{(k_1-2)j+2}(\underline{s})) < \cdots < k(\sigma^{j+2}(\underline{s})) < k(\sigma^2(\underline{s})) \\ &\cdots\cdots \\ &< k(\sigma^{k_1 j}(\underline{s})) < k(\sigma^{(k_1-1)j}(\underline{s})) < \cdots < k(\sigma^{2j}(\underline{s})) < k(\sigma^j(\underline{s})) \end{aligned}$$

holds.

Suppose $k(\underline{\alpha}) < k(\sigma^{j+1}(\underline{s}))$. If $k(\underline{s}) \leq k(\underline{\alpha}) < k(\sigma^{j+1}(\underline{s}))$ holds, $\underline{s} \sim \sigma^{j+1}(\underline{s})$ implies $0\underline{s} \sim 1\sigma^{j+1}(s)$ and $1\underline{s} \sim 0\sigma^{j+1}(\underline{s})$. By the assumption, $1\underline{s} = \sigma^{p-1}(\underline{s})$ holds, but $0\sigma^{j+1}(\underline{s})$ is different from any elements of $Q\underline{s}$ since the p-th character of $\sigma^{j+1}(\underline{s})$ is 1. So $0\sigma^{j+1}(\underline{s}) \notin Q\underline{s}$ holds and it contradicts the assumption. If $k(\underline{\alpha}) < k(\underline{s}) < k(\sigma^{j+1}(\underline{s}))(< k(\sigma^{p-1}(\underline{s})))$ holds, $\underline{s} \sim \sigma^{p-1}(\underline{s}) = 1\underline{s}$ implies $0\underline{s} \sim 01\underline{s}$ and $1\underline{s} \sim 11\underline{s}$. Then $C_{\underline{s},1\underline{s}}$ and $C_{0\underline{s},01\underline{s}}$ intersect each other. Hence $k(\sigma^{j+1}(\underline{s})) \leq k(\underline{\alpha})$ $---$ $(*1)$.

Suppose $k(\sigma(\underline{s})) \leq k(\underline{\alpha})$. Since the inequality $k(\underline{s}) < k(\sigma(\underline{s})) \leq k(\underline{\alpha})$ holds, $\underline{s} \sim \sigma(\underline{s})$ implies $0\underline{s} \sim 0\sigma(\underline{s})$ and $1\underline{s} \sim 1\sigma(\underline{s})$. By the assumption, $1\underline{s} = \sigma^{p-1}(\underline{s})$ holds, but $1\sigma^{j+1}(\underline{s})$ is different from any elemnts of Qs since the p-th character of $\sigma(\underline{s})$ is 0. So $1\sigma^{j+1}(\underline{s}) \notin Q\underline{s}$ holds and it contradicts the assumption. Hence $k(\underline{\alpha}) < k(\sigma(\underline{s}))$ $---$ $(*2)$.

By (*1) and (*2), $k(\sigma^{j+1}(\underline{s})) \leq k(\underline{\alpha}) < k(\sigma(\underline{s}))$ holds.

(ii) Let $u = \underbrace{0\cdots0}_{j-1}1 \in \{0,1\}^j$ and $w = 0u0u\cdots0uu \in \{0,1\}^{k_2(j+1)+j}$. Then $\underline{s} = \overline{w}$. It will be shown in a similar way to (i). □

If $\underline{s} = \overline{s_1 s_2 \cdots s_p}$ is p-periodic, then $\underline{t} = \overline{s'_1 s'_2 \cdots s'_p}$ $(s'_j = 1 - s_j)$ is also p-periodic. As for $\underline{t} = \overline{s'_1 s'_2 \cdots s'_p}$, a similar result to Theorem 4 is obtained as follows.

Corollary 1. *Let $\underline{s} \in \{0,1\}^\infty$ be p-periodic such that $k(\underline{s}) \geq k(\sigma^j(\underline{s}))$ for any $j \geq 0$, and suppose $Q\underline{s} = \{\underline{s}, \sigma(\underline{s}), \cdots, \sigma^{p-1}(\underline{s})\}$. Then $k(\underline{s}) = \frac{q}{2^p-1}$ $(q \in N)$ holds. As for the correspondence of a Julia set to a lamination, we have the following, according to the value of q.*

(i) *Suppose there exists $j \geq 1$ and $k_1 \geq 1$ such that $(2^p - 1) - \Sigma_{n=0}^{k_1-1} 2^{nj} = q$ and $p = jk_1 + 1$. If the lamination S_s^α correspondes to a Julia set, then $\underline{\alpha}$ satisfies the following*

$$k(\sigma(\underline{s})) \leq k(\underline{\alpha}) < k(\sigma^{j+1}(\underline{s})).$$

(ii) *Suppose there exists $j \geq 1$ and $k_2 \geq 1$ such that $(2^p-1)-(1+\Sigma_{n=1}^{k_2} 2^{nj+(n-1)}) = q$ and $p = (j+1)k_2+j$. If the lamination S_s^α corresponds to a Julia set, then $\underline{\alpha}$ satisfies the following*

$$k(\sigma(\underline{s})) \leq k(\underline{\alpha}) < k(\sigma^{(j+1)k_2}(\underline{s})).$$

As a special case of Theorem 4 and Corollary 1, we have the following.

Corollary 2. (i) *Let $\underline{s}$ be an element of $\{0,1\}^\infty$ satisfying $k(\underline{s}) = \dfrac{1}{2^p - 1}$ with some $p \geq 2$ and $Q\underline{s} = \{\underline{s}, \sigma(\underline{s}), \cdots, \sigma^{p-1}(\underline{s})\}$. If the lamination S_s^α corresponds to a Julia set, then α satisfies the following*

$$\frac{1}{2^p - 1} \leq k(\underline{\alpha}) < \frac{2}{2^p - 1}.$$

(ii) *Let $\underline{s}$ be an element of $\{0,1\}^\infty$ satisfying $k(\underline{s}) = \dfrac{2^p - 2}{2^p - 1}$ with some $p \geq 2$ and $Q\underline{s} = \{\underline{s}, \sigma(\underline{s}), \cdots, \sigma^{p-1}(\underline{s})\}$. If the lamination S_s^α corresponds to a Julia set, then α satisfies the following*

$$\frac{2^p - 3}{2^p - 1} \leq k(\underline{\alpha}) < \frac{2^p - 2}{2^p - 1}.$$

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Reference

[1] Bandt,C. and Keller,K., *Symbolic dynamics for angle-doubling on the circle,I.* Ergodic Theory and Related Topics III, Springer-Verlag, 1990.

[2] Bandt,C. and Keller,K., *Symbolic dynamics for angle-doubling on the circle,II.* Nonlinearity 6 (1993), 377-392.

[3] Falconer,K. *Fractal Geometry.* John Wiley & Sons, 1989.