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Note on the Schur multiplier of a certain semidirect product

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Let G, N and T be finite groups such that G is the semidirect product of N by T:

$$G \triangleright N, \quad G=NT, \quad N\cap T=\{1\}.$$

Let $\boldsymbol{Z}$ denote the additive group of the rational integers, and let $\boldsymbol{Q}$ denote the rational number field. The additive group of $\boldsymbol{Q}$ will also be denoted by $\boldsymbol{Q}$.

In this paper, we shall make simple remarks on the Schur multiplier of G, namely the second cohomology group $H^2(G, \boldsymbol{Q}/\boldsymbol{Z})$, under certain conditions on the structure of G. It is of course understood here that G acts trivialy on the additive group $\boldsymbol{Q}/\boldsymbol{Z}$. In the last part of the paper, we shall explain briefly how our result is related to the Hasse norm principle, over $\boldsymbol{Q}$, for an unramified abelian extension of a quadratic field in the narrow sense.

§1. Results and proofs.

We first prove the following

LEMMA. *Let A be a G-module on which N acts trivivally and let I denote the subgroup of* $\mathrm{Hom}(N, A)$ *such that $\tau(h(\nu))=h(\tau\nu\tau^{-1})$ for every $\tau\in T$ and every $\nu\in N$. Then*

$$H^1(G, A)\cong H^1(T, A)\oplus I.$$

PROOF. For each subgroup S of G, we denote by Z_S the additive group of maps $z: S\to A$ satisfying

$$\sigma_1(z(\sigma_2))-z(\sigma_1\sigma_2)+z(\sigma_1)=0, \quad \sigma_1, \sigma_2\in S.$$

Given any $f\in Z_G$, the restrictions $f|T$ and $f|N$ clearly belong to Z_T and Z_N, respectively. It then follows from $G=NT$ that the map $\iota: Z_G\to Z_T\oplus Z_N$ defined by $\iota(f)=(f|T, f|N)$ is an injective homomorphism. Since N acts trivially on A, we also have

$$f(\nu_1\nu_2)=\nu_1(f(\nu_2))+f(\nu_1)=f(\nu_1)+f(\nu_2)\,,\quad \nu_1,\nu_2\in N\,.$$

Furthermore, for each $\nu\in N$ and each $\tau\in T$,

$$\tau(f(\nu))=f(\tau\nu)-f(\tau)=f(\tau\nu\tau^{-1}\tau)-f(\tau)$$
$$=\tau\nu\tau^{-1}(f(\tau))+f(\tau\nu\tau^{-1})-f(\tau)=f(\tau\nu\tau^{-1})\,.$$

Thus, $f|N$ belongs to I.

Take next any $g\in Z_T$ and any $h\in I$. Noting that $T\cap N=\{1\}$, define the map $F: G\rightarrow A$ by

$$F(\nu\tau)=g(\tau)+h(\nu)\,,\quad \nu\in N,\ \tau\in T\,.$$

Then, for any $\nu_1,\nu_2\in N$ and any $\tau_1,\tau_2\in T$,

$$\nu_1\tau_1(F(\nu_2\tau_2))-F(\nu_1\tau_1\nu_2\tau_2)+F(\nu_1\tau_1)$$
$$=\nu_1\tau_1(g(\tau_2)+h(\nu_2))-g(\tau_1\tau_2)-h(\nu_1\tau_1\nu_2\tau_1^{-1})+g(\tau_1)+h(\nu_1)$$
$$=\tau_1(g(\tau_2))+\tau_1(h(\nu_2))-g(\tau_1\tau_2)-\nu_1(h(\tau_1\nu_2\tau_1^{-1}))-h(\nu_1)+g(\tau_1)+h(\nu_1)$$
$$=\tau_1(h(\nu_2))-h(\tau_1\nu_2\tau_1^{-1})=0\,.$$

Since $F|T=g$ and $F|N=h$, it follows that $\mathrm{Im}\,\iota=Z_T\oplus I$. Now, for each subgroup S of G, we let B_S denote the additive group of maps $b: S\rightarrow A$ such that

$$b(\sigma)=\sigma a-a\,,\quad \sigma\in S\,,$$

with some $a\in A$. By this definition, $B_S\subset Z_S$, $H^1(S,A)=Z_S/B_S$, and we easily see that

$$\iota(B_G)=B_T\oplus B_N\,,\quad B_N=\{0\}\,.$$

Consequently, ι induces an isomorphism from $H^1(G,A)$ onto $H^1(T,A)\oplus I$.

For each subgroup S of G, we let

$$S^*=H^1(S,\boldsymbol{Q}/\boldsymbol{Z})=\mathrm{Hom}\,(S,\boldsymbol{Q}/\boldsymbol{Z})\,.$$

PROPOSITION 1. *Assume that N is the direct product of its r cyclic subgroups $N_1,\cdots,N_r$ $(r\geqq 1)$ and that, for each $\tau\in T$, there exists an integer t such that $\tau\nu\tau^{-1}=\nu^t$ for every $\nu\in N$. Then*

$$H^2(G,\boldsymbol{Q}/\boldsymbol{Z})\cong H^2(T,\boldsymbol{Q}/\boldsymbol{Z})\oplus(N\wedge N)\oplus\bigoplus_{i=1}^{r}H^1(T,N_i^*)\,.$$

Here the action of T on each N_i^* is defined by

$$(\tau f)(\nu)=f(\tau\nu\tau^{-1})\,,\quad \tau\in T,\ f\in N_i^*,\ \nu\in N_i\,,$$

and $N\wedge N$ denotes as usual the exterior product of N.

PROOF. Let us prove the proposition by induction on r. Let s be any positive integer. Assuming that the proposition holds if $r<s$, we consider the case $r=s$. Let $N'=N_1\cdots N_{s-1}$ so that G is the semidirect product of N_s by TN'. Let R be the restriction map $H^2(G,\boldsymbol{Q}/\boldsymbol{Z})\to H^2(TN',\boldsymbol{Q}/\boldsymbol{Z})$. Then, by Theorem 2 of [2],

$$H^2(G,\boldsymbol{Q}/\boldsymbol{Z})\cong H^2(TN',\boldsymbol{Q}/\boldsymbol{Z})\oplus \operatorname{Ker} R$$

and there exists an exact sequence

$$0\longrightarrow H^1(TN',N_s^*)\longrightarrow \operatorname{Ker} R\longrightarrow H^2(N_s,\boldsymbol{Q}/\boldsymbol{Z}),$$

where the action of TN' on N_s^* is of course defined by

$$(\sigma f)(\mu)=f(\sigma\mu\sigma^{-1}),\quad \sigma\in TN',\ f\in N_s^*,\ \mu\in N_s.$$

However, $H^2(N_s,\boldsymbol{Q}/\boldsymbol{Z})\cong N_s\wedge N_s=\{1\}$ since N_s is a cyclic group. Thus

$$H^2(G,\boldsymbol{Q}/\boldsymbol{Z})\cong H^2(TN',\boldsymbol{Q}/\boldsymbol{Z})\oplus H^1(TN^1,N_s^*). \tag{1}$$

It further follows from our hypothesis of inductioh that

$$H^2(TN',\boldsymbol{Q}/\boldsymbol{Z})\cong H^2(T,\boldsymbol{Q}/\boldsymbol{Z})\oplus(N'\wedge N')\oplus\bigoplus_{i=1}^{s-1}H^1(T,N_i^*). \tag{2}$$

Take arbitrarily $h\in\operatorname{Hom}(N',N_s^*)$, $\tau\in T$, $\nu\in N'$, and $\mu\in N_s$. The assumption of the proposition then implies that

$$\tau\mu\tau^{-1}=\mu^t,\quad \tau\nu\tau^{-1}=\nu^t\quad \text{for some } t\in\boldsymbol{Z}.$$

Hence

$$\begin{aligned}(\tau(h(\nu))(\mu)&=(h(\nu))(\tau\mu\tau^{-1})=(h(\nu))(\mu^t)\\&=t(h(\nu)(\mu))=(th(\nu))(\mu)=(h(\tau\nu\tau^{-1}))(\mu).\end{aligned}$$

Therefore, $\tau(h(\nu))=h(\tau\nu\tau^{-1})$ so that, by the lemma,

$$H^1(TN',N_s^*)=H^1(T,N_s^*)\oplus\operatorname{Hom}(N',N_s^*).$$

Since $\operatorname{Hom}(N',N_s^*)\cong\oplus_{i=1}^{s-1}\operatorname{Hom}(N_i,N_s^*)\cong\oplus_{i=1}^{s-1}(N_i\otimes N_s)$, it follows from (1) and (2) that

$$\begin{aligned}&H^2(G,\boldsymbol{Q}/\boldsymbol{Z})\\&\cong H^2(T,\boldsymbol{Q}/\boldsymbol{Z})\oplus(N'\wedge N')\oplus\Big(\bigoplus_{i=1}^{s-1}H^1(T,N_i^*)\Big)\oplus H^1(T,N_s^*)\oplus\bigoplus_{i=1}^{s-1}(N_i\otimes N_s)\\&\cong H^2(T,\boldsymbol{Q}/\boldsymbol{Z})\oplus(N\wedge N)\oplus\bigoplus_{i=1}^{s}H^1(T,N_i^*).\end{aligned}$$

The proposition is therefore proved.

REMARK. Since N is abelian, $N\wedge N\cong H^2(N,\boldsymbol{Q}/\boldsymbol{Z})$ as is well known.

PROPOSITION 2. *Assume that N is abelian, $|T|=2$, and $\tau_0\nu\tau_0^{-1}=\nu^{-1}$ for every $\nu\in N$, τ_0 being the non-trivial element of T. Then*

$$H^2(G, \boldsymbol{Q}/\boldsymbol{Z}) \cong (N \wedge N) \oplus (\boldsymbol{Z}/2\boldsymbol{Z})^{\rho},$$

where ρ is the 2-rank of N.

PROOF. As T is cyclic, we first obtain $H^2(T, \boldsymbol{Q}/\boldsymbol{Z})=0$. Next, given any subgroup S of N, we have, by the assumption on τ_0,

$$\tau_0 f = -f \quad \text{for every } f \in S^* .$$

Hence $H^1(T, S^*) \cong S^*/2S^*$. The proof is now completed by Proposition 1.

§ 2. Relation to number theory.

Let k be a quadratic field and L an unramified abelian extension over k in the narrow sense. Then, by class field theory, L is a Galois extension over $\boldsymbol{Q}$, $\mathrm{Gal}(L/\boldsymbol{Q})$ is the semidirect product of the abelian group $\mathrm{Gal}(L/k)$ by J, an inertia group for $L/\boldsymbol{Q}$ of a prime ideal of L dividing a rational prime ramified in k, and $xyx^{-1}=y^{-1}$ holds for every $y \in \mathrm{Gal}(L/k)$, with the non-trivial element x of the group J of order 2. It therefore follows from Proposition 2 that

$$H^2(\mathrm{Gal}(L/\boldsymbol{Q}), \boldsymbol{Q}/\boldsymbol{Z}) \cong (\mathrm{Gal}(L/k) \wedge \mathrm{Gal}(L/k)) \oplus (\boldsymbol{Z}/2\boldsymbol{Z})^{\lambda} \tag{3}$$

where λ is the 2-rank of $\mathrm{Gal}(L/k)$. Now let g denote the number of rational primes ramified in k, so that $\lambda \leqq g-1$. Using (3), we can see that g does not exceed 3 if the Hasse norm principle holds for $L/\boldsymbol{Q}$, namely, if a rational number which is a norm for $L/\boldsymbol{Q}$ of some idele of L is always a norm for $L/\boldsymbol{Q}$ of some algebraic number in L. Furthermore, it follows from (3) (combined with classical results) that, in the case $g \leqq 2$, the Hasse norm principle holds for $L/\boldsymbol{Q}$ if and only if L is a cyclic extension over k (for the case where $g=1$ and L is the Hilbert class field over k in the narrow sense, see [1]).

The details of this section will be published elsewhere.

References

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