Natural Science Report, Ochanomizu University, Vol. 44, No. 2, 1993
お茶の水女子大学 自然科学報告 第44巻 第2号

On Connectedness of the Space of Harmonic 2-Spheres in Real Grassmann Manifolds of 2-Planes

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(Received October 5, 1993)

Introduction

In a previous paper [Mu], using the method of [GO], we studied the deformations of harmonic maps of a Riemann surface Σ into a quaternionic projective space $\boldsymbol{H}P^n$ which are strongly isotropic or quaternionic mixed pairs and as a consequence, we obtained the results on the connectedness of the space of such harmonic 2-spheres in $\boldsymbol{H}P^n$. In this paper, we deal with the case of harmonic 2-spheres in a real Grassmann manifold of 2-planes $Gr_2(\boldsymbol{R}^{n+2})$, or a complex hyperquadric $Q_n(\boldsymbol{C})$. According to [BW1] and [BW2], the construction theory of harmonic 2-spheres in $Gr_2(\boldsymbol{R}^{n+2})$ or $Q_n(\boldsymbol{C})$ have algebraic structure analogous to the case $\boldsymbol{H}P^n$. However, there are some differences between the case $\boldsymbol{H}P^n$ and the case $Gr_2(\boldsymbol{R}^{n+2})$; for instance, the group which acts on the twistor spaces over each space, the existence of stable harmonic 2-spheres, and so on. On the other hand, although $Q_n(\boldsymbol{C}) \to Gr_2(\boldsymbol{R}^{n+2})$ is the double universal covering, when we consider harmonic 2-sphere in $Q_n(\boldsymbol{C})$ and $Gr_2(\boldsymbol{R}^{n+2})$, the interesting differences between $Q_n(\boldsymbol{C})$ and $Gr_2(\boldsymbol{R}^{n+2})$ on topology and structure of manifolds attract our attention.

In Section 1, we shall discuss the standard twistor spaces over $Q_n(\boldsymbol{C})$ and $Gr_2(\boldsymbol{R}^{n+2})$. It is known that the standard twistor spaces over $Q_n(\boldsymbol{C})$ are $Q_n(\boldsymbol{C})$ itself and $\mathscr{Z}_m(\boldsymbol{C}^{2m+2})$ for $n=2m$, and those over $Gr_2(\boldsymbol{R}^{n+2})$ are $Q_n(\boldsymbol{C})$ (double covering) and $\mathscr{Z}_m(\boldsymbol{C}^{2m+2})$ for $n=2m$. In Section 2 and 3, applying the argument of [GO] to horizontal holomorphic maps into each twistor space, we shall prove results on Morse-Bott theoretic deformations for harmonic maps. In Section 4, we shall discuss the energy and degree for harmonic 2-spheres in $Q_n(\boldsymbol{C})$ and $Gr_2(\boldsymbol{R}^{n+2})$. In Section 5 and 6, we shall show main theorems. In Section 7, we shall remark the relation between the main theorem and the construction of harmonic 2-spheres in $Gr_2(\boldsymbol{R}^{n+2})$ and give some conjectures.

To state our theorems, we prepare the following notations. First, for a map $\phi: S^2 \to Q_n(\boldsymbol{C})$, in the case $n=2$, let $\mathrm{Harm}_{deg_{\mathcal{L}}, deg_{\mathcal{W}}}(S^2, Q_2(\boldsymbol{C}))$ be the space of harmonic maps with fixed bi-degree $(\deg_{\mathcal{L}}(\phi), \deg_{\mathcal{W}}(\phi))$. In the case $n \geqq 3$, let $\mathrm{Harm}_{\mathcal{E}, deg}(S^2, Q_n(\boldsymbol{C}))^{st.isot.}$ and $\mathrm{Hol}_{deg}(S^2, Q_n(\boldsymbol{C}))$ be the space of strongly isotropic harmonic maps with fixed energy and fixed degree and the space of holomorphic maps with fixed degree, respectively. Here we call a harmonic map $\phi: \Sigma \to Q_n(\boldsymbol{C})$ *strongly isotropic* if ϕ is strongly isotropic as a harmonic map into $Gr_2(\boldsymbol{C}^{n+2})$ when we regard a map ϕ as the composition of the maps $\Sigma \to Q_n(\boldsymbol{C})$, $Q_n(\boldsymbol{C}) \to Gr_2(\boldsymbol{R}^{n+2})$ (double covering) and $Gr_2(\boldsymbol{R}^{n+2}) \to Gr_2(\boldsymbol{C}^{n+2})$ (see [BW1]). Then the statement of theorem is as follows.

THEOREM A. (1) *The space* $\mathrm{Harm}_{deg_{\mathcal{L}}, deg_{\mathcal{W}}}(S^2, Q_2(\boldsymbol{C}))$ *is path-connected.*

(2) *If* $n \geqq 3$, *the space* $\mathrm{Harm}_{\mathcal{E}, deg}(S^2, Q_n(\boldsymbol{C}))^{st.isot.}$ *is path-connected.*

(3) *If* $n \geqq 3$, *the space* $\mathrm{Hol}_{deg}(S^2, Q_n(\boldsymbol{C}))$ *is path-connected.*

Next, for a map $\varphi: S^2 \to Gr_2(\boldsymbol{R}^{n+2})$, in the case $n=2$, let $\mathrm{Harm}_{d_{\mathcal{L}}, d_{\mathcal{W}}, \varepsilon}(S^2, Gr_2(\boldsymbol{R}^4))$ be the space of harmonic maps with fixed $d_{\mathcal{L}}(\varphi)$, fixed $d_{\mathcal{W}}(\varphi)$ and fixed signature $\varepsilon\,(=1$ or $-1)$ of $(\deg_{\mathcal{L}}(\phi) \cdot \deg_{\mathcal{W}}(\phi))$. Here $d_{\mathcal{L}}(\varphi) := |\deg_{\mathcal{L}}(\phi)|$ and $d_{\mathcal{W}}(\varphi) := |\deg_{\mathcal{W}}(\phi)|$ for a lift ϕ of φ to $Q_2(\boldsymbol{C})$. In the case $n \geqq 3$, let $\mathrm{Harm}_{\mathcal{E}, d}(S^2, Gr_2(\boldsymbol{R}^{n+2}))^{st.isot.}$ and $\mathrm{Harm}_d(S^2, Gr_2(\boldsymbol{R}^{n+2}))^{r.m.p.}$ be the space of strongly isotropic harmonic maps with fixed energy and fixed $d(\varphi)$ and the space of real mixed pairs (see Section 1) with fixed $d(\varphi)$, respectively. Here $d(\varphi) := |\deg(\phi)|$ for a lift ϕ of φ to $Q_n(\boldsymbol{C})$. Then our main result is as follows.

THEOREM B. (1) *The space* $\mathrm{Harm}_{d_{\mathcal{L}}, d_{\mathcal{W}}, \varepsilon}(S^2, Gr_2(\boldsymbol{R}^4))$ *is path-connected.*

(2) *If* $n \geqq 3$, *the space* $\mathrm{Harm}_{\mathcal{E}, d}(S^2, Gr_2(\boldsymbol{R}^{n+2}))^{st.isot.}$ *is path-connected.*

(3) *If* $n \geqq 3$, *the space* $\mathrm{Harm}_d(S^2, Gr_2(\boldsymbol{R}^{n+2}))^{r.m.p.}$ *is path-connected.*

ACKNOWLEDGEMENTS: The author would like to thank Professors Y. Ohnita and K. Tsukada for valuable discussions and useful suggestion.

1. Twistor spaces over $Gr_2(\boldsymbol{R}^{n+2})$ and $Q_n(\boldsymbol{C})$, and Harmonic 2-spheres

Let $\langle\ ,\ \rangle$ and $(\ ,\ )$ denote the standard Hermitian inner product and the standard complex symmetric bilinear form on $\boldsymbol{C}^{n+2}$ defined by

$$\langle v, w \rangle = v_1 \bar{w}_1 + \cdots + v_{n+2} \bar{w}_{n+2}, \qquad (v, w) = v_1 w_1 + \cdots + v_{n+2} w_{n+2}$$

where $v = (v_1, \cdots, v_{n+2})$, $w = (w_1, \cdots, w_{n+2}) \in \boldsymbol{C}^{n+2}$, and $\bar{\ }$ denotes complex conjugation.

For a k-dimensional complex subspace W of $\boldsymbol{C}^{n+2}$ invariant under the conjugation, we denote by $W_{\boldsymbol{R}}=\{w\in W \mid \bar{w}=w\}$ the real form of W, which is a k-dimensional real subspace of $\boldsymbol{R}^{n+2}$. Then we can identify a k-dimensional complex subspace of $\boldsymbol{C}^{n+2}$ invariant under the conjugation with a k-dimensional real subspace of $\boldsymbol{R}^{n+2}$; by corresponding W to $W_{\boldsymbol{R}}$ and conversely by corresponding a k-dimensional real subspace of $\boldsymbol{R}^{n+2}$ to its complexification.

DEFINITION. A complex subspace W of $\boldsymbol{C}^{n+2}$ is called *complex isotropic* if $(v, w)=0$ for $v, w\in W$, namely, $W\perp\overline{W}$ relative to $\langle\ ,\ \rangle$.

Let $Gr_2(\boldsymbol{R}^{n+2})$ denote the real Grassmann manifold of 2-planes in $\boldsymbol{R}^{n+2}$ with the standard Riemannian metric. Since we have an identification $Gr_2(\boldsymbol{R}^{n+2})\cong\{W\in Gr_2(\boldsymbol{C}^{n+2}) \mid W=\overline{W}\}$, we can regard $Gr_2(\boldsymbol{R}^{n+2})$ as a totally geodesic submanifold of $Gr_2(\boldsymbol{C}^{n+2})$. The universal double covering space of $Gr_2(\boldsymbol{R}^{n+2})$ is $\tilde{G}r_2(\boldsymbol{R}^{n+2})$, the real Grassmann manifold of oriented 2-planes in $\boldsymbol{R}^{n+2}$. Then $\tilde{G}r_2(\boldsymbol{R}^{n+2})$ may be identified with the complex hyperquadric

$$Q_n(\boldsymbol{C})=\{L\in \boldsymbol{C}P^{n+1} \mid (L, L)=0\},$$

as follows. The map $Q_n(\boldsymbol{C})\to\tilde{G}r_2(\boldsymbol{R}^{n+2})$ is given by $L=[Z]\mapsto[\mathrm{Re}(Z)\wedge\mathrm{Im}(Z)]$, where $[\mathrm{Re}(Z)\wedge\mathrm{Im}(Z)]$ denotes an oriented 2-plane in $\boldsymbol{R}^{n+2}$ spanned by the oriented pair of vectors $\{\mathrm{Re}(Z), \mathrm{Im}(Z)\}$. The inverse map $\tilde{G}r_2(\boldsymbol{R}^{n+2})\to Q_n(\boldsymbol{C})$ is given by $[X_1\wedge X_2]\mapsto[X_1+\sqrt{-1}X_2]$, where $[X_1\wedge X_2]$ denotes an oriented 2-plane with an orthonomal basis $\{X_1, X_2\}$ compatible with the orientation.

Throughout this paper, let G and G^C denote the special orthogonal group $SO(n+2)$ and its complexification $SO(n+2, \boldsymbol{C})$, namely,

$$\begin{aligned} G^C&=\{A\in SL(n+2, \boldsymbol{C}) \mid {}^tAA=I\} \\ &=\{A\in SL(n+2, \boldsymbol{C}) \mid (Av, Aw)=(v, w) \text{ for each } v, w\in\boldsymbol{C}^{n+2}\}. \end{aligned}$$

Now we introduce two twistor spaces over $Q_n(\boldsymbol{C})$: one is $Q_n(\boldsymbol{C})$ itself with the projection $\pi=id$ and another is

$$\mathscr{Z}_m(\boldsymbol{C}^{2m+2})=\{W\in Gr_m(\boldsymbol{C}^{2m+2}) \mid (W, W)=0\}$$

with the projection $\pi(W)=(W\oplus\overline{W})^\perp$ for $n=2m$. Here $\oplus$ denotes an Hermitian orthogonal direct sum with respect to $\langle\ ,\ \rangle$. Since G acts transitively on $Q_n(\boldsymbol{C})$ and $\mathscr{Z}_m(\boldsymbol{C}^{2m+2})$, we have $Q_n(\boldsymbol{C}) = SO(n+2)/SO(2)\times SO(n)$ and $\mathscr{Z}_m(\boldsymbol{C}^{2m+2}) = SO(2m+2)/SO(2)\times U(m)$. In particular, their complex dimensions are given by $\dim_C Q_n(\boldsymbol{C})=n$ and $\dim_C\mathscr{Z}_m(\boldsymbol{C}^{2m+2})=(m(m+3)/2)$.

The space $Gr_2(\boldsymbol{R}^{n+2})$ has two standard twistor spaces $Q_n(\boldsymbol{C})$ with the projection $\pi(L)=L\oplus\bar{L}$ (double covering) and $\mathscr{Z}_m(\boldsymbol{C}^{2m+2})$ with the projection $\pi(W)=(W\oplus\overline{W})^\perp$ for $n=2m$.

Let us discuss in detail the twistor space $\mathcal{Z}_m(\boldsymbol{C}^{2m+2})$. We define three tautological complex vector bundles $\mathcal{W}$, $\overline{\mathcal{W}}$ and $\mathcal{E}$ over $\mathcal{Z}_m(\boldsymbol{C}^{2m+2})$; for $W\in\mathcal{Z}_m(\boldsymbol{C}^{2m+2})$, $\mathcal{W}_W=W$, $(\overline{\mathcal{W}})_W=\overline{W}$ and $\mathcal{E}_W=E$, where $\boldsymbol{C}^{2m+2}=W\oplus\overline{W}\oplus E$. Using a natural inclusion map $\mathcal{Z}_m(\boldsymbol{C}^{2m+2})\hookrightarrow Gr_m(\boldsymbol{C}^{2m+2})$, we describe the holomorphic tangent bundle of $\mathcal{Z}_m(\boldsymbol{C}^{2m+2})$ as

$$\mathrm{T}\mathcal{Z}_m(\boldsymbol{C}^{2m+2})^{1,0}=\mathrm{Hom}(\mathcal{W},\overline{\mathcal{W}})^{isot.}\oplus\mathrm{Hom}(\mathcal{W},\mathcal{E})\,.$$

Here

$$\begin{aligned}&\mathrm{Hom}(\mathcal{W},\overline{\mathcal{W}})^{isot.}\\&\quad=\coprod_{W\in\mathcal{Z}_m(\boldsymbol{C}^{2m+2})}\{T\in\mathrm{Hom}(W,\overline{W})\,|\,(Tv,v)=0\ \text{for each}\ v\in W\}\end{aligned}$$

corresponds to *the vertical subspaces* of π and $\mathrm{Hom}(\mathcal{W},\mathcal{E})$ corresponds to *the horizontal subspaces* of π for $\pi:\mathcal{Z}_m(\boldsymbol{C}^{2m+2})\to Q_n(\boldsymbol{C})$ or $Gr_2(\boldsymbol{R}^{n+2})$. A smooth map $f:\Sigma\to\mathcal{Z}_m(\boldsymbol{C}^{2m+2})$ of a Riemann surface is said to be *a horizontal holomorphic map* if $df(\mathrm{T}\Sigma^{1,0})\subset\mathrm{Hom}(\mathcal{W},\mathcal{E})$. The holomorphicity and horizontality conditions are written respectively as

$$\partial''C^\infty(f^{-1}\mathcal{W})\subset C^\infty(f^{-1}\mathcal{W})\,,$$

$$\partial'C^\infty(f^{-1}\mathcal{W})\subset C^\infty(f^{-1}\mathcal{W})+C^\infty(f^{-1}\mathcal{E})\,.$$

We know that if a map $\phi:\Sigma\to Q_n(\boldsymbol{C})$ is of the form $\phi=\pi\circ f$, for a horizontal holomorphic map $f:\Sigma\to\mathcal{Z}_m(\boldsymbol{C}^{2m+2})$ with respect to $\pi:\mathcal{Z}_m(\boldsymbol{C}^{2m+2})\to Q_n(\boldsymbol{C})$, then ϕ is harmonic. Thus $\varphi:\Sigma\to Gr_2(\boldsymbol{R}^{n+2})$ is also harmonic, because $Q_n(\boldsymbol{C})\to Gr_2(\boldsymbol{R}^{n+2})$ is an universal double covering.

The group G^C acts transitively on $\mathcal{Z}_m(\boldsymbol{C}^{2m+2})$ in the natural way; for $A\in G^C$ and $W\in\mathcal{Z}_m(\boldsymbol{C}^{2m+2})$, we have $A(W)\in\mathcal{Z}_m(\boldsymbol{C}^{2m+2})$, because the group G^C preserves the complex symmetric bilinear form $(\ ,\ )$. Then we have the following.

LEMMA 1.1. (1) *This action of G^C on $\mathcal{Z}_m(\boldsymbol{C}^{2m+2})$ preserves the complex structure of $\mathcal{Z}_m(\boldsymbol{C}^{2m+2})$ and the horizontal subspaces with respect to $\pi:\mathcal{Z}_m(\boldsymbol{C}^{2m+2})\to Q_n(\boldsymbol{C})$ or $Gr_2(\boldsymbol{R}^{n+2})$.*

(2) *Let $A\in G^C$ and $f:\Sigma\to\mathcal{Z}_m(\boldsymbol{C}^{2m+2})$ be a horizontal holomorphic map. Then $A\circ f:\Sigma\to\mathcal{Z}_m(\boldsymbol{C}^{2m+2})$ is also a horizontal holomorphic map.*

PROOF. (1) Let $A\in G^C$ and $T\in\mathrm{Hom}(\mathcal{W},\mathcal{E})$. For any $s,s'\in C^\infty(\mathcal{W})$, then we have $\langle(AT)s,\bar{s}'\rangle=((AT)s,s')=(A(T(A^{-1}s)),s')=(T(A^{-1}s),A^{-1}s')=0$, because $T(A^{-1}s)\in\mathcal{E}$ and $A^{-1}s'\in\mathcal{W}$. Since $(AT)s\perp\bar{s}'$ and $\bar{s}'\in C^\infty(\overline{\mathcal{W}})$, we obtain $AT\in\mathrm{Hom}(\mathcal{W},\mathcal{E})$.

(2) For any $s,s'\in C^\infty(f^{-1}\mathcal{W})$, $A(s)$ is a section of $(A\circ f)^{-1}\mathcal{W}$. Note that $(A\circ f)^{-1}\mathcal{W}=A(f^{-1}\mathcal{W})$. Then we have

$$\partial'' A(s) = A(\partial'' s) \in AC^\infty(f^{-1}\mathcal{W}) = C^\infty((A\circ f)^{-1}\mathcal{W}).$$

Hence $A\circ f$ is holomorphic. Also we have

$$\langle \partial' A(s), \overline{A(s')}\rangle = (A(\partial' s), A(s')) = (\partial' s, s') = \langle \partial' s, \bar{s}'\rangle = 0,$$

because $\partial' s \in \partial' C^\infty(f^{-1}\mathcal{W})$ and $\bar{s}' \in C^\infty(f^{-1}\overline{\mathcal{W}})$. Hence $A\circ f$ is horizontal. □

Next let us discuss another twistor space $Q_n(\boldsymbol{C})$. We define three tautological complex vector bundles over $Q_n(\boldsymbol{C})$ for $L\in Q_n(\boldsymbol{C})$ as follows; $\mathcal{L}_L = L$, $(\overline{\mathcal{L}})_L = \bar{L}$ and $\mathcal{V}_L = V$, where $\boldsymbol{C}^{n+2} = L\oplus \bar{L}\oplus V$. The holomorphic tangent bundle of $Q_n(\boldsymbol{C})$ is given by

$$\mathrm{T}Q_n(\boldsymbol{C})^{1,0} = \mathrm{Hom}(\mathcal{L}, \overline{\mathcal{L}})^{isot.} \oplus \mathrm{Hom}(\mathcal{L}, \mathcal{V}).$$

Here

$$\mathrm{Hom}(\mathcal{L}, \overline{\mathcal{L}})^{isot.} = \coprod_{L\in Q_n(C)} \{T\in \mathrm{Hom}(L, \bar{L}) \mid (Tv, v) = 0 \text{ for each } v\in L\}$$

corresponds to *the vertical subspaces* of π and $\mathrm{Hom}(\mathcal{L}, \mathcal{V})$ corresponds to *the horizontal subspaces* of π for $\pi: Q_n(\boldsymbol{C})\to Gr_2(\boldsymbol{R}^{n+2})$. A smooth map $g: \Sigma\to Q_n(\boldsymbol{C})$ of a Riemann surface is said to be *a holomorphic map* if $dg(\mathrm{T}\Sigma^{1,0})\subset \mathrm{T}Q_n(\boldsymbol{C})^{1,0}$. It suffices to consider the holomorphicity condition, because the projection $Q_n(\boldsymbol{C})\to Gr_2(\boldsymbol{R}^{n+2})$ is a covering map. This condition is written as

$$\partial'' C^\infty(g^{-1}\mathcal{L})\subset C^\infty(g^{-1}\mathcal{L}).$$

Then the following fact holds.

LEMMA 1.2. (1) *This action of G^C on $Q_n(\boldsymbol{C})$ preserves the complex structure of $Q_n(\boldsymbol{C})$.*

(2) *Let $A\in G^C$ and $g: \Sigma\to Q_n(\boldsymbol{C})$ be a holomorphic map. Then $A\circ g: \Sigma\to Q_n(\boldsymbol{C})$ is a holomorphic map.*

DEFINITION ([BW1]). A map $\varphi: \Sigma\to Gr_2(\boldsymbol{R}^{n+2})$ is called *a real mixed pair* if there exists a holomorphic map $g: \Sigma\to Q_n(\boldsymbol{C})$ such that $\varphi = \pi\circ g$, namely, if $\underline{\varphi}$ denotes the corresponding subbundle to φ, then $\underline{\varphi} = \underline{g}\oplus \underline{\bar{g}}$.

Now we mention the relation between the classification of harmonic maps $\Sigma\to Gr_2(\boldsymbol{R}^{n+2})$ and the lift to the twistor space $\mathcal{Z}_m(\boldsymbol{C}^{2m+2})$ over $Gr_2(\boldsymbol{R}^{n+2})$.

In [Gl] and [BW1], it was shown that $\varphi: \Sigma\to Gr_2(\boldsymbol{R}^{n+2})$ is strongly isotropic if and only if there exists a horizontal holomorphic map $f: \Sigma\to \mathcal{Z}_m(\boldsymbol{C}^{2m+2})$ such that $\varphi = \pi\circ f$, namely $\underline{\varphi} = (\underline{f}\oplus \underline{\bar{f}})^\perp$. In this case, it is known that if a harmonic map $\varphi: \Sigma\to Gr_2(\boldsymbol{R}^{n+2})$ is strongly isotropic, then $\varphi(\Sigma)\subset Gr_2(\boldsymbol{R}^{2m+2})\subset Gr_2(\boldsymbol{R}^{n+2})$ for some $(2m+2)$-dimensional real subspace

$\boldsymbol{R}^{2m+2} \subset \boldsymbol{R}^{n+2}$.

REMARK. (1) Throughout this paper, for a k-dimensional complex subspace W of $\boldsymbol{C}^{n+2}$, we define

$$\boldsymbol{C}P^{k-1}(W)=\{L \mid L \text{ is a 1-dimensional subspace of } W\}.$$

(2) If we let F^{m+1} an $(m+1)$-dimensional complex isotropic subspace of $\boldsymbol{C}^{2m+2}$, then $\varphi: S^2 \to \{(L \oplus \bar{L})_{\boldsymbol{R}} \mid L \in \boldsymbol{C}P^m(F)\} \cong \boldsymbol{C}P^m \subset Gr_2(\boldsymbol{R}^{2m+2})$ is a holomorphic map if and only if φ is both a strongly isotropic harmonic map and a real mixed pair.

(3) If $\varphi: S^2 \to Gr_2(\boldsymbol{R}^4)$ is a harmonic map, then φ is strongly isotropic or a real mixed pair (see [BW1]).

2. Deformations of strongly isotropic harmonic maps into $Gr_2(\boldsymbol{R}^{n+2})$

(A) Morse-Bott theory over twistor space $\mathcal{Z}_m(\boldsymbol{C}^{2m+2})$. Let $G=SO(2m+2)$ and $\mathfrak{g}$ denote its Lie algebra. Then we can regard $\mathcal{Z}_m(\boldsymbol{C}^{2m+2})$ as an orbit of the adjoint representation of G as follows: If we let W_0 a fixed element of $\mathcal{Z}_m(\boldsymbol{C}^{2m+2})$ and set $\xi=\sqrt{-1}\pi_{W_0}-\sqrt{-1}\pi_{\bar{W}_0}$, then we have $\mathcal{Z}_m(\boldsymbol{C}^{2m+2}) \cong Ad(G)\xi$. Here π_{W_0} denotes the Hermitian projection in $\boldsymbol{C}^{2m+2}$ onto E_0.

Fix a element $L \in Q_{2m}(\boldsymbol{C})$ and put $P=\sqrt{-1}\pi_L-\sqrt{-1}\pi_{\bar{L}} \in \mathfrak{g}$. For $X=\sqrt{-1}\pi_W-\sqrt{-1}\pi_{\bar{W}} \in Ad(G)\xi$, with $W \in \mathcal{Z}_m(\boldsymbol{C}^{2m+2})$, we define the height function $h^P: Ad(G)\xi \to \boldsymbol{R}$ by

$$h^P(X)=\langle\langle X, P\rangle\rangle .$$

Here $\langle\langle\ ,\ \rangle\rangle$ is an $Ad(G)$-invariant inner product on $\mathfrak{g}$. Then it is known that h^P is a Morse-Bott function. Let $\operatorname{grad} h^P$ be a gradient vector field of h^P with respect to the Kähler metric. The following fact is due to Frankel; the flow of $-(\operatorname{grad} h^P)$ is given by the action of $\{\exp \sqrt{-1}tP\}$.

We shall describe non-degenerate critical manifolds of h^P. It is known that a point $X \in Ad(G)\xi$ is a critical point of h^P if and only if $[X, P]=0$, i. e.

$$[\sqrt{-1}\pi_W-\sqrt{-1}\pi_{\bar{W}},\quad \sqrt{-1}\pi_L-\sqrt{-1}\pi_{\bar{L}}]=0 .$$

Then a critical point X of h^P is characterized by $W=W_1 \oplus W_2 \oplus W_3$ with $W_1 \subseteq L$, $W_2 \subseteq \bar{L}$, $W_3 \subseteq (L \oplus \bar{L})^{\perp}$, where $\boldsymbol{C}^{2m+2}=L \oplus \bar{L} \oplus (L \oplus \bar{L})^{\perp}$. We obtain the following lemma.

LEMMA 2.1. *There are three connected non-degenerate critical manifolds of* h^P;

$$C_+=\{W\in\mathcal{Z}_m(\boldsymbol{C}^{2m+2})\,|\,L\subset W\}\cong\mathcal{Z}_{m-1}(\boldsymbol{C}^{2m}),$$

$$C_0=\{W\in\mathcal{Z}_m(\boldsymbol{C}^{2m+2})\,|\,W\subset(L\oplus\bar{L})^\perp\}\cong\mathcal{Z}_m(\boldsymbol{C}^{2m}),$$

$$C_-=\{W\in\mathcal{Z}_m(\boldsymbol{C}^{2m+2})\,|\,\bar{L}\subset W\}\cong\mathcal{Z}_{m-1}(\boldsymbol{C}^{2m}).$$

PROOF. Assume that $W_1\neq\{0\}$. We see that $W_1=L$ and so $W_2=\{0\}$. Hence we get the critical manifold C_+ of h^P. Next assume that $W_1=\{0\}$. Then $W=W_2\oplus W_3$. If we let $W_2=\{0\}$, then we get the critical manifold C_0, and if we let $W_2\neq\{0\}$, then we get the critical manifold C_- in the same way as C_+. Now it is easy to show that C_+, C_0 and C_- are diffeomorphic to $\mathcal{Z}_{m-1}(\boldsymbol{C}^{2m})$, $\mathcal{Z}_m(\boldsymbol{C}^{2m})$ and $\mathcal{Z}_{m-1}(\boldsymbol{C}^{2m})$, respectively. □

We set $G_P=\{A\in G^C\,|\,A(L)=L\}$. In general, we know that the stable manifold for a connected non-degenerate critical manifold N is given by $S^P(N)=G_PX$ for $X\in N$. In our case we shall determine the corresponding stable manifolds.

LEMMA 2.2. *For three non-degenerate critical manifolds in Lemma* 2.1, *the corresponding stable manifolds* $S^P(C_+)$, $S^P(C_0)$, $S^P(C_-)$ *are*

$$S_+=C_+,$$

$$S_0=\{W\in\mathcal{Z}_m(\boldsymbol{C}^{2m+2})\,|\,W\cap L=\{0\},\ W\subset\bar{L}^\perp\},$$

$$S_-=\{W\in\mathcal{Z}_m(\boldsymbol{C}^{2m+2})\,|\,W\cap L=\{0\},\ W\not\subset\bar{L}^\perp\},$$

respectively.

PROOF. It is clear that $S^P(C_+)$ coincides with S_+. For $A\in G_P$, we have $\langle W,\bar{L}\rangle=(W,L)=(A(W),A(L))=(A(W),L)=\langle A(W),\bar{L}\rangle$. Thus we get $A(W)\subset\bar{L}^\perp$ (respectively, $A(W)\not\subset\bar{L}^\perp$), because $W\perp\bar{L}$ (respectively, $W\not\subset\bar{L}^\perp$). On the other hand, since $W\perp L$, we have $A(W)\cap L=\{0\}$. Then we have $S^P(C_0)\subset S_0$ (respectively, $S^P(C_-)\subset S_-$). Since

$$\mathcal{Z}_m(\boldsymbol{C}^{2m+2})=S^P(C_+)\amalg S^P(C_0)\amalg S^P(C_-)$$
$$=S_+\amalg S_0\amalg S_-$$

are two decompositions of $\mathcal{Z}_m(\boldsymbol{C}^{2m+2})$, we obtain $S^P(C_+)=S_+$, $S^P(C_0)=S_0$ and $S^P(C_-)=S_-$. □

REMARK. For $S^P(C_-)$, we see that if $W\not\subset\bar{L}^\perp$, then $W\cap L=\{0\}$.

(B) Deformations of harmonic maps. Let $\varphi:\Sigma\to Gr_2(\boldsymbol{R}^{2m+2})$ be a strongly isotropic harmonic map, and $f:\Sigma\to\mathcal{Z}_m(\boldsymbol{C}^{2m+2})$ be a horizontal holomorphic map corresponding to φ. If $f(\Sigma)\subset S^P(C_-)$, then $\{(\exp\sqrt{-1}tP)\circ$

$f\}_{0\leq t\leq\infty}$ provides a continuous deformation to a horizontal holomorphic map into $\mathcal{C}_-$. We shall show that there exists some $L\in Q_{2m}(\boldsymbol{C})$ such that $f(\Sigma)\subset S^P(\mathcal{C}_-)$.

We set $\mathcal{Y}^f=\{L\in Q_{2m}(\boldsymbol{C})|f(z)\not\subset S^P(\mathcal{C}_-)$ for some $z\in\Sigma\}$. Then we have $\mathcal{Y}^f=\{L\in Q_{2m}(\boldsymbol{C})|\bar{L}\perp f(z)$ for some $z\in\Sigma\}$. It suffices to show that $\mathcal{Y}^f$ cannot be equal to $Q_{2m}(\boldsymbol{C})$. We define

$$\mathcal{Y}=\{(L,W)\in Q_{2m}(\boldsymbol{C})\times\mathcal{Z}_m(\boldsymbol{C}^{2m+2})|\bar{L}\perp W\}.$$

Let p_1 and p_2 be the projections to $Q_{2m}(\boldsymbol{C})$ and $\mathcal{Z}_m(\boldsymbol{C}^{2m+2})$, respectively. Then we get $\mathcal{Y}^f=p_1(p_2^{-1}f(\Sigma))$. We shall estimate the fibre $p_2^{-1}(W)=\{L\in Q_{2m}(\boldsymbol{C})|L\perp\bar{W}\}$.

Let us consider two cases; $L\subset W^\perp$ or $L\not\subset W^\perp$. Hence we get $p_2^{-1}(W)=\mathcal{H}_1\amalg\mathcal{H}_2$, where

$$\mathcal{H}_1=\{L\in Q_{2m}(\boldsymbol{C})|L\perp\bar{W},L\subset W^\perp\},$$

$$\mathcal{H}_2=\{L\in Q_{2m}(\boldsymbol{C})|L\perp\bar{W},L\not\subset W^\perp\}.$$

First we deal with the space $\mathcal{H}_1$.

LEMMA 2.3. *The space $\mathcal{H}_1$ is diffeomorphic to $O(2)/U(1)$. In particular,* $\dim_C\mathcal{H}_1$ *is equal to* 0.

PROOF. Since $L\perp W$ and $L\perp\bar{W}$, we have $L\subset(W\oplus\bar{W})^\perp$. Note that $(W\oplus\bar{W})^\perp$ is a 2-dimensional *real* subspace of $\boldsymbol{C}^{2m+2}$. Then we can write $(W\oplus\bar{W})^\perp=I\oplus\bar{I}$, where I is a 1-dimensional complex isotropic line. Thus $\mathcal{H}_1=\{I,\bar{I}\}\cong O(2)/U(1)$. □

Next we consider the space $\mathcal{H}_2$.

LEMMA 2.4. *The space $\mathcal{H}_2$ is diffeomorphic to the space attached along zero sections of two vector bundles*

$$\mathcal{B}:=\Big(\coprod_{V\in CP^{m-1}(W)}\mathrm{Hom}(V,I)\Big)\bigcup_{CP^{m-1}(W)}\Big(\coprod_{V\in CP^{m-1}(W)}\mathrm{Hom}(V,\bar{I})\Big)$$

over $\boldsymbol{C}P^{m-1}(W)$ with the fibres $\mathrm{Hom}(V,I)$ *and* $\mathrm{Hom}(V,\bar{I})$ *at* $V\in\boldsymbol{C}P^{m-1}(W)$ *respectively.*

REMARK. In particular, $\dim_C\mathcal{B}=(m-1)+1=m$.

PROOF. Let μ and ν be the Hermitian orthogonal projections from L to W and $(W\oplus\bar{W})^\perp$, respectively. Set $V=\mu(L)$ and so V is a line of W. We see that $\nu(L)$ is a complex isotropic subspace of $(W\oplus\bar{W})^\perp$, indeed, for any $v=\mu(v)+\nu(v)$, $w=\mu(w)+\nu(w)\in L$, we have $0=\langle v,\bar{w}\rangle=\langle\nu(v),\bar{\nu}(w)\rangle$.

Assume that $\nu(L)\neq\{0\}$. Then it must be $\nu(L)=I$ or $\bar{I}$ where $(W\oplus W)^{\perp}=I\oplus\bar{I}$. If $\nu(L)=I$, then we see that for all $x\in V$, there is unique $y\in I$ satisfying $x+y\in L$. Indeed, using a linear isomorphism $\mu: L\to W$, since we can write $z=x+\nu(z)\in L$ for any $x\in V$, then we take $\nu(z)=y$. Then we have a linear map $\delta_I: V\to I$ defined by $\delta_I(x)=y$ for $x\in V$. If $\nu(L)=\bar{I}$, then we have a linear map $\delta_{\bar{I}}: V\to\bar{I}$ similarly. Next assume that $\nu(L)=\{0\}$. Then $L=V$ is a 1-dimensional space of W. Hence we get a smooth map $\mathscr{H}_2\ni L\to(V,\delta_I)$ or $(V,\delta_{\bar{I}})\in\mathscr{B}$.

Now let us examine its inverse map. First for any $V\in \boldsymbol{C}P^{m-1}(W)$, we put $L=V$, then we get $L\in\mathscr{H}_2$. For any $(V,\delta)\in\coprod_{V\in CP^{m-1}(W)}\mathrm{Hom}(V,I)$, we put $L=\{x+\delta(x)|x\in V\}$. Then we show that $L\in\mathscr{H}_2$. Indeed, it is clear that $L\perp\bar{W}$ and $L\not\subset W^{\perp}$, and we see that L is complex isotropic because for any $x, x'\in V$ we get $\langle x+\delta(x), x'+\delta(x')\rangle=0$. For any $(V,\delta')\in\coprod_{V\in CP^{m-1}(W)}\mathrm{Hom}(V,\bar{I})$, if we put $L=\{x+\delta'(x)|x\in V\}$, then we see that $L\in\mathscr{H}_2$ as above. Thus we get the inverse map $\mathscr{B}\ni(V,\delta)$ or $(V,\delta')\to L\in\mathscr{H}_2$.

Hence we obtain a diffeomorphism $\mathscr{H}_2\longleftrightarrow\mathscr{B}$. □

It is sufficient to estimate the dimension of the fibre of p_2 from above by the larger dimension of $\mathscr{H}_1$ and $\mathscr{H}_2$. Hence we have

$$\dim_C\mathscr{Y}^f\leqq\dim_C p_2^{-1}f(\Sigma)\leqq\dim_C\mathscr{H}_2+\dim_C f(\Sigma)\leqq m+1\,. \tag{2.1}$$

From (2.1) and $\dim_C Q_{2m}(\boldsymbol{C})=2m$, if $m\geqq 2$, the space $\mathscr{Y}^f$ cannot be equal to $Q_{2m}(\boldsymbol{C})$. It suffices to choose $L\in Q_{2m}(\boldsymbol{C})\setminus\mathscr{Y}^f$. It follows that if $m\geqq 2$, then any horizontal holomorphic map into $\mathscr{Z}_m(\boldsymbol{C}^{2m+2})$ can be deformed continuously through horizontal holomorphic maps to a horizontal holomorphic map into $\mathscr{Z}_{m-1}(\boldsymbol{C}^{2m})$. Thus by induction on dimension m, we obtain the proposition.

PROPOSITION 2.5. *If $m\geqq 2$, then any horizontal holomorphic map $\Sigma\to\mathscr{Z}_m(\boldsymbol{C}^{2m+2})$ can be deformed continuously through horizontal holomorphic maps to a horizontal holomorphic map into $\mathscr{Z}_1(\boldsymbol{C}^4)$.*

Thus we obtain the following statement for harmonic maps.

THEOREM 2.6. *If $m\geqq 2$, then any strongly isotropic harmonic map $\varphi: \Sigma\to Gr_2(\boldsymbol{R}^{2m+2})$ can be deformed continuously through strongly isotropic harmonic maps to a strongly isotropic harmonic map $\Sigma\to Gr_2(\boldsymbol{R}^4)$.*

3. Deformations of harmonic maps of real mixed pairs

(A) Morse-Bott theory over twistor space $Q_n(C)$. Let $G=SO(n+2)$

and $\mathfrak{g}$ denote its Lie algebra. Then we can regard $Q_n(\boldsymbol{C})$ as an orbit of the adjoint representation of G. To consider the height function as Section 2, we fix an useful element P of $\mathfrak{g}$. However we note that the choice of P is different in each case when n is odd or even.

First we treat the case when $n=2l-1$. Let $\mathcal{F}_l=\{F\in Gr_l(\boldsymbol{C}^{n+2})|(F, F)=0\}$. Then we have $\mathcal{F}_l=SO(2l+1)/U(l)$ and the complex dimension of $\mathcal{F}_l$ is $(n+1)(n+3)/8$. Fix $F\in\mathcal{F}_l$ and put $P=\sqrt{-1}\pi_F-\sqrt{-1}\pi_{\bar{F}}\in\mathfrak{g}$. For $X=\sqrt{-1}\pi_L-\sqrt{-1}\pi_{\bar{L}}$ with $L\in Q_n(\boldsymbol{C})$, we define the height function $h^P(X)=\langle\langle X, P\rangle\rangle$.

Then a critical point X of h^P is characterized by $L=L_1\oplus L_2\oplus L_3$ with $L_1\subseteq F, L_2\subseteq\bar{F}, L_3\subseteq(F\oplus\bar{F})^\perp$, where $\boldsymbol{C}^{n+2}=F\oplus\bar{F}\oplus(F\oplus\bar{F})^\perp$. We obtain the following lemma.

LEMMA 3.1. *In the case $n=2l-1$, there are two connected non-degenerate critical manifold of h^P;*

$$C_+=\{L\in Q_n(\boldsymbol{C})|L\subset F\}\cong \boldsymbol{C}P^{l-1},$$

$$C_-=\{L\in Q_n(\boldsymbol{C})|L\subset \bar{F}\}\cong \boldsymbol{C}P^{l-1}.$$

We set $G_P=\{A\in G^C|A(F)=F\}$. In general, we know that the stable manifold for a connected non-degenerate critical manifold N is given by $S^P(N)=G_PX$ for $X\in N$. Then we determine the corresponding stable manifolds by the same way as Lemma 2.2.

LEMMA 3.2. *In the case $n=2l-1$, for two non-degenerate critical manifolds in Lemma 3.1, the corresponding stable manifolds are*

$$S^P(C_+)=C_+,$$

$$S^P(C_-)=\{L\in Q_n(\boldsymbol{C})|L\cap F=\{0\}\},$$

respectively.

Next we treat the case when $n=2l$. Let

$$\mathcal{K}_{l+1}=\{K\in Gr_{l+1}(\boldsymbol{C}^{n+2})|(K, K)=0\}.$$

Then we have $\mathcal{K}_{l+1}=SO(2l+2)/U(l+1)$ and the complex dimension of $\mathcal{K}_l$ is $n(n+2)/8$. Fix $K\in\mathcal{K}_{l+1}$ and put $P=\sqrt{-1}\pi_K-\sqrt{-1}\pi_{\bar{K}}\in\mathfrak{g}$. For $X=\sqrt{-1}\pi_L-\sqrt{-1}\pi_{\bar{L}}$, where $L\in Q_n(\boldsymbol{C})$, in this case, a critical point X of h^P is characterized by $L=L_1\oplus L_2$ with $L_1\subseteq K, L_2\subseteq\bar{K}$, where $\boldsymbol{C}^{n+2}=K\oplus\bar{K}$. We obtain the following two lemmas as above.

LEMMA 3.3. *In the case $n=2l$, there are two connected non-degenerate critical manifolds of h^P;*

$$\mathcal{C}_+=\{L\in Q_n(\boldsymbol{C})\,|\,L\subset K\}\cong \boldsymbol{C}P^l,$$

$$\mathcal{C}_-=\{L\in Q_n(\boldsymbol{C})\,|\,L\subset \bar{K}\}\cong \boldsymbol{C}P^l.$$

LEMMA 3.4. *In the case $n=2l$, for two non-degenerate critical manifolds in Lemma 3.3, the correspondtng stable manifolds are*

$$S^P(\mathcal{C}_+)=\mathcal{C}_+,$$

$$S^P(\mathcal{C}_-)=\{L\in Q_n(\boldsymbol{C})\,|\,L\cap K=\{0\}\},$$

respectively.

REMARK. In Lemma 3.1 and 3.3, the twistor fibration $\pi: Q_n(\boldsymbol{C})\to Gr_2(\boldsymbol{R}^{n+2})$ induces a biholomorphic diffeomorphism

$$\mathcal{C}_-\cong \boldsymbol{C}P^m \ni L \longmapsto (L\oplus \bar{L})_{\boldsymbol{R}}\in Gr_2(\boldsymbol{R}^{n+2})$$

where $m=(n-1)/2$ when n is odd and $m=n/2$ when n is even.

(B) **Deformations of harmonic maps.** Let $\varphi: \Sigma\to Gr_2(\boldsymbol{R}^{n+2})$ be a real mixed pair, and $g: \Sigma\to Q_n(\boldsymbol{C})$ be a holomorphic map correponding to φ. If $g(\Sigma)\subset S^P(\mathcal{C}_-)$, then $\{(\exp\sqrt{-1}tP)\circ g\}_{0\leq t\leq\infty}$ provides a continuous deformation to a holomorphic map into $\mathcal{C}_-$. We shall show that there exists some $F\in\mathcal{F}_l$ in the case $n=2l-1$ or $K\in\mathcal{K}_{l+1}$ in the case $n=2l$, such that $g(\Sigma)\subset S^P(\mathcal{C}_-)$.

In the case $n=2l-1$, we set $\mathcal{Y}_1^g=\{F\in\mathcal{F}_l\,|\,g(z)\not\subset S^P(\mathcal{C}_-)$ for some $z\in\Sigma\}$. Then we have $\mathcal{Y}_1^g=\{F\in\mathcal{F}_l\,|\,g(z)\in F$ for some $z\in\Sigma\}$. It suffices to show that $\mathcal{Y}_1^g$ cannot be equal to $\mathcal{F}_l$. We define $\mathcal{Y}_1=\{(F,L)\in\mathcal{F}_l\times Q_n(\boldsymbol{C})\,|\,L\subset F\}$. Let p_2 be the projection on the second factor of this product. Since the fibere of p_2 is $\mathcal{F}_{l-1}$, we have $\dim_{\boldsymbol{C}}\mathcal{Y}_1^g\leq((n-1)(n+1)/8)+1$. If n is odd greater than or equal to 3, then the space $\mathcal{Y}_1^g$ cannot be equal to $\mathcal{F}_l$. Therefore we can choose $F\in\mathcal{F}_l\setminus\mathcal{Y}_1^g$.

In the case $n=2l$, we set $\mathcal{Y}_2^g=\{K\in\mathcal{K}_{l+1}\,|\,g(z)\not\subset S^P(\mathcal{C}_-)$ for some $z\in\Sigma\}$. Then we have $\mathcal{Y}_2^g=\{K\in\mathcal{K}_{l+1}\,|\,g(z)\in K$ for some $z\in\Sigma\}$. We define $\mathcal{Y}_2=\{(K,L)\in\mathcal{K}_{l+1}\times Q_n(\boldsymbol{C})\,|\,L\subset K\}$. Let p_2 be the projection on the second factor of this product. Since the fibre of p_2 is $\mathcal{K}_l$, we have $\dim_{\boldsymbol{C}}\mathcal{Y}_2^g\leq(n-2)n/8+1$. If n is even greater than or equal to 4, then the space $\mathcal{Y}_2^g$ cannot be equal to $\mathcal{K}_{l+1}$. Then we can choose $K\in\mathcal{K}_{l+1}\setminus\mathcal{Y}_2^g$.

Hence we obtain the following proposition.

PROPOSITION 3.5. *If $n\geq 3$, then any holomorphic map $\phi: \Sigma\to Q_n(\boldsymbol{C})$ can be deformed continuously through holomorphic maps to a holomorphic map into $\boldsymbol{C}P^m$, where $m=(n-1)/2$ when n is odd and $m=n/2$ when n is even. Moreover, ϕ can be deformed continuously to a holomorphic map* *into* $\boldsymbol{C}P^1 \subset \boldsymbol{C}P^m$.

Thus we obtain the following statement for harmonic maps.

THEOREM 3.6. *If* $n \geqq 3$, *then any real mixed pair* $\varphi: \Sigma \to Gr_2(\boldsymbol{R}^{n+2})$ *can be deformed continuously through real mixed pairs to a holomorphic map* $\Sigma \to \boldsymbol{C}P^1 \subset Gr_2(\boldsymbol{R}^4)$.

4. Energy and Degree

We shall give the formula of the energy and the degree for a smooth map $\phi: \Sigma \to Q_n(\boldsymbol{C})$ of a compact Riemann surface. In the case $n \geqq 2$, we suppose that $Q_n(\boldsymbol{C})$ has the maximum c of the sectional curvatures.

First we consider the degree $\deg(\phi)$ of ϕ. There is a natural inclusion $i: Q_n(\boldsymbol{C}) \hookrightarrow \boldsymbol{C}P^{n+1} = Gr_1(\boldsymbol{C}^{n+2})$. We also denote by $\mathcal{L}$ a tautological bundle over $Q_n(\boldsymbol{C})$ pull-backed from the tautological bundle $\mathcal{L}$ over $\boldsymbol{C}P^{n+1}$ by i. In the case $n \geqq 3$, since $\pi_2(Q_n(\boldsymbol{C})) = \boldsymbol{Z}$, the degree of ϕ can be defined for a smooth map $\phi: S^2 \to Q_n(\boldsymbol{C})$ by

$$\deg(\phi) = -c_1(\phi^{-1}\mathcal{L}) \tag{4.1}$$

and that the $\deg(\phi)$ determines the homotopy class of ϕ, namely, for smooth maps $\phi, \phi': \Sigma \to Q_n(\boldsymbol{C})$, $\deg(\phi) = \deg(\phi')$ if and only if ϕ is homotopic to ϕ'.

Next we consider the energy of ϕ, in the case when ϕ is a holomorphic map and in the case when ϕ is a strongly isotropic harmonic map respectively.

Let g_{CP} be the Kähler metric on $\boldsymbol{C}P^{n+1} = Gr_1(\boldsymbol{C}^{n+2})$ induced from the standard Hermitian inner product $\langle\ ,\ \rangle$ through $(\mathrm{T}\boldsymbol{C}P^{n+1})^{1,0} \cong \mathrm{Hom}(\mathcal{L}, \mathcal{L}^\perp)$. Set $\omega_{CP}(X, Y) = g_{CP}(JX, Y)$ for $X, Y \in \mathrm{T}\boldsymbol{C}P^{n+1}$ where J is a complex structure tensor of $\boldsymbol{C}P^{n+1}$. Then it is known that the first Chern class of $\mathcal{L}$ is given by $c_1(\mathcal{L}) = [-(1/2\pi)\omega_{CP}]$. On the other hand, let $\omega_{c'}$ denote the Kähler form on $\boldsymbol{C}P^{n+1}$ of constant holomorphic sectional curvature c'. We remark that in the case $n \geqq 2$, the maximum of sectional curvatures of $Q_n(\boldsymbol{C})$ is the same value c' relative to the Riemannian metric induced through the inclusion $Q_n(\boldsymbol{C}) \subset \boldsymbol{C}P^{n+1}$. Then we restrict the Kähler form $\omega_{c'}$ on $Q_n(\boldsymbol{C})$. We know that the first Chern class of $\mathcal{L}$ is $c_1(\mathcal{L}) = [-(c'/4\pi)\omega_{c'}]$. Assume that ω_{CP} is equal to $\omega_{c'}$ on $Q_n(\boldsymbol{C})$. As $(1/2\pi)\omega_{CP}$ and $(c'/4\pi)\omega_{c'}$ are harmonic forms in the same cohomology class, we have $(1/2\pi)\omega_{CP} = (c'/4\pi)\omega_{c'}$. Then we get $c' = 2$.

Hence we obtain the energy of a holomorphic map ϕ

$$\mathcal{E}(\phi) = \frac{2}{c}\int_\Sigma \phi^*\omega_{CP} = -\frac{4\pi}{c}c_1(\phi^{-1}\mathcal{L}) = \frac{4\pi}{c}\deg(\phi).$$

Then we obtain the following.

PROPOSITION 4.1. *Let $\phi: \Sigma \to Q_n(\boldsymbol{C})$ be a holomorphic map of a compact Riemann surface. Then the energy of ϕ is*

$$\mathcal{E}(\phi)=\frac{4\pi}{c}\deg(\phi)\in\frac{4\pi}{c}\boldsymbol{Z}.$$

Next suppose that ϕ is a strongly isotropic harmonic map. There is a natural inclusion $j: \mathcal{Z}_m(\boldsymbol{C}^{2m+2}) \hookrightarrow Gr_m(\boldsymbol{C}^{2m+2})$. We also denote by $\mathcal{W}$ a tautological bundle over $\mathcal{Z}_m(\boldsymbol{C}^{2m+2})$ pull-backed from the tautological bundle $\mathcal{W}$ over $Gr_m(\boldsymbol{C}^{2m+2})$ by j. Let g_{Gr} be the Kähler metric induced from the standard Hermitian inner product $\langle\ ,\ \rangle$ through $\mathrm{T}Gr_m(\boldsymbol{C}^{2m+2})^{1,0} \cong \mathrm{Hom}(\mathcal{W}, \mathcal{W}^{\perp})$. Set $\omega_{Gr}(X, Y)=g_{Gr}(JX, Y)$. The Kähler metric g_{Gr} induces a Kähler metric on $\mathcal{Z}_1(\boldsymbol{C}^4)$ and we also denote by ω_{Gr} the Kähler form induced on $\mathcal{Z}_1(\boldsymbol{C}^4)$. It is known that the first Chern class of $\mathcal{W}$ is given by $c_1(\mathcal{W})=[-(1/2\pi)\omega_{Gr}]$. On the other hand, for $\mathcal{Z}_1(\boldsymbol{C}^4)\subset \boldsymbol{C}P^3$, let $\omega_{c'}$ denote the Kähler form on $\boldsymbol{C}P^3$ of constant holomorphic sectional curvatuлe c', and we restrict the Kähler form $\omega_{c'}$ on $\mathcal{Z}_1(\boldsymbol{C}^4)$. We know that the first Chern class of $\mathcal{W}$ is $c_1(\mathcal{W})=[-(c'/4\pi)\omega_{c'}]$. If we suppose that ω_{Gr} is equal to $\omega_{c'}$ on $\mathcal{Z}_1(\boldsymbol{C}^4)$, then by a similar argument we get $c'=2$.

Since $g: \Sigma \to \mathcal{Z}_m(\boldsymbol{C}^{2m+2})$ is horizontal holomorphic, we obtain the energy of a strongly isotropic harmonic map ϕ

$$\mathcal{E}(\phi)=\mathcal{E}(g)=\frac{2}{c}\int_{\Sigma} g^*\omega_{Gr}=-\frac{4\pi}{c}c_1(g^{-1}\mathcal{W}).$$

Then we obtain the following.

PROPOSITION 4.2. *Let $\phi: \Sigma\to Q_n(\boldsymbol{C})$ be a strongly isotropic harmonic map of a compact Riemann surface. Then the energy of ϕ is*

$$\mathcal{E}(\phi)=-\frac{4\pi}{c}c_1(g^{-1}\mathcal{W})\in\frac{4\pi}{c}\boldsymbol{Z}.$$

From the above results we can give the energy formula of a harmonic map $\varphi: \Sigma\to Gr_2(\boldsymbol{R}^{n+2})$ as follows.

PROPOSITION 4.3. (1) *Let $\varphi: \Sigma\to Gr_2(\boldsymbol{R}^{n+2})$ be a real mixed pair of a compact Riemann surface. Then the energy of φ is*

$$\mathcal{E}(\varphi)=\mathcal{E}(\phi)=\frac{4\pi}{c}\deg(\phi)\in\frac{4\pi}{c}\boldsymbol{Z},$$

where $\phi: \Sigma\to Q_n(\boldsymbol{C})$ is a lift of φ, which is a holomorphic map.

(2) *Let $\varphi: \Sigma\to Gr_2(\boldsymbol{R}^{n+2})$ be a strongly isotropic harmonic map of a compact Riemann surface. Then the energy of φ is*

$$\mathcal{E}(\varphi)=\mathcal{E}(\phi)=-\frac{4\pi}{c}c_1(g^{-1}\mathcal{W})\in\frac{4\pi}{c}\boldsymbol{Z},$$

where $\phi: \Sigma\to Q_n(\boldsymbol{C})$ *is a lift of* φ, *which is a strongly isotropic harmonic map.*

In the case $n=2m=2$, we have that $\mathcal{Z}_1(\boldsymbol{C}^4)\cong Q_2(\boldsymbol{C})\cong \boldsymbol{C}P^1\times\boldsymbol{C}P^1$. These identifications are as follows: $\boldsymbol{C}P^1\times\boldsymbol{C}P^1\ni(u,v)\mapsto(u,v)\in Q_2(\boldsymbol{C})$ and $\boldsymbol{C}P^1\times\boldsymbol{C}P^1\ni(u,v)\mapsto(u,\bar{v})\in\mathcal{Z}_1(\boldsymbol{C}^4)$. We note that for any $L\in Q_2(\boldsymbol{C})$, a 1-dimensional complex isotropic subspace W of $\boldsymbol{C}^4$ is determined uniquely where $\boldsymbol{C}^4=L\oplus\bar{L}\oplus W\oplus\bar{W}$. Then for $L\in Q_2(\boldsymbol{C})$, we may give two line bundles $\mathcal{L}$ and $\mathcal{W}$ over $Q_2(\boldsymbol{C})$ defined by $(\mathcal{L})_L=L$ and $(\mathcal{W})_L=W$. Denote $\mathcal{L}_1$ and $\mathcal{L}_2$ by the tautological bundle over each factor of $\boldsymbol{C}P^1\times\boldsymbol{C}P^1$. Then we see that $\mathcal{L}=\mathcal{L}_1\otimes\mathcal{L}_2$ and $\mathcal{W}=\mathcal{L}_1\otimes\overline{\mathcal{L}}_2$.

The bi-degree $(\deg_{\mathcal{L}}(\phi), \deg_{\mathcal{W}}(\phi))$ can be defined for a smooth map $\phi=(\phi_1,\phi_2): S^2\to Q_2(\boldsymbol{C})\cong\boldsymbol{C}P^1\times\boldsymbol{C}P^1$. If we define $\deg_{\mathcal{L}}(\phi)=-c_1(\phi^{-1}\mathcal{L})$ and $\deg_{\mathcal{W}}(\phi)=-c_1(\phi^{-1}\mathcal{W})$, then

$$\deg_{\mathcal{L}}(\phi)=\deg(\phi_1)+\deg(\phi_2),\qquad \deg_{\mathcal{W}}(\phi)=\deg(\phi_1)-\deg(\phi_2).$$

Since $\pi_2(Q_2(\boldsymbol{C}))=\pi_2(\boldsymbol{C}P^1\times\boldsymbol{C}P^1)=\boldsymbol{Z}\oplus\boldsymbol{Z}$, for smooth maps $\phi,\phi': \Sigma\to Q_2(\boldsymbol{C})$, $\deg_{\mathcal{L}}(\phi)=\deg_{\mathcal{L}}(\phi')$ and $\deg_{\mathcal{W}}(\phi)=\deg_{\mathcal{W}}(\phi')$ if and only if ϕ is homotopic to ϕ'.

5. On connectedness of the space of harmonic maps into $Q_n(\boldsymbol{C})$ (Proof of Theorem A)

Suppose that $\Sigma=S^2$. Before we mention the connectedness of the space of harmonic maps $S^2\to Gr_2(\boldsymbol{R}^{n+2})$, we shall treat the case of harmonic maps $S^2\to Q_n(\boldsymbol{C})$.

First we shall consider the case $n=2$. [BW1] showed that for a harmonic map $\phi=(\phi_1,\phi_2): S^2\to Q_2(\boldsymbol{C})$ with $\phi_1,\phi_2: S^2\to\boldsymbol{C}P^1$, since ϕ_1 and ϕ_2 are holomorphic or antiholomorphic, the pairs $\{\phi_1,\phi_2\}$ or $\{\phi_1,\bar{\phi}_2\}$ are both holomorphic or both antiholomorphic. If we fix $\deg_{\mathcal{L}}(\phi)$ and $\deg_{\mathcal{W}}(\phi)$, then they determine $\deg(\phi_1)$ and $\deg(\phi_2)$. Then we obtain Theorem A(1).

We know that in general the energy of ϕ is $\mathcal{E}(\phi)\geqq|(4\pi/c)\deg(\phi)|$ and the equality holds if and only if ϕ is holomorphic or antiholomorphic.

Now we consider the case $n\geqq 3$. Suppose that $\phi: S^2\to Q_n(\boldsymbol{C})$ is a holomorphic map. Then combining Proposition 3.5 and the fact the space of holomorphic maps of S^2 into $\boldsymbol{C}P^1$ with fixed degree is path-connected, we obtain Theorem A(3).

Next suppose that $\phi: S^2\to Q_n(\boldsymbol{C})$ is a strongly isotropic harmonic map. We may assume that ϕ is neither holomorphic nor antiholomorphic. Let $g: S^2\to\mathcal{Z}_m(\boldsymbol{C}^{2m+2})$ corresponding to φ be a horizontal holomorphic map and

$g'=(g_1', g_2'): S^2 \to \mathcal{Z}_1(\boldsymbol{C}^4) \cong Q_2(\boldsymbol{C}) \cong \boldsymbol{C}P^1 \times \boldsymbol{C}P^1$ be a horizontal holomorphic map which is obtained by continuous deformation of g in Proposition 2.5.

Then we can restate the energy $\mathcal{E}(\phi)$ and the degree $\deg(\phi)$ of ϕ in (4.1) and Proposition 4.2, using $g'=(g_1', g_2')$ as follows:

$$\mathcal{E}(\phi) = -\frac{4\pi}{c} c_1(g^{-1}\mathcal{W}) = -\frac{4\pi}{c} c_1(g'^{-1}\mathcal{W}) = \frac{4\pi}{c}(\deg(g_1') - \deg(g_2'))$$

and

$$\deg(\phi) = -c_1(\phi^{-1}\mathcal{L}) = -c_1(g^{-1}\pi^{-1}\mathcal{L}) = -c_1(g'^{-1}\pi^{-1}\mathcal{L}) = \deg(g_1') + \deg(g_2').$$

From the assumption of ϕ, $\mathcal{E}(\phi) \neq |(4\pi/c)\deg(\phi)|$. Then the energy $\mathcal{E}(\phi)$ and the degree $\deg(\phi)$ determine $\deg(g_1')$ and $\deg(g_2')$. Thus Theorem A(2) follows from Theorem A(1).

This completes the proof of Theorem A.

6. On connectedness of the space of harmonic maps into $Gr_2(\boldsymbol{R}^{n+2})$ (Proof of Theorem B)

In this section, we consider connectedness of the space of harmonic maps $\varphi: S^2 \to Gr_2(\boldsymbol{R}^{n+2})$ which are strongly isotropic or real mixed pairs.

We remark that any Kähler form cannot be defined entirely on $Gr_2(\boldsymbol{R}^{n+2})$, because the second cohomology group of $Gr_2(\boldsymbol{R}^{n+2})$ is $H^2(Gr_2(\boldsymbol{R}^{n+2}), \boldsymbol{Z}) = \boldsymbol{Z}_2$. Then to introduce a homotopy invariant as *degree* for φ, using a lift ϕ of φ to $Q_n(\boldsymbol{C})$ which is a Kähler manifold, we define $d(\varphi) := |\deg(\phi)|$.

Now we must note the following. For a smooth map $\phi: S^2 \to Q_n(\boldsymbol{C})$ with $\deg(\phi) \neq 0$, $\bar{\phi}$ cannot be homotopic to ϕ, because $\deg(\phi) = -\deg(\bar{\phi})$. However, for smooth maps $\varphi = \pi \circ \phi$ and $\varphi' = \pi \circ \phi'$ with $\phi, \phi': S^2 \to Q_n(\boldsymbol{C})$, if $\deg(\phi) = \pm \deg(\phi')$, i.e., $d(\varphi) = d(\varphi')$, then φ is homotopic to φ'.

In the case $n=2$, if we fix

$$d_{\mathcal{L}}(\varphi) = |\deg_{\mathcal{L}}(\phi)| = |\deg(\phi_1) + \deg(\phi_2)|,$$

$$d_{\mathcal{W}}(\varphi) = |\deg_{\mathcal{W}}(\phi)| = |\deg(\phi_1) - \deg(\phi_2)|,$$

and signature ε ($=1$ or -1) of $(\deg_{\mathcal{L}}(\phi) \cdot \deg_{\mathcal{W}}(\phi))$, then in the above sense, we can characterize the homotopy class for a smooth map $S^2 \to Gr_2(\boldsymbol{R}^4)$. Thus we obtain Theorem B(1).

In the case $n \geqq 3$, Theorem B(2) and (3) follow from Theorem A(2) and (3).

Hence we have completed the proof of Theorem B.

7. Remarks and Conjectures

We shall remark the relation of our results with the construction theory of all harmonic maps $\varphi: S^2 \to Gr_2(\boldsymbol{R}^{n+2})$. According to the classification theory of [BW1], there are four classes of harmonic 2-spheres in $Gr_2(\boldsymbol{R}^{n+2})$ as follows; (I) strongly isotropic and ∂'-reducible, (II) strongly isotropic and ∂'-irreducible, (III) finite isotropy order and ∂'-reducible, (IV) finite isotropy order and ∂'-irreducible.

Bahy-El-Dien and Wood [BW1] showed that if a harmonic map $\varphi: S^2 \to Gr_2(\boldsymbol{R}^{n+2})$ is of class (III), then φ is a real mixed pair. If a harmonic map $\varphi: S^2 \to Gr_2(\boldsymbol{R}^{n+2})$ is of class (IV), then φ can be lifted to a horizontal holomorphic map into neither $Q_n(\boldsymbol{C})$ nor $\mathscr{Z}_m(\boldsymbol{C}^{2m+2})$. However they showed that φ of class (IV) can be transformed to a map of class (III) after a finite number of forward and backward replacements.

Theorem B implies the connectedness of the space of harmonic 2-spheres in $Gr_2(\boldsymbol{R}^{n+2})$ of class (I), (II) or (III). Then it is very interesting to investigate the deformations of harmonic 2-spheres in $Gr_2(\boldsymbol{R}^{n+2})$ of class (IV) and to determine the connectedness problem of the space of *all* harmonic 2-spheres in $Gr_2(\boldsymbol{R}^{n+2})$.

Conjecture. Is the space of all harmonic 2-spheres in $Gr_2(\boldsymbol{R}^{n+2})$ (resp. $Q_n(\boldsymbol{C})$), $n \geqq 3$, with fixed energy and fixed d (resp. degree) path-connected?

The similar results have already obtained by [Mu] in the case $\boldsymbol{H}P^n$.

Conjecture. Is the space of all harmonic 2-spheres in $\boldsymbol{H}P^n$, $n \geqq 2$, with fixed energy path-connected?

More generally let M be a compact Riemannian symmetric space of inner type and $\mathscr{T}(M)$ be the standard twistor space over M with the projection $\pi: \mathscr{T}(M) \to M$, which was classified by Bryant [Br] and Salamon [Sa]. We call a harmonic map $\varphi: \Sigma \to M$ which can be lifted to a horizontal holomorphic map into $\mathscr{T}(M)$ *isotropic harmonic map, strongly pseudo-holomorphic map* or *branched superminimal immersion*. Especially, we obtained similar results on the connectedness of the *isotropic* harmonic 2-spheres in the classical Riemannian symmetric spaces M of inner type. We shall discuss these results elsewhere.

Conjecture. Let M be a Riemannian symmetric space of compact type. Is the space of harmonic 2-spheres in M with fixed energy and fixed homotopy class path-connected?

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