COMMUTING SQUARES FROM A PAIR OF COSPECTRAL GRAPHS

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Abstract

We construct the commuting squares of finite dimensional von Neumann algebras based on a pair of cospectral finte graphs. And we have a new doubly-indexed squence of the indices of irreducible subfactors of the hyperfinite factor of type II_1 which are greater than 4.

0. Introduction

The subfactor theory was initiated by V. Jones in [J], since then the theory of subfactors has been related to many fields of mathematics and physics. Especially, the connections between lattice models in statistical mechanics, a knot theory of low dimensional topology and subfactor theory of operator algebras were soonly established by many researchers after his work.

The classification of subfactors of the hyperfinite factor is one of the most important and exciting problems in the theory of operator algebras. In the classification, one of the most interesting cases of subfactors is in the case of irreducible subfactors, those for which the relative commutant is reduced to the scalars. V. Jones proved in [J] that any subfactor with an index less than 4 is automatically irreducible. However, his example of subfactor with index greater than 4 is not irreducible. The problem of characterizing the values of indices of irreducible subfactors which are greater than 4, still remains open in the case of hyperfinite.

The concept of a commuting square of finite von Neumann algebras was first considered in [Po1] by S. Popa as an orthogonal pair of subalgebras, and was largely used in [PP]. It plays an important role in subfactor theory as seen from [Po2], [Po3] etc. For example, by using suitable commuting squares of finite dimensional von Neumann algebras, several authors have constructed irreducible subfactors of the hyperfinite factor of type II_1 [GHJ],[HS],[Sc],[Su1],[Su2],[Y] etc. U. Haagerup and J. Schou in [HS] established a criterion for the existence of symmetric commuting squares of finite dimensional von Neumann algebras, which is deeply related to the axioms of Ocneanu's biunitary connections on the pair of bipartite graphs.

The tower of higher relative commutants is a very useful invariant for the classification of subfactors of the hyperfinite factor of type II_1. A. Ocneanu introduced paragroups [Oc1] which is a certain quantization of finite groups, for analyzing this invariant. A paragroup is a pair of bipartite graphs with a certain complex-valued function defined on squares constituted from four edges of the graphs, which is called connection. Moreover in the study of paragroups, finite graphs are regarded as discrete compact manifolds and the above complex-valued function on squares can be regarded as an analogue of a connection. The key notion of Ocneanu's theory is the flatness of this connection. From another point of view, a connection resembles the Boltzmann weights of an IRF model of statistical mechanics. The axioms of Ocneanu's biunitary connections correspond to the basic relations of Boltzmann weights [DWA] and the flatness of a connection is related to the requirement of Yang-Baxter equation [EK].

A. Ocneanu announced the complete classfication, up to conjugacy, for subfactors of indices less than 4 in the terms of Dynkin diagrams [Oc1]. However his proof has been unavailable. But S. Popa gave the proof for stronger form in [Po3] and also gave the classification table for subfactor with index 4 [Po4].

The discrete version of the famous open question, "*Can one hear the shape of a drum?*" [Kac] can be considered as an inverse problem between the spectra of graphs and the structure of graphs. And in the graph theory, we have many negative examples of topologically non-isomorphic cospectral graphs for this question . Here the term *cospectral* means that all the eigenvalues of the Laplacians (the adjacency matrices) of the graphs coincide including the multiplicities. In this paper, we use such a pair of cospectral bipartite graphs as underlying graphs for our biunitary connections. And we give a new doubly-indexed sequence of the indices of irreducible subfactors of the hyperfinite factor of type II_1 by constructing biunitary connections, that is by constructing the commuting squares of finite dimensional von Neumann algebras which are based on the cospectral graphs.

The contents of each section are as follows. In the first section, we recall the notions and terminology of commuting squares and Ocneanu's biunitary connections of graphs. And the resemblance between Ocneanu's biunitary connections and the Boltzmann weights of an IRF model is also described. In the subsequent section, the description of the pair of the cospectral graphs on which we construct a biunitary connection, will be given. In the final section, we examplify the construction of a biunitary connection on the pair of the cospectral graphs and list the values of the indices of our irreducible subfactors.

1. Commuting squares, biunitary connections and Boltzmann weights

Let H be a Hilbert space, we denote by $B(H)$ the $*-$algebra of all bounded operators on H, with x^* the adjoint of the operator $x \in B(H)$. A von Neumann algebra acting on H is a weakly closed $*-$subalgebra of $B(H)$ which contains the identity.

A factor is a von Neumann algebra M with the center $Z(M)$ reduced to the scalar multiples of the identity. The factor has been classified into types I, II_1, II_∞ and III by its projections. Among these types, a factor of type II_1 M can be characterized as infinite dimensional factor which admits the unique normalized trace τ such that

$$\tau(1) = 1,$$
$$\tau(xy) = \tau(yx) \quad \text{for } x, y \in M,$$
$$\tau(x^*x) \geq 0 \quad \text{for } x \in M.$$

A finite factor is a von Neumann algebra which is either a factor of type II_1, or isomorphic to $B(H)$ for some H of finite dimension.

Let M be a finite factor and let N be a subfactor of M. V. Jones introduced the index $[M, N]$ associated with the inclusion $N \subset M$ in [J]. V. Jones proved that the possible values of the index must be in the set $\{4\cos^2 \frac{\pi}{n} : n \geq 3\} \cup [4, \infty]$ and that each of these values can be realized in the case of the hyperfinite factor of type II_1 which is approximated by ascending squence of finite dimensional $*-$subalgebras. A subfactor called irreducible if the relative commutant $N' \cap M$ is reduced to the scalars. V. Jones showed that any subfactor with an index smaller than 4 is automatically irreducible. It remains open to characterize the possible values greater than 4 of the indices of irreducible subfactors in the case of the hyperfinite factor of type II_1.

In the analysis of the structure of von Neumann algebras, the conditional expectation plays a very important role. Let M be a finite von Neumann algebra with faithful normal normalized trace τ and let N be a von Neumann subalgebra of M. Then there is the $\tau-$ preserving conitional expectation $E_N : M \longrightarrow N$ defined by the relation

$$\tau(E_N(x)y) = \tau(xy) \quad \text{for } x \in M \text{ and } y \in N.$$

The map E_N is normal and has the following algebraic properties:

$$E_N(axb) = aE_N(x)b \quad \text{for } x \in M, a, b \in N$$
$$E_N(x^*) = E_N(x)^* \quad \text{for } x \in M$$
$$E_N(x^*)E_N(x) \leq E_N(x*x) \quad \text{and} \quad E_N(x^*x) = 0 \quad \text{implies } x = 0.$$

Of course, this conditional expectation map corresponds to the condtional expectation for a sub $\sigma-$ field in the probability theory defined by the Radon - Nikodým derivative.

The concept of a commuting square of finite von Neumann algebras has first been introduced in [Po1] and [Po2] by Popa as an orthogonal pair of algebras, which corresponds to the independence of two $\sigma-$ fields in the probability theory. It was later generalized and largely used in [PP]. Here, we recall the notions of a commuting square of finite von Neumann algebras.

A commuting square is a quadruple (A, B, C, D) of finite von Neumann algebras with the following inclusion relations

$$A \subseteq B \subseteq D \quad \text{and} \quad A \subseteq C \subseteq D,$$

together with a faithful normal trace τ on the largest algebra D, such that the τ - preserving conditional expectations E_A, E_B and E_C of D on A, B and C, respectively, satisfy the conditions,

$$E_A = E_B E_C = E_C E_B.$$

Equivalent definitions for a commuting square are found in [GHJ, Chap. IV]. A finite dimensional von Neumann algebra, which is clearly finite von Neumann algebra, is $*-$ isomorphic to a finite direct sum of matrix algebras (multimatrix algebras). And an inclusion of finite dimensional von Neumann algebras can be given by the inclusion matrix which has non-negative integer entries, and also can be described diagramatically [GHJ].

Thus, if the algebras A, B, C and D are finite dimensional von Neumann algebras in above then the four inclusions $A \subseteq B, \quad B \subseteq D, \quad A \subseteq C$ and $C \subseteq D$ are given by the inclusion matrices G, L, K and H, respectively.

$$
\begin{array}{ccc}
C & \xrightarrow{\;\;H\;\;} & D \\[4pt]
K \uparrow & & \uparrow L \\[4pt]
A & \xrightarrow[\;\;G\;\;]{} & B
\end{array}
$$

And then these matrices must satisfy the condition

$$GL = KH.$$

In [HS], a commuting square (A, B, C, D) of finite dimensional von Neumann algebras with inclusion matrices G, L, K and H, given above, is called a *symmetric* commuting square, if the inclusion matrices satisfy the additional condition,

$$({}^t G)K = L({}^t H).$$

In generally, given a non-negative integer entried $n \times m$ matrix $X = (x_{ij})$, we denote by Γ_X the oriented bipartite graph (the vertices are divided into even and odd ones), which has $(n + m)$ many vertices, say $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_m$, and x_{ij} many oriented edges from a_i to b_j. The bipartite graph $\Gamma_{({}^t X)}$, (the graph corresponding to the transposed matrix ${}^t X$) has the same figure as Γ_X, but each edge of it has the reversed orientation.

The criterion for the existence of a commuting square of finite dimensional von Neumann algebras with inclusions described by the given matrices, was obtained in [HS] and reformulated in [Su1, §2].

We write the criterion here again. Given the non-negative integer-entried $n \times m$ matrix G, $m \times q$ matrix L, $n \times p$ matrix K, and $p \times q$ matrix H, with $GL = KH$ and $({}^t G)K = L({}^t H)$, we consider the inclusion diagram

$$
\begin{array}{ccc}
C & \xrightarrow{\;\;H\;\;} & D \\[4pt]
K \uparrow & & \uparrow L \\[4pt]
A & \xrightarrow[\;\;G\;\;]{} & B
\end{array}
$$

of finite dimensional von Neumann algebras, where n, m, p and q are the dimensions of the centers of algebras A, B, C and D, respectively. We take bipartite graphs Γ_G, Γ_L, Γ_K and Γ_H as mentioned before, and we relabel the vertices of these graphs as follows.

The vertices of Γ_G : $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_m$.

The vertices of Γ_L : $b_1, b_2, \cdots, b_m, d_1, d_2, \cdots, d_q$.

The vertices of Γ_K : $a_1, a_2, \cdots, a_n, c_1, c_2, \cdots, c_p$.

The vertices of Γ_H : $c_1, c_2, \cdots, c_p, d_1, d_2, \cdots, d_q$.

In the above situation, the criterion can be written as follows.

Proposition 1.1.   ([HS], [Su1])   *In order to become a symmetric commuting square with inclusion matrices G, L, K and H, as above,*

$$
\begin{array}{ccc}
C & \xrightarrow{\ H\ } & D \\
{\scriptstyle K}\uparrow & & \uparrow{\scriptstyle L} \\
A & \xrightarrow[\ G\ ]{} & B
\end{array}\ ,
$$

it is necessary and sufficient that the following conditions are satisfied.

(1)   *There exist positive-entried vectors*

$$
\mathbf{a} = (\mu(a_i))_{i=1,2,\cdots,n}\ , \qquad \mathbf{b} = (\mu(b_j))_{j=1,2,\cdots,m}\ ,
$$
$$
\mathbf{c} = (\mu(c_l))_{l=1,2,\cdots,p}\ , \quad and \quad \mathbf{d} = (\mu(d_k))_{k=1,2,\cdots,q}
$$

satisfying

$$
\mathbf{a} = \frac{1}{\|G\|}G\mathbf{b} = \frac{1}{\|K\|}K\mathbf{c}, \quad \mathbf{b} = \frac{1}{\|L\|}L\mathbf{d} \quad and \quad \mathbf{c} = \frac{1}{\|H\|}H\mathbf{d}.
$$

(2)   *For each a_i ($1 \le i \le n$) and d_k ($1 \le k \le q$), there exists a unitary matrix $U^{(i,k)}$ of size $(GL)_{(i,k)}$ with rows indexed by paths $(a_i \to b_j \to d_k)$, where b_j is arbitrary, and columns indexed by paths $(a_i \to c_l \to d_k)$, where c_l is arbitrary. Of course, the edges $(a_i \to b_j)$, $(b_j \to d_k)$, $(a_i \to c_l)$ and $(c_l \to d_k)$ are in Γ_G, Γ_L, Γ_K and Γ_H, respectively. And the (i,k) - th entries of the matrices GL and KH are the same by assumption.*

(3)   *For each b_j ($1 \le j \le m$) and c_l ($1 \le l \le p$), there exists a unitary matrix $V^{(j,l)}$ of size $(({}^tG)K)_{(j,l)}$ with rows indexed by paths $(b_j \to a_i \to c_l)$, where a_i is arbitrary, and columns indexed by paths $(b_j \to d_k \to c_l)$, where d_k is arbitrary. Of course, the edges $(b_j \to a_i)$, $(a_i \to c_l)$, $(b_j \to d_k)$ and $(d_k \to c_l)$ are in $\Gamma_{({}^tG)}$, Γ_K, Γ_L and $\Gamma_{({}^tH)}$, respectively. And the (j,l) - th entries of the matrices $({}^tG)K$ and $L({}^tH)$ are the same by assumption.*

(4)   *The matrics $U^{(i,k)}$, $V^{(j,l)}$ are related by the requirement that*

$$
\sqrt{\mu(a_i)\mu(d_k)}\left(U^{(i,k)}\right)_{(a_i \to b_j \to d_k),(a_i \to c_l \to d_k)}
$$
$$
= \sqrt{\mu(b_j)\mu(c_l)}\left(V^{(j,l)}\right)_{(b_j \to a_i \to c_l),(b_j \to d_k \to c_l)} \quad .
$$

The conditions in Proposition 1.1 are closely related to the axioms of Ocneanu's biunitary connections for a pair of graphs. Condition (1) corresponds to the Perron-Frobenius eigenvectors. Conditions (2) and (3) are called the *biunitarity*. And condition (4) is nothing but the *renormalization rule*. Moreover, this biunitary connection resembles the Boltzmann weights of the IRF (Interaction Round a Face) model in statistical mechanics.

First, let us fix the notation and terminology about connections used in this paper. Let $\mathcal{G}$ and $\mathcal{H}$ be finite bipartite graphs with vertices $\mathcal{G}^{(0)}$ and $\mathcal{H}^{(0)}$, and edges $\mathcal{G}^{(1)}$ and $\mathcal{H}^{(1)}$, respectively. We assume that these graphs have the same Perron - Frobenius eigenvalues of the adjacency matrices, and that even vertices of the graphs coincide.

For an edge $\xi \in \mathcal{G}^{(1)} \cup \mathcal{H}^{(1)}$, let $s(\xi)$ and $r(\xi) \in \mathcal{G}^{(0)} \cup \mathcal{H}^{(0)}$ denote its source and range, respectively. We denote $\tilde{\xi}$ the edge with the reversed orientation.

Consider a diagram

$$
\begin{array}{c}
\xi_2 \nearrow \overset{b}{} \searrow \xi_3 \\
a d \\
\xi_1 \searrow \underset{c}{} \nearrow \xi_4
\end{array},
$$

where the ξ_i's satisfy one of the following conditions,

$$\xi_1 \in \mathcal{G}^{(1)}, \xi_2 \in \mathcal{G}^{(1)}, \xi_3 \in \mathcal{H}^{(1)}, \xi_4 \in \mathcal{H}^{(1)}, \quad or$$

$$\xi_1 \in \mathcal{G}^{(1)}, \xi_2 \in \mathcal{H}^{(1)}, \xi_3 \in \mathcal{H}^{(1)}, \xi_4 \in \mathcal{G}^{(1)}, \quad or$$

$$\xi_1 \in \mathcal{H}^{(1)}, \xi_2 \in \mathcal{H}^{(1)}, \xi_3 \in \mathcal{G}^{(1)}, \xi_4 \in \mathcal{G}^{(1)}, \quad or$$

$$\xi_1 \in \mathcal{H}^{(1)}, \xi_2 \in \mathcal{G}^{(1)}, \xi_3 \in \mathcal{G}^{(1)}, \xi_4 \in \mathcal{H}^{(1)},$$

and $a = s(\xi_1) = s(\xi_2)$, $b = r(\xi_2) = s(\xi_3)$, $c = r(\xi_1) = s(\xi_4)$, $d = r(\xi_3) = r(\xi_4)$.

We call such a diagram an admissible cell. A connection W is a map which associates a complex number to any admissible cell. For each cell, we write the complex number assigned by W,

$$
W \begin{pmatrix} \xi_2 \nearrow \overset{b}{} \searrow \xi_3 \\ a d \\ \xi_1 \searrow \underset{c}{} \nearrow \xi_4 \end{pmatrix} \in \mathbf{C},
$$

and call it a cell weight. The map W requires the following properties.

(1) Rotation symmetry.

$$
W \begin{pmatrix} \xi_2 \nearrow \overset{b}{} \searrow \xi_3 \\ a d \\ \xi_1 \searrow \underset{c}{} \nearrow \xi_4 \end{pmatrix} = \overline{W \begin{pmatrix} \xi_2 \nwarrow \overset{a}{} \searrow \xi_1 \\ b c \\ \xi_3 \searrow \underset{d}{} \swarrow \xi_4 \end{pmatrix}}
$$

$$
= \overline{W \begin{pmatrix} \xi_4 \nearrow \overset{d}{} \nwarrow \xi_3 \\ c b \\ \xi_1 \searrow \underset{a}{} \nearrow \xi_2 \end{pmatrix}} = W \begin{pmatrix} \xi_4 \swarrow \overset{c}{} \nwarrow \xi_1 \\ d a \\ \xi_3 \nwarrow \underset{b}{} \swarrow \xi_2 \end{pmatrix}.
$$

(2) Renormalization rule.

$$
W \begin{pmatrix} \xi_2 \nearrow \overset{b}{} \searrow \xi_3 \\ a d \\ \xi_1 \searrow \underset{c}{} \nearrow \xi_4 \end{pmatrix} = \sqrt{\frac{\mu(b)\mu(c)}{\mu(a)\mu(d)}} W \begin{pmatrix} \tilde{\xi}_2 \nwarrow \overset{b}{} \searrow \xi_3 \\ a d \\ \xi_1 \searrow \underset{c}{} \nwarrow \tilde{\xi}_4 \end{pmatrix}
$$

$$
= \sqrt{\frac{\mu(b)\mu(c)}{\mu(a)\mu(d)}} W \begin{pmatrix} \xi_4 \swarrow \overset{c}{} \searrow \tilde{\xi}_1 \\ d a \\ \tilde{\xi}_3 \searrow \underset{b}{} \swarrow \xi_2 \end{pmatrix},
$$

where $\mu(\cdot)$ denotes an entry of the Perron - Frobenius eigenvector of the adjacency matrix of each graph.

(3) Biunitarity

For each fixed pairs of vertices (a, d) and (b, c), both matrices

$$W \begin{pmatrix} & \nearrow^{*}\searrow & \\ a & & d \\ & \searrow_{*}\nearrow & \end{pmatrix} \qquad \text{and} \qquad W \begin{pmatrix} & \swarrow^{b}\searrow & \\ * & & * \\ & \searrow_{c}\swarrow & \end{pmatrix}$$

are unitary. That is,

$$\sum_{e, \xi_1, \xi_4} W \begin{pmatrix} & \xi_2 \nearrow^{b}\searrow \xi_3 & \\ a & & d \\ & \xi_1 \searrow_{e}\nearrow \xi_4 & \end{pmatrix} \overline{W \begin{pmatrix} & \eta_2 \nearrow^{c}\searrow \eta_3 & \\ a & & d \\ & \xi_1 \searrow_{e}\nearrow \xi_4 & \end{pmatrix}} = \delta_{\xi_2,\eta_2} \delta_{\xi_3,\eta_3} \delta_{b,c},$$

$$\sum_{e, \xi_2, \xi_3} W \begin{pmatrix} & \xi_1 \swarrow^{b}\searrow \xi_2 & \\ a & & e \\ & \xi_4 \searrow_{c}\swarrow \xi_3 & \end{pmatrix} \overline{W \begin{pmatrix} & \eta_1 \swarrow^{b}\searrow \xi_2 & \\ d & & e \\ & \eta_4 \searrow_{c}\swarrow \xi_3 & \end{pmatrix}} = \delta_{\xi_1,\eta_1} \delta_{\xi_4,\eta_4} \delta_{a,d}.$$

Compairing Proposition 1.1 and the axioms of a biunitary connection given above, the criterion for symmetric commuting squares can be written in the terms of connections as follows.

If a pair of graphs $(\mathcal{G}, \mathcal{H})$ admits a connection then there exists a symmetric commuting square of the form

$$\begin{array}{ccc} C & \xrightarrow{H} & D \\ G\uparrow & & \uparrow H \\ A & \xrightarrow{G} & B \end{array},$$

where the inclusion matrices G and H is given in the correspondence $\mathcal{G} = \Gamma_G$ and $\mathcal{H} = \Gamma_H$, respectively.

Remark 1.2. If the graphs $\mathcal{G}$ and $\mathcal{H}$ are the same, the contragredient map is trivial which means that the inclusion matrices G and H are mutually transposed (that is the graphs $\mathcal{G}$ and $\mathcal{H}$ are glued up in the mirror image), the graph has a triple point and has no square cycles, and the norm of the graph (the Perron-Frobenius eigenvalue) is bigger than 2 then there are no connections on such a pair [Oc2]. This fact would be called the triple point obstruction.

The connection on the pair of graphs is closely related to the Boltzmann weights for the IRF model [DWA], [B] as we mentioned before. However, in our case, we have no spectral parameters. The biunitary conditions (3) of a connection correspond to inversion relations. The crossing symmetry of the IRF model is interpreted to the renormalization rule (2) of a connection. And the rotation symmetry corresponds to the reflection symmetry of the IRF medel. The charge (or spin) conservation condition in the IRF model corresponds to the admissible cell. That is the cell weight values zero except on the admissible cells. The crossing multipliers in an IRF model can be regarded as the Perron-Frobenius eigen-weights at the vertices of the graphs.

Once we get a symmetric commuting square of finite dimensional von Neumann algebras, we would obtain the pair of the hyperfinite factor and subfactor of type II_1 by iterating the fundamental constructions.

This procedure is called the path model or the string algebra construction. And if the initial symmetric commuting square has the certain condition as descriebled in the next proposition, we can obtain the

irreducible pair of the hyperfinite factor and subfactor of type II_1 and can determine the value of index for this pair.

Proposition 1.3. ([HS])   *Let*

$$C \xrightarrow{\quad H \quad} D$$
$$K \uparrow \qquad \uparrow L$$
$$A \xrightarrow{\quad G \quad} B$$

be a symmetric commuting square of finite dimensional von Neumann algebras. Assume that Γ_G, Γ_L, Γ_k and Γ_H are connected graphs, and Γ_G or Γ_H has a vertex which is connected to only one other vertex of the graph. Then there exists an irreducible subfactor N of the hyperfinite factor of type II_1 R with index

$$[R : N] = ||G||^2 = ||H||^2.$$

We use the criterion for symmetric commuting squares in a connection version on the special pair of graphs. Thus we try to construct connections on it. The method of constructing connections is also found in [Sc], [Y], [Oc2] and [IK] etc. And for more details about biunitary connections, see [Oc1], [EK], [Oc2] and [Kaw].

2. The pair of cospectral graphs

In this paper, we will use some special pairs of graphs, called *cospectral*, as the underlying graphs $\mathcal{G}$ and $\mathcal{H}$ for biunitary connections. And we would obtain the new series for the indices of irreducible subfactors.

Definition 2.1.   Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be topologically non-isomorphic graphs. And let $P_{\mathcal{G}_1}(t)$ and $P_{\mathcal{G}_2}(t)$ be the characteristic polynomials of the adjacency matrices, Laplacians of the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. If $P_{\mathcal{G}_1}(t) \equiv P_{\mathcal{G}_2}(t)$ then the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are called *cospectral.*

Several examples of the cospectral graphs can be found in [CDS]. Especially, we can find the following pairs of graphs among them :

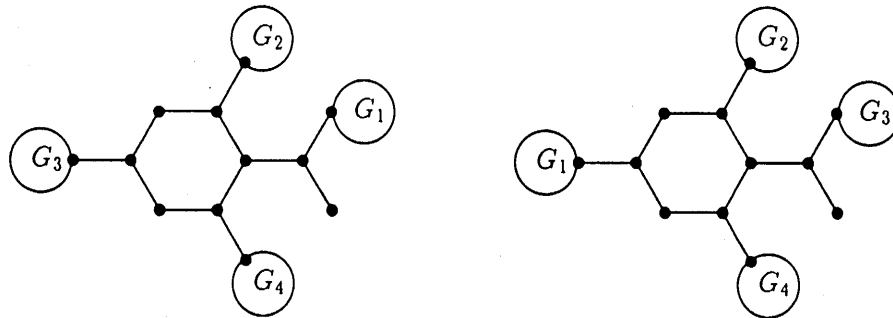If G_1, G_2, G_3 and G_4 are arbitary graphs then the graphs in Fig. 2.1 are cospectral.

Fig. 2.1

Then we shall give a description of the graphs used in this paper which are the special cases of Fig. 2.1, and note some properties about them.

Definition 2.2. For $n, m \in N \cup \{0\}$ with $n \geq m$, let $\mathcal{G}(n, m)$ be the bipartite graph of the form
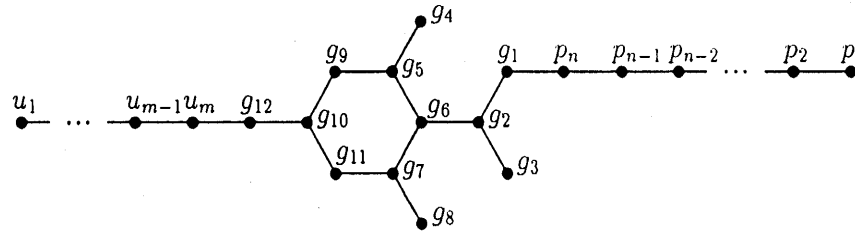

Fig. 2.2

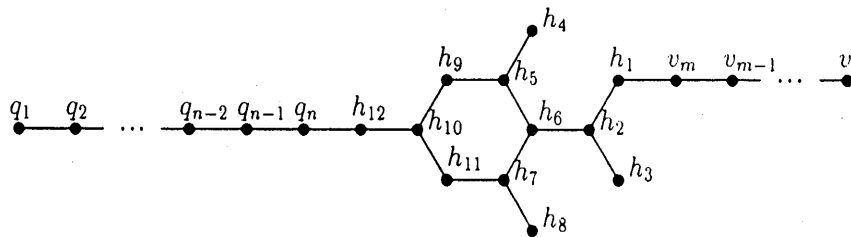and $\mathcal{H}(n, m)$ be the bipartite graph of the form

Fig. 2.3

that is, the graph $\mathcal{H}(n, m)$ is obtained by exchanging the right line graph (A_n part) and the left line graph (A_m part) of the graph $\mathcal{G}(n, m)$.

Clearlly the graphs $\mathcal{G}(n, m)$ and $\mathcal{H}(n, m)$ are the special case of Fig.2.1 obtained by putting A_n and A_m as G_1 and G_3, respectively, and letting G_2 and G_4 be empty. For a moment, we lable the virtices of the graphs as in Fig.2.2 and Fig.2.3. Of course, the graphs $\mathcal{G}(n, m)$ and $\mathcal{H}(n, m)$ are cospectral so the characteristic polynomials of them are the same.

Here let us recall the procedures which enable characteristic polynomials to be determined by simple calculation.

Lemma 2.3. ([GHJ], [CDS] etc)   *Let x_1 be a vertex of degree 1 in the graph $\mathcal{G}$ and let x_2 be the vertex adjacent to x_1. Let $\mathcal{G}_1$ be the induced subgraph obtained from $\mathcal{G}$ by deleting the vertex x_1. If x_1 and x_2 are deleted, the induced graph $\mathcal{G}_2$ is obtained. Then we have*

$$(2.1) \qquad\qquad\qquad P_{\mathcal{G}}(t) = t P_{\mathcal{G}_1}(t) - P_{\mathcal{G}_2}(t)$$

More generally, let $\mathcal{G}$ be the graph obtained by joining the vertex x of the graph $\mathcal{G}_1$ to the vertex y of the graph $\mathcal{G}_2$ by an edge. Let $\mathcal{G}'_1$ and $\mathcal{G}'_2$ be the induced subgraph of $\mathcal{G}_1$ and $\mathcal{G}_2$ obtained by deleting the vertex x and y from $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Then we have

$$(2.2) \qquad\qquad\qquad P_{\mathcal{G}}(t) = P_{\mathcal{G}_1}(t) P_{\mathcal{G}_2}(t) - P_{\mathcal{G}'_1}(t) P_{\mathcal{G}'_2}(t)$$

Having in mind the equality (2.1), we introduce the sequence of polynomials which is rather well-known, for describing our characteristic polynomial.

Definition 2.4. We define polynomials $R_n(t)$ ($n = 0, 1, 2, \cdots$) recursively in the following manner

$$R_0(t) = 1, \quad R_1(t) = t \quad \text{and}$$

$$(2.3) \qquad R_{n+2}(t) = tR_{n+1}(t) - R_n(t) \quad (\text{for} \quad n \geq 0).$$

This series of the polynomials $\{R_n(t)\}$ has the following properties.

Propostion 2.5. ([Sc])

(i)   For $m \geq 1$, $n > 0$ and $\tau > 2$, $\dfrac{R_n(\tau)}{R_{n+m}(\tau)}$ has the following properties :

(1) $\dfrac{R_n(\tau)}{R_{n+m}(\tau)}$ is decreasing in τ.

(2) $\dfrac{R_n(\tau)}{R_{n+m}(\tau)}$ is increasing in n and decreasing in m.

(ii)   For $\tau > 2$, if we put $\tau = 2\cosh x$ $(x > 0)$ then we have

$$R_n(\tau) = \frac{\sinh(n+1)x}{\sinh x} \qquad (n = 0, 1, 2, \cdots),$$

and hence

$$\lim_{n \to \infty} \frac{R_n(\tau)}{R_{n+m}(\tau)} = e^{-mx}.$$

The following facts about the graphs $\mathcal{G}(n, m)$ and $\mathcal{H}(n, m)$, Lemma 2.6 and Lemma 2.7, might be obtained from the definition of eigensystem of the adjacency matrices of the graphs. And it is easy to check them by direct calculation from Lemma 2.3 so we should like to omite the details.

Lemma 2.6.   *The characteristic polynomial $P(t)$ for the ajacency matrices of the graphs $\mathcal{G}(n, m)$ and $\mathcal{H}(n, m)$ can be written in the form*

$$(2.4) \qquad \begin{aligned} P(t) = {}& t^2(t^2 - 2) \left[\{(t^2 - 4)R_{m+3}(t) - (t^2 - 2)R_{m+1}(t)\}R_{n+3}(t) \right. \\ & \left. - \{(t^2 - 2)R_{m+3}(t) - t^2 R_{m+1}(t)\}R_{n+1}(t) \right] \end{aligned}$$

or equivalently,

$$(2.5) \qquad \begin{aligned} P(t) = {}& t^2(t^2 - 2) \left[\{(t^2 - 4)R_{n+3}(t) - (t^2 - 2)R_{n+1}(t)\}R_{m+3}(t) \right. \\ & \left. - \{(t^2 - 2)R_{n+3}(t) - t^2 R_{n+1}(t)\}R_{m+1}(t) \right], \end{aligned}$$

where $\{R_n(t)\}$ is the sequence of polynomials defined in Definition 2.4.

Since the graphs $\mathcal{G}(n, m)$ and $\mathcal{H}(n, m)$ are connected, their adjacency matrices are primitive and the Perron - Frobenius eigenvalue λ is the largest root of the polynomial equation $P(t) = 0$. On the other hand, it is clear that the Perron - Frobenius eigenvalue λ which coincides with the norm of the adjacency matrices, is greater than 2. Thus we can assert that the Perron - Frobenius eigenvalue λ might be the largest root of the polynomial equation

$$(2.6) \qquad \begin{aligned} & \{(t^2 - 4)R_{m+3}(t) - (t^2 - 2)R_{m+1}(t)\}R_{n+3}(t) \\ & \quad - \{(t^2 - 2)R_{m+3}(t) - t^2 R_{m+1}(t)\}R_{n+1}(t) = 0, \end{aligned}$$

and hence λ satisfies the equality,

(2.7)
$$\lambda^2 = \frac{4R_{m+3}(\lambda)R_{n+3}(\lambda) - 2R_{m+3}(\lambda)R_{n+1}(\lambda) - 2R_{m+1}(\lambda)R_{n+3}(\lambda)}{(R_{m+3}(\lambda) - R_{m+1}(\lambda))(R_{n+3}(\lambda) - R_{n+1}(\lambda))}.$$

We denote by $\mu(x)$ the entry of the Perron - Frobenius eigenvector corresponding to the vertex x and call it the Perron - Frobenius wight at the vertex x. Of course, the Perron - Frobenius eigenvector is determined only up to a positive scalar multiple, however in our applications, these multiples appear on denominators and numerators at the same time. Hence it does not matter how we may normalize it.

Lemma 2.7.   *The Perron - Frobenius wights on each vertex of the graphs $\mathcal{G}(n, m)$ and $\mathcal{H}(n, m)$ are as follows.*
On the graph $\mathcal{G}(n, m)$:

$$\mu(p_i) = R_{i-1}(\lambda) \qquad (i = 1, 2, \cdots, n),$$

$$\mu(u_j) = \kappa R_{j-1}(\lambda) \qquad (j = 1, 2, \cdots, m),$$

$$\mu(g_1) = R_n(\lambda),$$

$$\mu(g_2) = R_{n+1}(\lambda),$$

$$\mu(g_3) = \frac{1}{\lambda}R_{n+1}(\lambda),$$

$$\mu(g_4) = \frac{1}{2\lambda}(R_{n+3}(\lambda) - R_{n+1}(\lambda)) = \frac{\kappa}{2\lambda}(R_{m+3}(\lambda) - R_{m+1}(\lambda)),$$

$$\mu(g_5) = \frac{1}{2}(R_{n+3}(\lambda) - R_{n+1}(\lambda)),$$

$$\mu(g_6) = \frac{1}{\lambda}R_{n+3}(\lambda),$$

$$\mu(g_7) = \frac{1}{2}(R_{n+3}(\lambda) - R_{n+1}(\lambda)),$$

$$\mu(g_8) = \frac{1}{2\lambda}(R_{n+3}(\lambda) - R_{n+1}(\lambda)) = \frac{\kappa}{2\lambda}(R_{m+3}(\lambda) - R_{m+1}(\lambda)),$$

$$\mu(g_9) = \frac{1}{2\lambda}\{(\lambda^2 - 3)R_{n+3}(\lambda) - (\lambda^2 - 1)R_{n+1}(\lambda)\} = \frac{\kappa}{2\lambda}(R_{m+3}(\lambda) + R_{m+1}(\lambda)),$$

$$\mu(g_{10}) = \kappa R_{m+1}(\lambda),$$

$$\mu(g_{11}) = \frac{1}{2\lambda}\{(\lambda^2 - 3)R_{n+3}(\lambda) - (\lambda^2 - 1)R_{n+1}(\lambda)\} = \frac{\kappa}{2\lambda}(R_{m+3}(\lambda) + R_{m+1}(\lambda)),$$

$$\mu(g_{12}) = \kappa R_m(\lambda).$$

On the graph $\mathcal{H}(n, m)$:

$$\mu(q_i) = R_{i-1}(\lambda) \qquad (i = 1, 2, \cdots, n),$$

$$\mu(v_j) = \kappa R_{j-1}(\lambda) \qquad (j = 1, 2, \cdots, m),$$

$$\mu(h_1) = \kappa R_m(\lambda),$$

$$\mu(h_2) = \kappa R_{m+1}(\lambda),$$

$$\mu(h_3) = \frac{\kappa}{\lambda}R_{m+1}(\lambda),$$

$$\mu(h_4) = \frac{\kappa}{2\lambda}(R_{m+3}(\lambda) - R_{m+1}(\lambda)) = \frac{1}{2\lambda}(R_{n+3}(\lambda) - R_{n+1}(\lambda)),$$

$$\mu(h_5) = \frac{1}{2}(R_{n+3}(\lambda) - R_{n+1}(\lambda)),$$

$$\mu(h_6) = \frac{\kappa}{\lambda} R_{m+3}(\lambda),$$

$$\mu(h_7) = \frac{1}{2}(R_{n+3}(\lambda) - R_{n+1}(\lambda)),$$

$$\mu(h_8) = \frac{\kappa}{2\lambda}(R_{m+3}(\lambda) - R_{m+1}(\lambda)) = \frac{1}{2\lambda}(R_{n+3}(\lambda) - R_{n+1}(\lambda)),$$

$$\mu(h_9) = \frac{\kappa}{2\lambda}\{(\lambda^2 - 3)R_{m+3}(\lambda) - (\lambda^2 - 1)R_{m+1}(\lambda)\} = \frac{1}{2\lambda}(R_{n+3}(\lambda) + R_{n+1}(\lambda)),$$

$$\mu(h_{10}) = R_{n+1}(\lambda),$$

$$\mu(h_{11}) = \frac{\kappa}{2\lambda}\{(\lambda^2 - 3)R_{m+3}(\lambda) - (\lambda^2 - 1)R_{m+1}(\lambda)\} = \frac{1}{2\lambda}(R_{n+3}(\lambda) + R_{n+1}(\lambda)),$$

$$\mu(h_{12}) = R_n(\lambda),$$

$$\text{where } \kappa = \frac{R_{n+3}(\lambda) - R_{n+1}(\lambda)}{R_{m+3}(\lambda) - R_{m+1}(\lambda)}.$$

Remark 2.8.   We have normalized the Perron-Frobenius weights in Lemma 2.7 so that the smallest weights in each graph equal to 1. There are many other equivalent expressions for each weight, for example $\frac{\kappa}{2\lambda}(R_{n+3}(\lambda) - R_{n+1}(\lambda))$ can be represented as $\frac{\kappa}{2\lambda}R_{n+2}(\lambda)$ and so on. However, in the previous Lemma, we have managed to use $R_{m+3}(\lambda)$, $R_{m+1}(\lambda)$, $R_{n+3}(\lambda)$ and $R_{n+1}(\lambda)$ as far as we could. Moreover, in the expressions of the weights $\mu(g_9)$, $\mu(g_{11})$, $\mu(h_9)$ and $\mu(h_{11})$, we have used the equality (2.7).

Next we shall glue up the two bipartite graphs $\mathcal{G}(n,m)$ and $\mathcal{H}(n,m)$ for a biunitary connection. For the biunitary axiom, the number of paths for making rows and columns are equal, and the Perron - Frobenius weights must match at common vertices. According to Lemma 2.7, we shall fold up the graphs $\mathcal{G}(n,m)$ and $\mathcal{H}(n,m)$ as shown in Fig. 2.4, so that the Perron-Frobenius weights on the middle lines coincide.
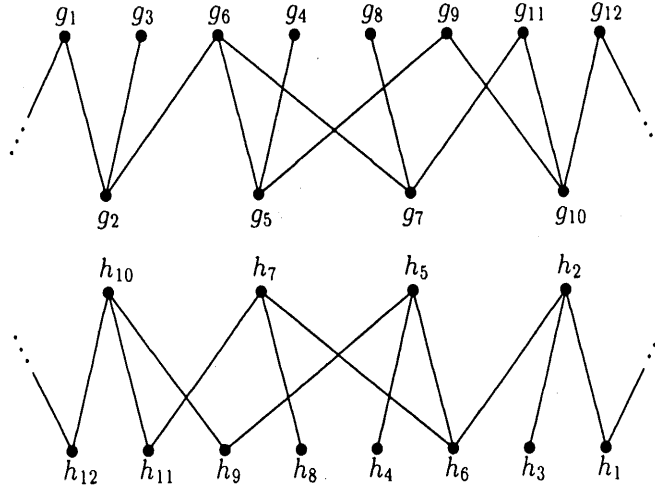

Fig. 2.4

Of course, the vertices remained on the graphs (the vertices on A_n and A_m parts) are fold up in order. Then identify the vertices on the middle lines, and we gule up the two graphs and relabel the vertices around the central hexagon as illustrated in Fig. 2.5.
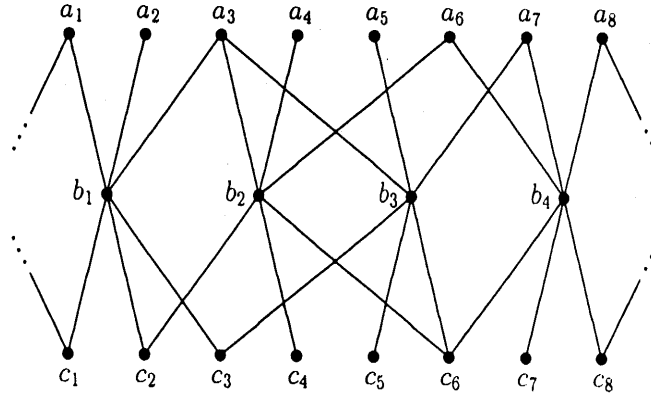
Fig. 2.5

Remark 2.9. In the case of $m = n$, the two graphs $\mathcal{G}(n,n)$ and $\mathcal{H}(n,n)$ are the same one. And then there are two way for gluing, one way is in the π - rotation image which we have just done in above, the other one is in the mirror image. However if we glue up them in the mirror image then the triple point obstruction will not allow us to construct any biunitary connection on this system as we mentioned in Remark 1.2.

At the end of this section, we will list again the Perron - Frobenius weights of the vertices labeled as in Fig. 2.5, for our later convenience.

$$\mu(a_1) = R_n(\lambda),$$

$$\mu(a_2) = \frac{1}{\lambda} R_{n+1}(\lambda),$$

$$\mu(a_3) = \frac{1}{\lambda} R_{n+3}(\lambda),$$

$$\mu(a_4) = \frac{1}{2\lambda}(R_{n+3}(\lambda) - R_{n+1}(\lambda)) = \frac{\kappa}{2\lambda}(R_{m+3}(\lambda) - R_{m+1}(\lambda)),$$

$$\mu(a_5) = \frac{1}{2\lambda}(R_{n+3}(\lambda) - R_{n+1}(\lambda)) = \frac{\kappa}{2\lambda}(R_{m+3}(\lambda) - R_{m+1}(\lambda)),$$

$$\mu(a_6) = \frac{1}{2\lambda}\{(\lambda^2 - 3)R_{n+3}(\lambda) - (\lambda^2 - 1)R_{n+1}(\lambda)\} = \frac{\kappa}{2\lambda}(R_{m+3}(\lambda) + R_{m+1}(\lambda)),$$

$$\mu(a_7) = \frac{1}{2\lambda}\{(\lambda^2 - 3)R_{n+3}(\lambda) - (\lambda^2 - 1)R_{n+1}(\lambda)\} = \frac{\kappa}{2\lambda}(R_{m+3}(\lambda) + R_{m+1}(\lambda)),$$

$$\mu(a_8) = \kappa R_m(\lambda),$$

$$\mu(b_1) = R_{n+1}(\lambda),$$

$$\mu(b_2) = \frac{1}{2}(R_{n+3}(\lambda) - R_{n+1}(\lambda)),$$

$$\mu(b_3) = \frac{1}{2}(R_{n+3}(\lambda) - R_{n+1}(\lambda)),$$

$$\mu(b_4) = \kappa R_{m+1}(\lambda),$$

$$\mu(c_1) = R_n(\lambda),$$

$$\mu(c_2) = \frac{\kappa}{2\lambda}\{(\lambda^2 - 3)R_{m+3}(\lambda) - (\lambda^2 - 1)R_{m+1}(\lambda)\} = \frac{1}{2\lambda}(R_{n+3}(\lambda) + R_{n+1}(\lambda)),$$

$$\mu(c_3) = \frac{\kappa}{2\lambda}\{(\lambda^2 - 3)R_{m+3}(\lambda) - (\lambda^2 - 1)R_{m+1}(\lambda)\} = \frac{1}{2\lambda}(R_{n+3}(\lambda) + R_{n+1}(\lambda)),$$

$$\mu(c_4) = \frac{\kappa}{2\lambda}(R_{m+3}(\lambda) - R_{m+1}(\lambda)) = \frac{1}{2\lambda}(R_{n+3}(\lambda) - R_{n+1}(\lambda)),$$

$$\mu(c_5) = \frac{\kappa}{2\lambda}(R_{m+3}(\lambda) - R_{m+1}(\lambda)) = \frac{1}{2\lambda}(R_{n+3}(\lambda) - R_{n+1}(\lambda)),$$

$$\mu(c_6) = \frac{\kappa}{\lambda}R_{m+3}(\lambda),$$

$$\mu(c_7) = \frac{\kappa}{\lambda}R_{m+1}(\lambda),$$

$$\mu(c_8) = \kappa R_m(\lambda),$$

where $\kappa = \dfrac{R_{n+3}(\lambda) - R_{n+1}(\lambda)}{R_{m+3}(\lambda) - R_{m+1}(\lambda)}$.

3. Connections on cospectral graphs

In this section, we construct a biunitary connection on the pair of the cospectral graphs that we have introduced in the previous section.

Here, let us concentrate our interests upon the cell weights around the central hexagon illustrated in Fig 2.5. Because the cell weights at the vertices on the line (Dynkin $A-$ type) graphs are determined by the routine argument (See [Y], [Oc2], [IK] and [Kaw] etc).

At the vertex b_1 in Fig. 2.5, we get 9 weights of the cells,

$$W\left(\begin{array}{c} \nearrow^{a_i}\searrow \\ b_1 \qquad b_1 \\ \searrow_{c_j}\nearrow \end{array}\right), \qquad i = 1,2,3; \; j = 1,2,3.$$

The absolute values of each weight of them can be determined as follows.

There is the only one path from a_2 to c_1 which passes through the vertex b_1. This implies that $\left|W\left(\begin{array}{c}\nearrow^{a_2}\searrow\\ b_1 \quad b_1\\ \searrow_{c_1}\nearrow\end{array}\right)\right| = 1$ because the weight $W\left(\begin{array}{c}\nwarrow^{a_2}\nearrow\\ b_1 \quad b_1\\ \searrow_{c_1}\swarrow\end{array}\right)$ forms a 1×1 unitary. And then using renormalization rule, we have

$$\left|W\left(\begin{array}{c}\nearrow^{a_2}\searrow\\ b_1 \quad b_1\\ \searrow_{c_1}\nearrow\end{array}\right)\right| = \frac{\sqrt{\mu(a_2)\mu(c_1)}}{\mu(b_1)} \qquad \left|W\left(\begin{array}{c}\nwarrow^{a_2}\nearrow\\ b_1 \quad b_1\\ \searrow_{c_1}\swarrow\end{array}\right)\right| = \frac{\sqrt{\mu(a_2)\mu(c_1)}}{\mu(b_1)}.$$

By the same reason, we obtain that

$$\left|W\left(\begin{array}{c}\nearrow^{a_2}\searrow\\ b_1 \quad b_1\\ \searrow_{c_2}\nearrow\end{array}\right)\right| = \frac{\sqrt{\mu(a_2)\mu(c_2)}}{\mu(b_1)}, \qquad \left|W\left(\begin{array}{c}\nearrow^{a_2}\searrow\\ b_1 \quad b_1\\ \searrow_{c_3}\nearrow\end{array}\right)\right| = \frac{\sqrt{\mu(a_2)\mu(c_3)}}{\mu(b_1)},$$

$$\left|W\left(\begin{array}{c}\nearrow^{a_1}\searrow\\ b_1 \quad b_1\\ \searrow_{c_2}\nearrow\end{array}\right)\right| = \frac{\sqrt{\mu(a_1)\mu(c_2)}}{\mu(b_1)}, \qquad \left|W\left(\begin{array}{c}\nearrow^{a_1}\searrow\\ b_1 \quad b_1\\ \searrow_{c_3}\nearrow\end{array}\right)\right| = \frac{\sqrt{\mu(a_1)\mu(c_3)}}{\mu(b_1)},$$

$$\left|W\left(\begin{array}{c}\nearrow^{a_3}\searrow\\ b_1 \quad b_1\\ \searrow_{c_1}\nearrow\end{array}\right)\right| = \frac{\sqrt{\mu(a_3)\mu(c_1)}}{\mu(b_1)}.$$

By the way, we can determine the weights of the cells on the line graphs one by one from the end point until we encounter the triple poit. And in this proocedure, we can find a suitable starting gauge which makes that

$$W \begin{pmatrix} & \nearrow^{a_1} \searrow & \\ b_1 & & b_1 \\ & \searrow_{c_1} \nearrow & \end{pmatrix} = \frac{1}{\mu(b_1)} e^{i\theta},$$

for any given $\theta \in [0, 2\pi)$. Hence the absolute value of this cell weight is $\frac{1}{\mu(b_1)}$. Next we shall see the

absolute value of the weight $W \begin{pmatrix} & \nearrow^{a_3} \searrow & \\ b_1 & & b_1 \\ & \searrow_{c_2} \nearrow & \end{pmatrix}$. There are two paths from a_3 to c_2. One way is via the

vertex b_1 and the other path is via the vertex b_2. By the biunitary axiom, we have the 2×2 matrix made

from the wights $W \begin{pmatrix} & \nearrow^{a_3} \searrow & \\ * & & * \\ & \searrow_{c_2} \nearrow & \end{pmatrix}$ such that

(3.1)

$$\begin{bmatrix} W \begin{pmatrix} & \swarrow^{a_3} \searrow & \\ b_1 & & b_1 \\ & \searrow_{c_2} \swarrow & \end{pmatrix} & W \begin{pmatrix} & \swarrow^{a_3} \searrow & \\ b_1 & & b_2 \\ & \searrow_{c_2} \swarrow & \end{pmatrix} \\ W \begin{pmatrix} & \swarrow^{a_3} \searrow & \\ b_2 & & b_1 \\ & \searrow_{c_2} \swarrow & \end{pmatrix} & W \begin{pmatrix} & \swarrow^{a_3} \searrow & \\ b_2 & & b_2 \\ & \searrow_{c_2} \swarrow & \end{pmatrix} \end{bmatrix}.$$

Since the weight $W \begin{pmatrix} & \nearrow^{a_3} \searrow & \\ b_2 & & b_1 \\ & \searrow_{c_2} \nearrow & \end{pmatrix}$ forms a 1×1 unitary, the absolute value of the weight

$W \begin{pmatrix} & \swarrow^{a_3} \searrow & \\ b_1 & & b_2 \\ & \searrow_{c_2} \swarrow & \end{pmatrix}$ can be determined by the remormalization rule that

$$\left| W \begin{pmatrix} & \swarrow^{a_3} \searrow & \\ b_1 & & b_2 \\ & \searrow_{c_2} \swarrow & \end{pmatrix} \right| = \sqrt{\frac{\mu(b_1)\mu(b_2)}{\mu(a_3)\mu(c_2)}} \left| W \begin{pmatrix} & \nearrow^{a_3} \searrow & \\ b_1 & & b_2 \\ & \searrow_{c_2} \nearrow & \end{pmatrix} \right| = \sqrt{\frac{\mu(b_1)\mu(b_2)}{\mu(a_3)\mu(c_2)}}.$$

Then the unitarity of the 2×2 matrix (3.1) implies that

$$\left| W \begin{pmatrix} & \swarrow^{a_3} \searrow & \\ b_1 & & b_1 \\ & \searrow_{c_2} \swarrow & \end{pmatrix} \right| = \sqrt{1 - \frac{\mu(b_1)\mu(b_2)}{\mu(a_3)\mu(c_2)}}$$

and the renormalization rule leads us that

$$\left| W \begin{pmatrix} & \nearrow^{a_3} \searrow & \\ b_1 & & b_1 \\ & \searrow_{c_2} \nearrow & \end{pmatrix} \right| = \frac{\sqrt{\mu(a_3)\mu(c_2)}}{\mu(b_1)} \sqrt{1 - \frac{\mu(b_1)\mu(b_2)}{\mu(a_3)\mu(c_2)}}.$$

Replace c_2 and b_2 by c_3 and b_3 respectively, and we obtain that

$$\left| W\left(\begin{array}{c} a_3 \\ b_1 \quad\quad b_1 \\ c_3 \end{array} \right) \right| = \frac{\sqrt{\mu(a_3)\mu(c_3)}}{\mu(b_1)}\sqrt{1 - \frac{\mu(b_1)\mu(b_3)}{\mu(a_3)\mu(c_3)}}$$

by the same argument.

From these observatons, we can say that what we would like to have is a 3×3 unitary matrix $U = (u_{ij})$ satisfing the following norm conditions such that

$$(3.2) \qquad (|u_{ij}|) = \begin{bmatrix} \frac{\sqrt{\mu(a_3)\mu(c_3)}}{\mu(b_1)}\sqrt{1 - \frac{\mu(b_1)\mu(b_3)}{\mu(a_3)\mu(c_3)}} & \frac{\sqrt{\mu(a_2)\mu(c_3)}}{\mu(b_1)} & \frac{\sqrt{\mu(a_1)\mu(c_3)}}{\mu(b_1)} \\[2ex] \frac{\sqrt{\mu(a_3)\mu(c_2)}}{\mu(b_1)}\sqrt{1 - \frac{\mu(b_1)\mu(b_2)}{\mu(a_3)\mu(c_2)}} & \frac{\sqrt{\mu(a_2)\mu(c_2)}}{\mu(b_1)} & \frac{\sqrt{\mu(a_1)\mu(c_2)}}{\mu(b_1)} \\[2ex] \frac{\sqrt{\mu(a_3)\mu(c_1)}}{\mu(b_1)} & \frac{\sqrt{\mu(a_2)\mu(c_1)}}{\mu(b_1)} & \frac{1}{\mu(b_1)} \end{bmatrix}$$

which will ensure the unitarity of the matrix constituted from the weights $W\left(\begin{array}{c} * \\ b_1 \quad\quad b_1 \\ * \end{array} \right)$ of the cells

at the vertex b_1 in Fig. 2.5.

For this purpose, we first introduce the lemma which is very useful to assert the existance of a unitary matrix satisfying the some norm conditions. And it is found in [Sc, Lemma 1.2.11] and also the same arguments can be found in [Oc2, Section IV.2].

Lemma 3.1. ([Sc])   *Let the 3×3 matrix*

$$D = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}$$

be doubly stochastic and we put

$$\alpha = d_{11}d_{21}, \quad \beta = d_{12}d_{22}, \quad \gamma = d_{13}d_{23}.$$

Then there exists a unitary matrix $U = (u_{ij})$ with $|u_{ij}|^2 = d_{ij}$ $(i,j = 1,2,3,)$ if and only if $\sqrt{\alpha}, \sqrt{\beta}$ and $\sqrt{\gamma}$ satisfy the triangle inequality. The last condition is equivalent to

$$\alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta - 2\beta\gamma - 2\gamma\alpha \leq 0.$$

Remark 3.2.   If such a unitry exists, we can take it to be of the form

$$\begin{pmatrix} \sqrt{d_{11}} & \sqrt{d_{12}} & \sqrt{d_{13}} \\ \sqrt{d_{21}} & \sqrt{d_{22}}e^{i\theta_1} & \sqrt{d_{23}}e^{i\theta_2} \\ \sqrt{d_{31}} & \sqrt{d_{32}}e^{i\theta_3} & \sqrt{d_{33}}e^{i\theta_4} \end{pmatrix}$$

for some θ_j $(j = 1,2,3,4)$.

Applying the above Lemma, let us show the existence of the unitary matrix U with the norm conditions (3.2).

Proposition 3.3. *Let* $\mu(\cdot)$ *'s be the Perron-Frobenius weights listed in the previous section. Then the* 3×3 *matrix of the form*

$$D = \begin{bmatrix} \dfrac{\mu(a_3)\mu(c_3)}{\mu(b_1)^2}\left(1 - \dfrac{\mu(b_1)\mu(b_3)}{\mu(a_3)\mu(c_3)}\right) & \dfrac{\mu(a_2)\mu(c_3)}{\mu(b_1)^2} & \dfrac{\mu(a_1)\mu(c_3)}{\mu(b_1)^2} \\[4mm] \dfrac{\mu(a_3)\mu(c_2)}{\mu(b_1)^2}\left(1 - \dfrac{\mu(b_1)\mu(b_2)}{\mu(a_3)\mu(c_2)}\right) & \dfrac{\mu(a_2)\mu(c_2)}{\mu(b_1)^2} & \dfrac{\mu(a_1)\mu(c_2)}{\mu(b_1)^2} \\[4mm] \dfrac{\mu(a_3)\mu(c_1)}{\mu(b_1)^2} & \dfrac{\mu(a_2)\mu(c_1)}{\mu(b_1)^2} & \dfrac{1}{\mu(b_1)^2} \end{bmatrix}$$

is doubly stochastic and there exists a unitary $U = (u_{ij})$ *such that* $|u_{ij}|^2 = d_{ij}$, *where* d_{ij} *denotes the* $(i,j)-th$ *entry of the matrix* D.

Proof. First we shall see the doubly stochastic property of the matrix D. From the definition of the Perron-Frobenius eigenvector, we obtain the relations $\lambda\mu(b_1) = \mu(a_1) + \mu(a_2) + \mu(a_3)$ and $\lambda\mu(c_3) = \mu(b_1) + \mu(b_3)$. Using these relations, we have that

$$\frac{\mu(a_3)\mu(c_3)}{\mu(b_1)^2}\left(1 - \frac{\mu(b_1)\mu(b_3)}{\mu(a_3)\mu(c_3)}\right) + \frac{\mu(a_2)\mu(c_3)}{\mu(b_1)^2} + \frac{\mu(a_1)\mu(c_3)}{\mu(b_1)^2}$$

$$= \frac{1}{\mu(b_1)^2}\{(\mu(a_3) + \mu(a_2) + \mu(a_1))\mu(c_3) - \mu(b_1)\mu(b_3)\}$$

$$= \frac{1}{\mu(b_1)}(\lambda\mu(c_3) - \mu(b_3)) = 1,$$

as the sum of the entries of the first row. It is checked in the same way that the sum of the entries of the second row equals to 1. And on the third row, the sum of the entries would be

$$(3.3) \qquad \frac{\mu(a_3)\mu(c_1)}{\mu(b_1)^2} + \frac{\mu(a_2)\mu(c_1)}{\mu(b_1)^2} + \frac{1}{\mu(b_1)^2}$$

$$= \frac{1}{\mu(b_1)^2}(\mu(a_3)\mu(c_1) + \mu(a_2)\mu(c_1) + 1)$$

$$= \frac{1}{\mu(b_1)^2}\{(\mu(a_3) + \mu(a_2) + \mu(a_1))\mu(c_1) - \mu(a_1)\mu(c_1) + 1\}$$

$$= \frac{1}{\mu(b_1)^2}(\lambda\mu(b_1)\mu(c_1) - \mu(a_1)\mu(c_1) + 1).$$

By substituting the weights listed in the previous section, the rightest hand of (3.3) equals to

$$(3.4) \qquad \frac{1}{R_{n+1}^2}(\lambda R_{n+1}R_n - R_n^2 + 1).$$

From now on, we abbreviate $R_k(\lambda)$ as R_k for simplicity provided that there is no confusion. On the other hand, the recursive relations for the series of polynomials $\{R_n(x)\}$ and the definition of the eigenvalue imply the relations

$$(3.5) \qquad 1 = R_n^2 - R_{n-1}R_{n+1} \quad (n = 1, 2, \cdots).$$

So the expression (3.4) would be

$$\frac{1}{R_{n+1}^2}(\lambda R_{n+1}R_n - R_{n-1}R_{n+1}) = \frac{1}{R_{n+1}^2}\{R_{n+1}(\lambda R_n - R_{n-1})\} = 1.$$

It can be checked by similar way which we have done on the rows, that the sum of the entries of each column equals to 1.

Next we shall see the existance of a unitary matrix $U = (u_{ij})$ satisfying the conditions $|u_{ij}|^2 = d_{ij}$. To this end, we would apply Lemma 3.1, so we put

$$\begin{cases} \alpha = \dfrac{\mu(a_3)\mu(c_3)}{\mu(b_1)^2}\left(1 - \dfrac{\mu(b_1)\mu(b_3)}{\mu(a_3)\mu(c_3)}\right)\dfrac{\mu(a_3)\mu(c_2)}{\mu(b_1)^2}\left(1 - \dfrac{\mu(b_1)\mu(b_2)}{\mu(a_3)\mu(c_2)}\right) \\[2mm] \quad = \dfrac{1}{\mu(b_1)^4}\{\mu(a_3)\mu(c_3) - \mu(b_1)\mu(b_3)\}\{\mu(a_3)\mu(c_2) - \mu(b_1)\mu(b_2)\}, \\[3mm] \beta = \dfrac{\mu(a_2)\mu(c_3)}{\mu(b_1)^2}\dfrac{\mu(a_2)\mu(c_2)}{\mu(b_1)^2} = \dfrac{1}{\mu(b_1)^4}\mu(a_2)^2\mu(c_2)\mu(c_3), \\[3mm] \gamma = \dfrac{\mu(a_1)\mu(c_3)}{\mu(b_1)^2}\dfrac{\mu(a_1)\mu(c_2)}{\mu(b_1)^2} = \dfrac{1}{\mu(b_1)^4}\mu(a_1)^2\mu(c_2)\mu(c_3). \end{cases}$$

Then all that we have to do is to check the inequality

$$(3.6) \qquad \alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta - 2\beta\gamma - 2\gamma\alpha \le 0.$$

We can cancel the common factor $\dfrac{1}{\mu(b_1)^4} > 0$ in order to check the inequality (3.6). By substituting the Perron-Frobenius weights in the previous section, and we have

$$\begin{cases} \alpha = \dfrac{1}{4\lambda^4}\{R_{n+3}(R_{n+3} + R_{n+1}) - \lambda^2 R_{n+1}(R_{n+3} - R_{n+1})\}^2, \\[3mm] \beta = \dfrac{1}{4\lambda^4}R_{n+1}{}^2(R_{n+3} + R_{n+1})^2, \\[3mm] \gamma = \dfrac{1}{4\lambda^2}R_n{}^2(R_{n+3} + R_{n+1})^2. \end{cases}$$

By multiplying $4\lambda^4 > 0$, we may assume that

$$\begin{cases} \alpha = \{R_{n+3}(R_{n+3} + R_{n+1}) - \lambda^2 R_{n+1}(R_{n+3} - R_{n+1})\}^2, \\[2mm] \beta = R_{n+1}^2(R_{n+3} + R_{n+1})^2, \\[2mm] \gamma = \lambda^2 R_n^2(R_{n+3} + R_{n+1})^2 \end{cases}$$

in order to show the inequality (3.6). And by direct computations, we obtain that

$$(3.7) \qquad \alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta - 2\beta\gamma - 2\gamma\alpha$$
$$= \{\lambda R_n(R_{n+3} + R_{n+1}) - \lambda^2 R_{n+1}(R_{n+3} - R_{n+1}) + (R_{n+3} + R_{n+1})^2\}\times$$
$$\{\lambda R_n(R_{n+3} + R_{n+1}) + \lambda^2 R_{n+1}(R_{n+3} - R_{n+1}) - (R_{n+3} + R_{n+1})^2\}\times$$
$$\{\lambda R_n(R_{n+3} + R_{n+1}) - \lambda^2 R_{n+1}(R_{n+3} - R_{n+1}) + (R_{n+3}{}^2 - R_{n+1}{}^2)\}\times$$
$$\{\lambda R_n(R_{n+3} + R_{n+1}) + \lambda^2 R_{n+1}(R_{n+3} - R_{n+1}) - (R_{n+3}{}^2 - R_{n+1}{}^2)\}.$$

The equalities (3.5), $R_n^2 - R_{n-1}R_{n+1} = 1$ $(n = 1, 2, \cdots)$, lead that

$$\begin{cases} \lambda R_n(R_{n+3} + R_{n+1}) = \lambda^2 R_n R_{n+2} = \lambda^2(R_{n+1}{}^2 - 1), \\[2mm] (R_{n+3} + R_{n+1})^2 = \lambda^2 R_{n+2}{}^2 = \lambda^2(1 + R_{n+1}R_{n+3}). \end{cases}$$

Using these relations, we can obtain the followings for each factor of (3.7). On the first factor, we have

(3.8)
$$\lambda R_n(R_{n+3} + R_{n+1}) - \lambda^2 R_{n+1}(R_{n+3} - R_{n+1}) + (R_{n+3} + R_{n+1})^2$$

$$= \lambda^2\{(R_{n+1}{}^2 - 1) - R_{n+1}R_{n+3} + R_{n+1}{}^2 + (1 + R_{n+3}R_{n+1})\}$$

$$= 2\lambda^2 R_{n+1}{}^2 > 0.$$

On the second factor,

(3.9)
$$\lambda R_n(R_{n+3} + R_{n+1}) + \lambda^2 R_{n+1}(R_{n+3} - R_{n+1}) - (R_{n+3} - R_{n+1})^2$$

$$= \lambda^2\{(R_{n+1}{}^2 - 1) + R_{n+1}R_{n+3} - R_{n+1}{}^2 - (1 + R_{n+3}R_{n+1})\}$$

$$= -2\lambda^2 < 0.$$

On the third factor,

(3.10)
$$\lambda R_n(R_{n+3} + R_{n+1}) - \lambda^2 R_{n+1}(R_{n+3} - R_{n+1}) + (R_{n+3}{}^2 - R_{n+1}{}^2)$$

$$= \text{(The 1st factor)} - 2R_{n+3}R_{n+1} - 2R_{n+1}{}^2$$

$$= 2\lambda^2 R_{n+1}{}^2 - 2R_{n+1}(R_{n+3} + R_{n+1})$$

$$= 2\lambda^2 R_{n+1}{}^2 - 2\lambda R_{n+1}R_{n+2}$$

$$= 2\lambda R_{n+1}(\lambda R_{n+1} - R_{n+2}) = 2\lambda R_{n+1}R_n > 0,$$

and on the fourth factor, we obtain that

(3.11)
$$\lambda R_n(R_{n+3} + R_{n+1}) + \lambda^2 R_{n+1}(R_{n+3} - R_{n+1}) - (R_{n+3}{}^2 - R_{n+1}{}^2)$$

$$= \text{(The 2nd factor)} + 2R_{n+3}R_{n+1} + 2R_{n+1}{}^2$$

$$= -2\lambda^2 + 2R_{n+1}(R_{n+3} + R_{n+1})$$

$$= -2\lambda^2 + 2\lambda R_{n+1}R_{n+2} = 2\lambda(R_{n+1}R_{n+2} - \lambda) > 0.$$

Combine the equality (3.7) and the inequalities (3.8) to (3.11), and we get the inequality (3.6) which completes the proof.

$$\square$$

By the similar argument that we have explained at the beginning of this section, it can be said that the next proposition will ensure the unitarity of the matrix constituted from the weights of the form

$$W\left(\begin{array}{c} \nearrow^* \searrow \\ b_2 \qquad b_2 \\ \searrow_* \nearrow \end{array}\right).$$

Proposition 3.4. *Let $\mu(\cdot)$'s be the Perron-Frobenius weights listed in the previous section. Then the 3×3 matrix of the form*

$$D = \begin{bmatrix} \dfrac{\mu(a_3)\mu(c_6)}{\mu(b_2)^2}\left(1 - \dfrac{\mu(b_2)\mu(b_3)}{\mu(a_3)\mu(c_6)}\right) & \dfrac{\mu(a_4)\mu(c_6)}{\mu(b_2)^2} & \dfrac{\mu(a_6)\mu(c_6)}{\mu(b_2)^2}\left(1 - \dfrac{\mu(b_2)\mu(b_4)}{\mu(a_6)\mu(c_6)}\right) \\[3mm] \dfrac{\mu(a_3)\mu(c_4)}{\mu(b_2)^2} & \dfrac{\mu(a_4)\mu(c_4)}{\mu(b_2)^2} & \dfrac{\mu(a_6)\mu(c_4)}{\mu(b_2)^2} \\[3mm] \dfrac{\mu(a_3)\mu(c_2)}{\mu(b_2)^2}\left(1 - \dfrac{\mu(b_1)\mu(b_2)}{\mu(a_3)\mu(c_2)}\right) & \dfrac{\mu(a_4)\mu(c_2)}{\mu(b_2)^2} & \dfrac{\mu(a_6)\mu(c_2)}{\mu(b_2)^2} \end{bmatrix}$$

is doubly stochastic and there exists a unitary $U = (u_{ij})$ such that $|u_{ij}|^2 = d_{ij}$, where d_{ij} denotes the $(i,j)-$ th entry of the matrix D.

Proof.   It is not so hard to check that the matrix D is doubly stochastic by using the relations obtained from the definition of the Perron-Frobenius eigenvector. So we shall concentrate our interesting upon the existance of a unitary matrix $U = (u_{ij})$ satisfying the conditions $|u_{ij}|^2 = d_{ij}$. We put

$$\begin{cases} \alpha = \dfrac{\mu(a_3)\mu(c_6)}{\mu(b_2)^2} \left(1 - \dfrac{\mu(b_2)\mu(b_3)}{\mu(a_3)\mu(c_6)}\right) \dfrac{\mu(a_3)\mu(c_4)}{\mu(b_2)^2} \\[2ex] \quad = \dfrac{\mu(c_4)}{\mu(b_2)^4} \left\{\mu(a_3)(\mu(a_3)\mu(c_6) - \mu(b_2)\mu(b_3))\right\}, \\[2ex] \beta = \dfrac{\mu(a_4)\mu(c_6)}{\mu(b_2)^2} \dfrac{\mu(a_4)\mu(c_4)}{\mu(b_2)^2} = \dfrac{\mu(c_4)}{\mu(b_2)^4} \mu(a_4)^2\mu(c_6), \\[2ex] \gamma = \dfrac{\mu(a_6)\mu(c_6)}{\mu(b_2)^2} \left(1 - \dfrac{\mu(b_2)\mu(c_4)}{\mu(a_6)\mu(c_6)}\right) \dfrac{\mu(a_6)\mu(b_4)}{\mu(b_2)^2} \\[2ex] \quad = \dfrac{\mu(c_4)}{\mu(b_2)^4} \left\{\mu(a_6)(\mu(a_6)\mu(c_6) - \mu(b_2)\mu(b_4))\right\}. \end{cases}$$

According to Lemma 3.1, all that we have to do is to show the inequality that

$$(3.12) \qquad\qquad \alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta - 2\beta\gamma - 2\gamma\alpha \le 0.$$

Since $\dfrac{\mu(c_4)}{\mu(b_2)^4} > 0$ is the common factor, we can assume that

$$\begin{cases} \alpha = \mu(a_3)\left(\mu(a_3)\mu(c_6) - \mu(b_2)\mu(b_3)\right), \\[1.5ex] \beta = \mu(a_4)^2\mu(c_6), \\[1.5ex] \gamma = \mu(a_6)\left(\mu(a_6)\mu(c_6) - \mu(b_2)\mu(b_4)\right) \end{cases}$$

in order to show the inequality (3.12).

If we substitute the Perron-Frobenius weights listed in the previous section into these α, β and γ then we have

$$(3.13) \qquad \begin{cases} \alpha \;= \dfrac{1}{4\lambda^3} R_{n+3} \left\{4\kappa R_{m+3}R_{n+3} - \lambda^2(R_{n+3} - R_{n+1})^2\right\} \\[2ex] \quad = \dfrac{\kappa}{4\lambda^3} R_{n+3} \left\{4R_{m+3}R_{n+3} - \lambda^2(R_{m+3} - R_{m+1})(R_{n+3} - R_{n+1})\right\}, \\[2ex] \beta \;= \dfrac{\kappa}{4\lambda^3} R_{m+3}(R_{n+3} - R_{n+1})^2, \\[2ex] \gamma \;= \dfrac{\kappa}{4\lambda^3} \left\{\lambda^2(R_{n+3} - R_{n+1}) - 3R_{n+3} + R_{n+1}\right\} \times \\[2ex] \qquad\qquad \left\{\lambda^2(R_{m+3} - R_{m+1})(R_{n+3} - R_{n+1}) - 3R_{m+3}R_{n+3} + R_{m+3}R_{n+1}\right\}. \end{cases}$$

We can cancel the common factor $\dfrac{\kappa}{4\lambda^3} > 0$ again in these α, β and γ. Using the equality,

$$\lambda^2 = \frac{4R_{m+3}R_{n+3} - 2R_{m+3}R_{n+1} - 2R_{m+1}R_{n+3}}{(R_{m+3} - R_{m+1})(R_{n+3} - R_{n+1})},$$

we can rewrite (3.13) as followings.

(3.14)
$$\begin{cases} \alpha & = R_{n+3}\left\{4R_{m+3}R_{n+3} - \lambda^2(R_{m+3} - R_{m+1})(R_{n+3} - R_{n+1})\right\} \\ & = 2R_{n+3}(R_{m+3}R_{n+1} + R_{m+1}R_{n+3}), \\ \beta & = R_{m+3}(R_{n+3} - R_{n+1})^2, \\ \gamma & = \left\{\lambda^2(R_{n+3} - R_{n+1}) - 3R_{n+3} + R_{n+1}\right\} \times \\ & \qquad \left\{\lambda^2(R_{m+3} - R_{m+1})(R_{n+3} - R_{n+1}) - 3R_{m+3}R_{n+3} + R_{m+3}R_{n+1}\right\} \\ & = \kappa(R_{m+3} + R_{m+1})(R_{m+3}R_{n+3} - R_{m+3}R_{n+1} - 2R_{m+1}R_{n+3}) \end{cases}$$

where $\kappa = \dfrac{R_{n+3}(\lambda) - R_{n+1}(\lambda)}{R_{m+3}(\lambda) - R_{m+1}(\lambda)}$.

We use α, β and γ in (3.14) on the subsquent calculation. By direct but just a little complicated computations, we obtain that

(3.15)
$$\begin{aligned} & \alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta - 2\beta\gamma - 2\gamma\alpha \\ &= \frac{4}{(R_{m+3} - R_{m+1})^2}(R_{m+3}R_{n+1} + R_{m+1}R_{n+3}) \times \\ & \qquad (2R_{m+3}R_{n+3} - R_{m+3}R_{n+1} - R_{m+1}R_{n+3}) \times \\ & \qquad (2R_{m+3}{}^2R_{n+1}R_{n+3} + 2R_{m+1}R_{m+3}R_{n+3}{}^2 - R_{m+3}{}^2R_{n+3}{}^2 \\ & \qquad - 2R_{m+3}R_{m+1}R_{n+3}R_{n+1} - R_{m+1}{}^2R_{n+1}{}^2) \\ &= 2\kappa(R_{m+3}R_{n+1} + R_{m+1}R_{n+3}) \times \\ & \qquad (2R_{m+3}R_{n+3} - R_{m+3}R_{n+1} - R_{m+1}R_{n+3}) \times \\ & \qquad \left\{(\lambda^2 - 4)(R_{m+3}R_{n+1} + R_{m+1}R_{n+3}) - 2(R_{m+3} - R_{m+1})(R_{n+3} - R_{n+1})\right\}. \end{aligned}$$

Here we have applied equality (2.7) for the last equality in the above expression. Since we know that

$$R_{k+3} > R_{k+1} > 0 \quad \text{for any } k \geq 0,$$

it is obvious from Proposition 2.5 that

(3.16)
$$\begin{aligned} & 2R_{m+3}R_{n+3} - R_{m+3}R_{n+1} - R_{m+1}R_{n+3} \\ &= (R_{m+3} - R_{m+1})R_{n+3} + (R_{n+3} - R_{n+1})R_{m+3} > 0. \end{aligned}$$

From the equalities (3.15) and (3.16), it is enough to show the inequality that

(3.17)
$$\begin{aligned} & (\lambda^2 - 4)(R_{m+3}R_{n+1} + R_{m+1}R_{n+3}) - 2(R_{m+3} - R_{m+1})(R_{n+3} - R_{n+1}) \\ &= \lambda^2(R_{m+3}R_{n+1} + R_{m+1}R_{n+3}) - 2(R_{m+3} + R_{m+1})(R_{n+3} + R_{n+1}) \leq 0, \end{aligned}$$

for the inequality (3.12). The above inequality (3.17) is eqvivalent to

(3.18)
$$\frac{(R_{m+3} + R_{m+1})(R_{n+3} + R_{n+1})}{(R_{m+3}R_{n+1} + R_{m+1}R_{n+3})} \geq \frac{\lambda^2}{2}.$$

If we denote $A_{m,n}$ the left hand of the inequality (3.18) then we have

$$A_{m,n} = \frac{(1 + x_m)(1 + x_n)}{x_m + x_n},$$

where we set that

$$x_k = \frac{R_{k+1}}{R_{k+3}} < 1.$$

Since x_k is increasing in k as we said in Proposition 2.5, it follows that $A_{m,n}$ is decreasing both in m and n. Indeed, for fixed n, we can rewrite

$$A_{m,n} = (1 + x_n) + (1 - x_n{}^2)\frac{1}{x_m + x_n}.$$

Since $1 - x_n{}^2 > 0$ and $\{x_m\}$ is positive increasing sequence, we can conclude that $A_{m,n}$ is decreasing in m. Similary $A_{m,n}$ is also decreasing in n. Moreover from Proposition 2.5, we have $\lim\limits_{k\to\infty} x_k = e^{-2t}$ where $\lambda = e^t + e^{-t}$. Therefore we have

$$A_{m,n} \geq \lim_{n,m\to\infty} A_{m,n} = \frac{(1 + e^{-2t})^2}{2e^{-2t}} = \frac{(e^t + e^{-t})^2}{2} = \frac{\lambda^2}{2}.$$

Now the inequality (3.18) has been proved and it makes the proof be complete.

$\square$

Proposition 3.5. *Let $\mu(\cdot)$'s be the Perron-Frobenius weights listed in the previous section. Then the 3×3 matrix of the form*

$$D = \begin{bmatrix} \dfrac{\mu(a_3)\mu(c_6)}{\mu(b_3)^2}\left(1 - \dfrac{\mu(b_2)\mu(b_3)}{\mu(a_3)\mu(c_6)}\right) & \dfrac{\mu(a_5)\mu(c_6)}{\mu(b_3)^2} & \dfrac{\mu(a_7)\mu(c_6)}{\mu(b_3)^2}\left(1 - \dfrac{\mu(b_3)\mu(b_4)}{\mu(a_7)\mu(c_6)}\right) \\[3mm] \dfrac{\mu(a_3)\mu(c_5)}{\mu(b_3)^2} & \dfrac{\mu(a_5)\mu(c_5)}{\mu(b_3)^2} & \dfrac{\mu(a_7)\mu(c_5)}{\mu(b_3)^2} \\[3mm] \dfrac{\mu(a_3)\mu(c_3)}{\mu(b_3)^2}\left(1 - \dfrac{\mu(b_1)\mu(b_3)}{\mu(a_3)\mu(c_3)}\right) & \dfrac{\mu(a_5)\mu(c_3)}{\mu(b_3)^2} & \dfrac{\mu(a_7)\mu(c_3)}{\mu(b_3)^2} \end{bmatrix}$$

is doubly stochastic and there exists a unitary $U = (u_{ij})$ such that $|u_{ij}|^2 = d_{ij}$, where d_{ij} denotes the $(i, j)-$ th entry of the matrix D.

From the table of the Perron-Frobenius weights, it follows that $\mu(a_4) = \mu(a_5)$, $\mu(a_6) = \mu(a_7)$, $\mu(b_2) = \mu(b_3)$, $\mu(c_3) = \mu(c_2)$ and $\mu(c_4) = \mu(c_5)$. So the matrix D in this proposition is the same one in Proposition 3.4, so we do not need a proof.

Switching m and n, the next proposition can be proved by the similar way of Proposition 3.3. Thus we can skip the details of the proof.

Proposition 3.6. *Let $\mu(\cdot)$'s be the Perron-Frobenius weights listed in the previous section. Then the 3×3 matrix of the form*

$$D = \begin{bmatrix} \dfrac{\mu(c_6)\mu(a_6)}{\mu(b_4)^2}\left(1 - \dfrac{\mu(b_4)\mu(b_2)}{\mu(c_6)\mu(a_6)}\right) & \dfrac{\mu(c_7)\mu(a_6)}{\mu(b_4)^2} & \dfrac{\mu(c_8)\mu(a_6)}{\mu(b_4)^2} \\[3mm] \dfrac{\mu(c_6)\mu(a_7)}{\mu(b_4)^2}\left(1 - \dfrac{\mu(b_4)\mu(b_3)}{\mu(c_6)\mu(a_7)}\right) & \dfrac{\mu(c_7)\mu(a_7)}{\mu(b_4)^2} & \dfrac{\mu(c_8)\mu(a_7)}{\mu(b_4)^2} \\[3mm] \dfrac{\mu(c_6)\mu(a_8)}{\mu(b_4)^2} & \dfrac{\mu(c_7)\mu(a_8)}{\mu(b_4)^2} & \dfrac{1}{\mu(b_4)^2} \end{bmatrix}$$

is doubly stochastic and there exists a unitary $U = (u_{ij})$ such that $|u_{ij}|^2 = d_{ij}$, where d_{ij} denotes the (i,j)−th entry of the matrix D.

Next we shall determine each the cell weight satisfying the biunitary condition. From Propositions 3.3 to 3.6 and Remark 3.2, we have the following four unitary matrices by taking suitable gauges $e^{i\theta_k}$.

$$U_1 = \begin{bmatrix} \sqrt{\frac{\mu(a_3)\mu(c_3)}{\mu(b_1)^2}}\left(1 - \frac{\mu(b_1)\mu(b_3)}{\mu(a_3)\mu(c_3)}\right) & \sqrt{\frac{\mu(a_2)\mu(c_3)}{\mu(b_1)^2}} & \sqrt{\frac{\mu(a_1)\mu(c_3)}{\mu(b_1)^2}} \\ \sqrt{\frac{\mu(a_3)\mu(c_2)}{\mu(b_1)^2}}\left(1 - \frac{\mu(b_1)\mu(b_2)}{\mu(a_3)\mu(c_2)}\right) & \sqrt{\frac{\mu(a_2)\mu(c_2)}{\mu(b_1)^2}}e^{i\theta_1} & \sqrt{\frac{\mu(a_1)\mu(c_2)}{\mu(b_1)^2}}e^{i\theta_2} \\ \sqrt{\frac{\mu(a_3)\mu(c_1)}{\mu(b_1)^2}} & \sqrt{\frac{\mu(a_2)\mu(c_1)}{\mu(b_1)^2}}e^{i\theta_3} & \frac{1}{\mu(b_1)}e^{i\theta_4} \end{bmatrix}$$

$$U_2 = \begin{bmatrix} \sqrt{\frac{\mu(a_3)\mu(c_6)}{\mu(b_2)^2}}\left(1 - \frac{\mu(b_2)\mu(b_3)}{\mu(a_3)\mu(c_6)}\right) & \sqrt{\frac{\mu(a_4)\mu(c_6)}{\mu(b_2)^2}} & \sqrt{\frac{\mu(a_6)\mu(c_6)}{\mu(b_2)^2}}\left(1 - \frac{\mu(b_2)\mu(b_4)}{\mu(a_6)\mu(c_6)}\right) \\ \sqrt{\frac{\mu(a_3)\mu(c_4)}{\mu(b_2)^2}} & \sqrt{\frac{\mu(a_4)\mu(c_4)}{\mu(b_2)^2}}e^{i\theta_5} & \sqrt{\frac{\mu(a_6)\mu(c_4)}{\mu(b_2)^2}}e^{i\theta_6} \\ \sqrt{\frac{\mu(a_3)\mu(c_2)}{\mu(b_2)^2}}\left(1 - \frac{\mu(b_1)\mu(b_2)}{\mu(a_3)\mu(c_2)}\right) & \sqrt{\frac{\mu(a_4)\mu(c_2)}{\mu(b_2)^2}}e^{i\theta_7} & \sqrt{\frac{\mu(a_6)\mu(c_2)}{\mu(b_2)^2}}e^{i\theta_8} \end{bmatrix}$$

$$U_3 = \begin{bmatrix} \sqrt{\frac{\mu(a_3)\mu(c_6)}{\mu(b_3)^2}}\left(1 - \frac{\mu(b_2)\mu(b_3)}{\mu(a_3)\mu(c_6)}\right) & \sqrt{\frac{\mu(a_5)\mu(c_6)}{\mu(b_3)^2}} & \sqrt{\frac{\mu(a_7)\mu(c_6)}{\mu(b_3)^2}}\left(1 - \frac{\mu(b_3)\mu(b_4)}{\mu(a_7)\mu(c_6)}\right) \\ \sqrt{\frac{\mu(a_3)\mu(c_5)}{\mu(b_3)^2}} & \sqrt{\frac{\mu(a_5)\mu(c_5)}{\mu(b_3)^2}}e^{i\theta_9} & \sqrt{\frac{\mu(a_7)\mu(c_5)}{\mu(b_3)^2}}e^{i\theta_{10}} \\ \sqrt{\frac{\mu(a_3)\mu(c_3)}{\mu(b_3)^2}}\left(1 - \frac{\mu(b_1)\mu(b_3)}{\mu(a_3)\mu(c_3)}\right) & \sqrt{\frac{\mu(a_5)\mu(c_3)}{\mu(b_3)^2}}e^{i\theta_{11}} & \sqrt{\frac{\mu(a_7)\mu(c_3)}{\mu(b_3)^2}}e^{i\theta_{12}} \end{bmatrix}$$

$$U_4 = \begin{bmatrix} \sqrt{\frac{\mu(c_6)\mu(a_6)}{\mu(b_4)^2}}\left(1 - \frac{\mu(b_4)\mu(b_2)}{\mu(c_6)\mu(a_6)}\right) & \sqrt{\frac{\mu(c_7)\mu(a_6)}{\mu(b_4)^2}} & \sqrt{\frac{\mu(c_8)\mu(a_6)}{\mu(b_4)^2}} \\ \sqrt{\frac{\mu(c_6)\mu(a_7)}{\mu(b_4)^2}}\left(1 - \frac{\mu(b_4)\mu(b_3)}{\mu(c_6)\mu(a_7)}\right) & \sqrt{\frac{\mu(c_7)\mu(a_7)}{\mu(b_4)^2}}e^{i\theta_{13}} & \sqrt{\frac{\mu(c_8)\mu(a_7)}{\mu(b_4)^2}}e^{i\theta_{14}} \\ \sqrt{\frac{\mu(c_6)\mu(a_8)}{\mu(b_4)^2}} & \sqrt{\frac{\mu(c_7)\mu(a_8)}{\mu(b_4)^2}}e^{i\theta_{15}} & \frac{1}{\mu(b_4)}e^{i\theta_{16}} \end{bmatrix}$$

Remainding the above unitaries U_1 to U_4, we put the weights of the admissible cells of the form

$$W\begin{pmatrix} & a_i & \\ b_k & & b_l \\ & c_j & \end{pmatrix} \text{ as follows :}$$

$$W\left(\begin{smallmatrix} & a_1 & \\ b_1 & & b_1 \\ & c_1 & \end{smallmatrix}\right) = \frac{1}{\sqrt{\mu(a_1)\mu(c_1)}}e^{i\theta_4}, \qquad W\left(\begin{smallmatrix} & a_1 & \\ b_1 & & b_1 \\ & c_2 & \end{smallmatrix}\right) = e^{i\theta_2},$$

$$W\left(\begin{smallmatrix} & a_1 & \\ b_1 & & b_1 \\ & c_3 & \end{smallmatrix}\right) = 1, \qquad W\left(\begin{smallmatrix} & a_2 & \\ b_1 & & b_1 \\ & c_1 & \end{smallmatrix}\right) = e^{i\theta_3},$$

$$W\left(\begin{smallmatrix} & a_2 & \\ b_1 & & b_1 \\ & c_2 & \end{smallmatrix}\right) = e^{i\theta_1}, \qquad W\left(\begin{smallmatrix} & a_2 & \\ b_1 & & b_1 \\ & c_3 & \end{smallmatrix}\right) = 1,$$

$$W\left(\begin{smallmatrix} & a_3 & \\ b_1 & & b_1 \\ & c_1 & \end{smallmatrix}\right) = 1, \qquad W\left(\begin{smallmatrix} & a_3 & \\ b_1 & & b_1 \\ & c_2 & \end{smallmatrix}\right) = \sqrt{1-\frac{\mu(b_1)\mu(b_2)}{\mu(a_3)\mu(c_2)}},$$

$$W\left(\begin{smallmatrix} & a_3 & \\ b_1 & & b_2 \\ & c_2 & \end{smallmatrix}\right) = i\sqrt{\frac{\mu(b_1)\mu(b_2)}{\mu(a_3)\mu(c_2)}}, \qquad W\left(\begin{smallmatrix} & a_3 & \\ b_2 & & b_1 \\ & c_2 & \end{smallmatrix}\right) = i\sqrt{\frac{\mu(b_1)\mu(b_2)}{\mu(a_3)\mu(c_2)}},$$

$$W\left(\begin{smallmatrix} & a_3 & \\ b_2 & & b_2 \\ & c_2 & \end{smallmatrix}\right) = \sqrt{1-\frac{\mu(b_1)\mu(b_2)}{\mu(a_3)\mu(c_2)}}, \qquad W\left(\begin{smallmatrix} & a_3 & \\ b_1 & & b_1 \\ & c_3 & \end{smallmatrix}\right) = \sqrt{1-\frac{\mu(b_1)\mu(b_3)}{\mu(a_3)\mu(c_3)}},$$

$$W\left(\begin{smallmatrix} & a_3 & \\ b_1 & & b_3 \\ & c_3 & \end{smallmatrix}\right) = i\sqrt{\frac{\mu(b_1)\mu(b_3)}{\mu(a_3)\mu(c_3)}}, \qquad W\left(\begin{smallmatrix} & a_3 & \\ b_3 & & b_1 \\ & c_3 & \end{smallmatrix}\right) = i\sqrt{\frac{\mu(b_1)\mu(b_3)}{\mu(a_3)\mu(c_3)}},$$

$$W\left(\begin{smallmatrix} & a_3 & \\ b_3 & & b_3 \\ & c_3 & \end{smallmatrix}\right) = \sqrt{1-\frac{\mu(b_1)\mu(b_3)}{\mu(a_3)\mu(c_3)}}, \qquad W\left(\begin{smallmatrix} & a_3 & \\ b_2 & & b_2 \\ & c_4 & \end{smallmatrix}\right) = 1,$$

$$W\left(\begin{smallmatrix} & a_3 & \\ b_3 & & b_3 \\ & c_5 & \end{smallmatrix}\right) = 1, \qquad W\left(\begin{smallmatrix} & a_3 & \\ b_2 & & b_2 \\ & c_6 & \end{smallmatrix}\right) = \sqrt{1-\frac{\mu(b_2)\mu(b_3)}{\mu(a_3)\mu(c_6)}},$$

$$W\left(\begin{smallmatrix} & a_3 & \\ b_2 & & b_3 \\ & c_6 & \end{smallmatrix}\right) = i\sqrt{\frac{\mu(b_2)\mu(b_3)}{\mu(a_3)\mu(c_6)}}, \qquad W\left(\begin{smallmatrix} & a_3 & \\ b_3 & & b_2 \\ & c_6 & \end{smallmatrix}\right) = i\sqrt{\frac{\mu(b_2)\mu(b_3)}{\mu(a_3)\mu(c_6)}},$$

$$W\left(\begin{smallmatrix} & a_3 & \\ b_3 & & b_3 \\ & c_6 & \end{smallmatrix}\right) = \sqrt{1-\frac{\mu(b_2)\mu(b_3)}{\mu(a_3)\mu(c_6)}}, \qquad W\left(\begin{smallmatrix} & a_4 & \\ b_2 & & b_2 \\ & c_2 & \end{smallmatrix}\right) = e^{i\theta_3},$$

$$W\left(\begin{smallmatrix} & a_4 & \\ b_2 & & b_2 \\ & c_4 & \end{smallmatrix}\right) = e^{i\theta_5}, \qquad W\left(\begin{smallmatrix} & a_4 & \\ b_2 & & b_2 \\ & c_6 & \end{smallmatrix}\right) = 1,$$

$$W\left(\begin{smallmatrix} & a_5 & \\ b_3 & & b_3 \\ & c_3 & \end{smallmatrix}\right) = e^{i\theta_{11}}, \qquad W\left(\begin{smallmatrix} & a_5 & \\ b_3 & & b_3 \\ & c_5 & \end{smallmatrix}\right) = e^{i\theta_{9}},$$

$$W\left(\begin{smallmatrix} & a_5 & \\ b_3 & & b_3 \\ & c_6 & \end{smallmatrix}\right) = 1, \qquad W\left(\begin{smallmatrix} & a_6 & \\ b_2 & & b_2 \\ & c_2 & \end{smallmatrix}\right) = e^{i\theta_{8}},$$

$$W\left(\begin{smallmatrix} & a_6 & \\ b_2 & & b_2 \\ & c_4 & \end{smallmatrix}\right) = e^{i\theta_{6}}, \qquad W\left(\begin{smallmatrix} & a_6 & \\ b_2 & & b_2 \\ & c_6 & \end{smallmatrix}\right) = \sqrt{1 - \frac{\mu(b_2)\mu(b_4)}{\mu(a_6)\mu(c_6)}},$$

$$W\left(\begin{smallmatrix} & a_6 & \\ b_2 & & b_4 \\ & c_6 & \end{smallmatrix}\right) = i\sqrt{\frac{\mu(b_2)\mu(b_4)}{\mu(a_6)\mu(c_6)}}, \qquad W\left(\begin{smallmatrix} & a_6 & \\ b_4 & & b_2 \\ & c_6 & \end{smallmatrix}\right) = i\sqrt{\frac{\mu(b_2)\mu(b_4)}{\mu(a_6)\mu(c_6)}},$$

$$W\left(\begin{smallmatrix} & a_6 & \\ b_4 & & b_4 \\ & c_6 & \end{smallmatrix}\right) = \sqrt{1 - \frac{\mu(b_2)\mu(b_4)}{\mu(a_6)\mu(c_6)}}, \qquad W\left(\begin{smallmatrix} & a_6 & \\ b_2 & & b_2 \\ & c_7 & \end{smallmatrix}\right) = 1,$$

$$W\left(\begin{smallmatrix} & a_6 & \\ b_2 & & b_2 \\ & c_8 & \end{smallmatrix}\right) = 1, \qquad W\left(\begin{smallmatrix} & a_7 & \\ b_3 & & b_3 \\ & c_3 & \end{smallmatrix}\right) = e^{i\theta_{12}},$$

$$W\left(\begin{smallmatrix} & a_7 & \\ b_3 & & b_3 \\ & c_5 & \end{smallmatrix}\right) = e^{i\theta_{10}}, \qquad W\left(\begin{smallmatrix} & a_7 & \\ b_3 & & b_3 \\ & c_6 & \end{smallmatrix}\right) = \sqrt{1 - \frac{\mu(b_3)\mu(b_4)}{\mu(a_7)\mu(c_6)}},$$

$$W\left(\begin{smallmatrix} & a_7 & \\ b_3 & & b_4 \\ & c_6 & \end{smallmatrix}\right) = i\sqrt{\frac{\mu(b_3)\mu(b_4)}{\mu(a_7)\mu(c_6)}}, \qquad W\left(\begin{smallmatrix} & a_7 & \\ b_4 & & b_3 \\ & c_6 & \end{smallmatrix}\right) = i\sqrt{\frac{\mu(b_3)\mu(b_4)}{\mu(a_7)\mu(c_6)}},$$

$$W\left(\begin{smallmatrix} & a_7 & \\ b_4 & & b_4 \\ & c_6 & \end{smallmatrix}\right) = \sqrt{1 - \frac{\mu(b_3)\mu(b_4)}{\mu(a_7)\mu(c_6)}}, \qquad W\left(\begin{smallmatrix} & a_7 & \\ b_4 & & b_4 \\ & c_7 & \end{smallmatrix}\right) = e^{i\theta_{13}},$$

$$W\left(\begin{smallmatrix} & a_7 & \\ b_4 & & b_4 \\ & c_8 & \end{smallmatrix}\right) = e^{i\theta_{14}}, \qquad W\left(\begin{smallmatrix} & a_8 & \\ b_4 & & b_4 \\ & c_6 & \end{smallmatrix}\right) = 1,$$

$$W\left(\begin{smallmatrix} & a_8 & \\ b_4 & & b_4 \\ & c_7 & \end{smallmatrix}\right) = e^{i\theta_{15}}, \qquad W\left(\begin{smallmatrix} & a_8 & \\ b_4 & & b_4 \\ & c_8 & \end{smallmatrix}\right) = \frac{1}{\sqrt{\mu(a_8)\mu(c_8)}}e^{i\theta_{16}}.$$

Then the weights of the form $W\left(\begin{array}{c} a_3 \\ * \quad * \\ c_2 \end{array}\right)$ make the matrix

$$\begin{bmatrix} \sqrt{1 - \dfrac{\mu(b_1)\mu(b_2)}{\mu(a_3)\mu(c_2)}} & i\sqrt{\dfrac{\mu(b_1)\mu(b_2)}{\mu(a_3)\mu(c_2)}} \\[3mm] i\sqrt{\dfrac{\mu(b_1)\mu(b_2)}{\mu(a_3)\mu(c_2)}} & \sqrt{1 - \dfrac{\mu(b_1)\mu(b_2)}{\mu(a_3)\mu(c_2)}} \end{bmatrix}$$

and this matrix is obviously unitary. Similarly, it is clear that the 2×2 matrices constituted from the weights of the form $W\left(\begin{array}{c} a_3 \\ * \quad * \\ c_3 \end{array}\right)$, $W\left(\begin{array}{c} a_3 \\ * \quad * \\ c_6 \end{array}\right)$, $W\left(\begin{array}{c} a_6 \\ * \quad * \\ c_6 \end{array}\right)$ and $W\left(\begin{array}{c} a_7 \\ * \quad * \\ c_6 \end{array}\right)$

are all unitary. Of course, the weights of the from $W\left(\begin{array}{c} a_i \\ * \quad * \\ c_j \end{array}\right)$ remained in the above list, except

$W\left(\begin{array}{c} a_1 \\ b_1 \quad b_1 \\ c_1 \end{array}\right)$ and $W\left(\begin{array}{c} a_8 \\ b_4 \quad b_4 \\ c_8 \end{array}\right)$, might be 1×1 unitaries.

Next we see the other unitarity of the weights. Applying the renormalization rule, we can easyly check the unitarity of the matrix constituted from the weights of the admissible cells of the form $W\left(\begin{array}{c} * \\ b_k \quad b_l \\ * \end{array}\right)$. For instance, renormalization rule implies that

$$W\left(\begin{array}{c} a_2 \\ b_1 \quad b_1 \\ c_1 \end{array}\right) = \frac{\sqrt{\mu(a_2)\mu(c_1)}}{\mu(b_1)} \qquad W\left(\begin{array}{c} a_2 \\ b_1 \quad b_1 \\ c_1 \end{array}\right) = \frac{\sqrt{\mu(a_2)\mu(c_1)}}{\mu(b_1)} e^{i\theta_3}.$$

Apply the renormalization rule to other cells and then we have the unitary matrices U_k $(k = 1, 2, 3, 4)$

as the matrices of the weights $W\left(\begin{array}{c} * \\ b_k \quad b_k \\ * \end{array}\right)$ $(k = 1, 2, 3, 4)$, respectively. And all the weights of

the form $W\left(\begin{array}{c} * \\ b_k \quad b_l \\ * \end{array}\right)$ $(k \neq l)$ equal to imaginary unit i which are trivially 1×1 unitary matrices.

Here we can assert our theorem.

Theorem 3.7. *The pair of graphs $\mathcal{G}(n, m)$ and $\mathcal{H}(n, m)$ has a biunitary connection. And we have irreducible subfactors of the hyperfinite factor of type II_1 with the indices $||\mathcal{G}(n, m)||^2$ $(= ||\mathcal{H}(n, m)||^2)$.*

At the end of this section, we give the table of the values of the indices, the square values $||\mathcal{G}(n, m)||^2$ of the graph norms (see table 3.1).

Finally we would like to make a comment on the possibility of the pairs of cospectral graphs. As we said in section 2, we can obtain the pair of the cospectral graphs by taking any other graphs as the graphs G_i ($i = 1, 2, 3, 4$) in Fig. 2.1. And the auther believes that there are many possibilities for biunitary connections on such a pairs of the cospectral graphs. However we can see that if we put line (Dynkin A-type) graph as G_2 or G_4 then we can not construct any connections in the way that we have demonstrated in this paper.

The table of the indicies, $||\mathcal{G}(n,m)||^2$

m	0	1	2	3	4
n					
0	5.56155281				
1	5.60519839	5.64575131			
2	5.61803399	5.65757513	5.66907909		
3	5.62185035	5.66105201	5.67245146	5.67579031	
4	5.62299297	5.66208047	5.67344557	5.67677356	5.67775332
5	5.62333617	5.66238553	5.67373938	5.67706385	5.67804251
6	5.62343938	5.66247612	5.67382631	5.67714965	5.67812795
7	5.62347044	5.66250303	5.67385204	5.67717502	5.67815320
8	5.62347978	5.66251103	5.67385966	5.67718252	5.67816067
9	5.62348260	5.66251340	5.67386191	5.67718474	5.67816287
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
limit	5.62348381	5.66251441	5.67386286	5.67718567	5.67816380

m	5	6	7	8	9
n					
5	5.67833133				
6	5.67841666	5.67850195			
7	5.67844188	5.67852716	5.67855236		
8	5.67844933	5.67853461	5.67855981	5.67856726	
9	5.67845153	5.67853681	5.67856201	5.67856946	5.67857166
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
limit	5.67845246	5.67853773	5.67856294	5.67857038	5.67857259

Table 3.1

Remark 3.8. Using the eigenvalue equation in Lemma 2.6, we get the following facts.

(1) In the above table, for fixed m, the limit $\lim\limits_{n \to \infty} ||\mathcal{G}(n,m)||^2$ is given by

$$\left(\nu_m + \frac{1}{\nu_m}\right)^2,$$ where ν_m is the largest root of the polynomial equation

$$x^{2m+12} - 3x^{2m+10} - x^{2m+8} - x^{2m+6} + x^6 - x^4 + x^2 - 1 = 0.$$

(2) $\lim\limits_{n \to \infty} ||\mathcal{G}(n,m)||^2 < ||\mathcal{G}(m+1, m+1)||^2.$

(3) The smallest value is $||\mathcal{G}(0,0)||^2 = \dfrac{7 + \sqrt{17}}{2}.$

(4) $||\mathcal{G}(\infty, \infty)||^2 \sim 5.67857351$

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