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Connections on the Special 5-star Graphs

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0. Introduction

Since V. Jones in [7] introduced the index theory for subfactors of finite factors, the theory of subfactors has been related to many fields of mathematics and physics. One of the most interesting cases of subfactors is the case of irreducible subfactors, those for which the relative commutant is reduced to the scalars. Here V. Jones proved that any subfactor with an index smaller than 4 is automatically irreducible. It remains open to characterize the possible values >4 of the index of irreducible subfactors of the hyperfinite II_1-factor.

A. Ocneanu introduced paragroups, which is a certain quantization of finite groups, for the classification of subfactors, [10], [11]. He defined some structures on finite graphs, called *connections*, regarding paragroups as discrete manifolds. The flatness of a connection, which is the directe analogue from differential geometry, plays an important role in the classification theory of subfactors [6], [8], [9], [11].

The commuting square is the key idea in subfactor theory as seen from [12], [13] etc. By using suitable commuting squares of finite dimensional von Neumann algebras, several authors have constructed irreducible subfactors of the hyperfinite II_1-factor [5], [3], [16], [17], [18]. Especially, U. Haagerup and J. Schou established a criterion for the existence of symmetric commuting squares of finite dimensional von Neumann algebras, which is deeply concerned with the axioms of Ocneanu's paragroups on finite graphs. They derived the following useful result: If a connected bipartite graph with a vertex which is connected to only one other vertex of the graph has a connection, then there exists an irreducible subfactor of the hyperfinite II_1-factor whose index is the square of the norm of this graph. Applying this result to the 4-star graphs and to some kite graphs, they produced infinitely many irreducible subfactors with indices in the interval $(4, 5)$ [5], [16]. And V. Sunder in [17] obtained a new series of irreducible subfactors by using their criterion. The reader can find more interesting results on the symmetric commuting squares in J. Schou's thesis [16].

Ocneanu's theory of paragroups is closely related to solvable lattice models in statistical mechanics where the connection corresponds to Boltzmann weight in lattice models [9], [15]. Recently the idea of an orbifold of a solvable lattice models has been used in several papers. Especially, Y. Kawahigashi in [8] gave a poof for the flatness of Ocneanu's connection on the Dynkin diagram of type D_n by using this idea, and he and D. Evans in [2] obtained a new series of subfactors by applying the notion of orbifold models of $SU(N)$ to the Hecke algebra subfactors of Wenzl. Moreover, in [6] M. Izumi and Y. Kawahigashi filled Popa's table of classification for subfactor with index 4 in [14] by considering the orbifold of the extended Dynkin diagram $A^{(1)}_{2n-5}$. And Ph. Roche investigated the symmetry between two graphs and gave a procedure for constructing a new lattice model from a given lattice model associated with the graph, where the new lattice model is associated with the new graph obtained from the symmetry of the original graph. The symmetry between two bipartite graphs in the sense of Roche [15] is closely related to the duality described by M. Choda in [1].

In this paper, we apply Roche's procedure to some extended kite graphs $B(n, m, n, m)$ where $n \geqq m \geqq 0$, used in the previous paper [19], and we constructed connections on the 5-star graphs $S(n, m+1, m+1, m+1, m+1)$, obtained by a $\boldsymbol{Z}_2$-symmetry of the original graphs. In general, we consider the symmetric graph and obtain the new graph by its symmetry. We find the cell system, which gives us an embedding of the string algebra of the original graph into that of the new graph. To find the cell system is generally difficult, but since in our case the symmetry is coming from the graphical symmetry it is not so difficult. The connection on the original graph is affiliated to the string algebras of the original graph. We define the connection on the new graph so that it is compatible with the embedding is given by the cell system. In this construction, the star triangle relations (Yang-Baxter equations) give us the link between these two connections. In the last section of this paper, we list some remarks on the symmetries of Sunders's graphs in [17].

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1. The symmetry between graphs and the cell system

We shall begin with recalling the definition of the symmetry between graphs and the cell system in [15]. For a given connected bipartite graph

$\mathcal{G}$, we denote its vertices and edges by $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(1)}$, respectively.

DEFINITION 1.1. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two bipartite graphs with distinguished vertices $*_1$ and $*_2$. Let G_1 and G_2 be their adjacency matrices, respectively. We say that there exist a *symmetry* from $\mathcal{G}_1$ to $\mathcal{G}_2$ if there exists a non-negative integer entried $^{\#}(\mathcal{G}_1^{(0)})\times{}^{\#}(\mathcal{G}_2^{(0)})$ matrix C satisfying the following conditions.

(1) $G_1C=CG_2$,

(2) $c_{*_1,i}=1$ if and only if $i=*_2$,

(3) For any i, there exists j such that $c_{ij}\neq 0$.

REMARK 1.2. If there exists a symmetry from $\mathcal{G}_1$ to $\mathcal{G}_2$ then two matrices G_1 and G_2 have the same Perron-Frobenius eigenvalue.

We define the graph $\mathcal{H}$ as $\mathcal{H}^{(0)}=\mathcal{G}_1^{(0)}\cup\mathcal{G}_2^{(0)}$ and $\mathrm{Path}^1_{x,y}(\mathcal{H})=\phi$ if x and y are both in $\mathcal{G}_1^{(0)}$ or $\mathcal{G}_2^{(0)}$, and $\mathrm{Path}^1_{x,y}(\mathcal{H})=c_{x,y}$, where $\mathrm{Path}^1_{x,y}(\mathcal{H})$ is the set of paths of length 1 from x to y.

Next we recall the *cell system* between $\mathcal{G}_1$ and $\mathcal{G}_2$, which gives us the embedding of the string algebra of $\mathcal{G}_1$ into that of $\mathcal{G}_2$ [10, 15].

DEFINITION 1.3. The cell system is a family of complex numbers associated with the following cells,

$$\begin{array}{ccc} p_1 & \xrightarrow{\;\eta\;} & p_2 \\ {\scriptstyle \xi_1}\big\downarrow & & \big\downarrow{\scriptstyle \xi_2} \\ q_1 & \xrightarrow[\;\eta'\;]{} & q_2 \end{array},$$

where $\xi_1\in\mathcal{G}_1^{(1)}$, $\xi_2\in\mathcal{G}_2^{(1)}$ and $\eta,\eta'\in\mathcal{H}^{(1)}$, and $p_1=s(\eta)=s(\xi_1)$, $p_2=r(\eta)=s(\xi_2)$, $q_1=r(\xi_1)=s(\eta')$, $q_2=r(\xi_2)=r(\eta')$. Here $s(\chi)$ and $r(\chi)$ denote the source and the range of the edge χ, respectively.

And this family of complex numbers satisfies the following conditions.

(1) Unitarity

$$\sum_{p_2,\eta_1,\xi_2}\left(\begin{array}{ccccc} & & p_1 & & \\ & \xi_1\swarrow & & \searrow\eta_1 & \\ q_1 & & & & p_2 \\ & \eta_2\searrow & & \swarrow\xi_2 & \\ & & q_2 & & \end{array}\right)\left(\begin{array}{ccccc} & & p_1 & & \\ & \eta_1\swarrow & & \searrow\xi_1' & \\ p_2 & & & & q_1' \\ & \xi_2\searrow & & \swarrow\eta_3 & \\ & & q_2 & & \end{array}\right)=\delta_{q_1q_1'}\delta_{\xi_1\xi_1'}\delta_{\eta\eta,\eta_3}$$

(2) Trace preserving

$$\sum_{q_2,\eta_2,\xi_2} \left(\begin{array}{ccccc} & & q_1 & & \\ & \xi_1 \nearrow & & \searrow \eta_2 & \\ p_1 & & & & q_2 \\ & \eta_1 \searrow & & \nearrow \xi_2 & \\ & & p_2 & & \end{array}\right) \left(\begin{array}{ccccc} & & q_1 & & \\ & \eta_2 \swarrow & & \nwarrow \bar{\xi}_1 & \\ q_2 & & & & \bar{p}_1 \\ & \xi_2 \nwarrow & & \swarrow \eta_4 & \\ & & p_2 & & \end{array}\right) \frac{\mu(p_1)\mu(q_2)}{\mu(q_1)\mu(p_2)} = \delta_{p_1\bar{p}_1}\delta_{\xi_1\bar{\xi}_1}\delta_{\eta_1,\eta_4},$$

where $\mu(x)$ is the Perron-Frobenius weight at the vertex x. In the above relations, we use the conventions

$$\begin{array}{ccc} p_1 & \xrightarrow{\ \eta\ } & p_2 \\ \xi_1 \big\downarrow & & \big\downarrow \xi_2 \\ q_1 & \xrightarrow[\ \eta'\ ]{} & q_2 \end{array} = \begin{array}{ccccc} & & p_1 & & \\ & \xi_1 \swarrow & & \searrow \eta & \\ q_1 & & & & p_2 \\ & \eta' \searrow & & \swarrow \xi_2 & \\ & & q_2 & & \end{array}$$

and

$$\begin{array}{ccc} p_2 & \xleftarrow{\ \eta\ } & p_1 \\ \xi_2 \big\downarrow & & \big\downarrow \xi_1 \\ q_2 & \xleftarrow[\ \eta'\ ]{} & q_1 \end{array} = \overline{\begin{array}{ccc} p_1 & \xrightarrow{\ \eta\ } & p_2 \\ \xi_1 \big\downarrow & & \big\downarrow \xi_2 \\ q_1 & \xrightarrow[\ \eta'\ ]{} & q_2 \end{array}}\ .$$

As we mentioned, we can embed the string algebra of $\mathscr{G}_1$ into that of $\mathscr{G}_2$. And we would like to have the identification of strings, which is compatible with the embedding, by connection.

For this purpose, it is enough to require so called the star triangle relations are satisfied [8, Lemma 5.1], [15, Proposition 5]. That is, for any hexagon of the form,

$$\begin{array}{ccccc} & p_1 & \longrightarrow & p_2 & \\ \swarrow & & & & \searrow \\ q_1 & & & & q_2 \\ \searrow & & & & \swarrow \\ & r_1 & \longrightarrow & r_2 & \end{array}$$

where the left side is in the graph $\mathscr{G}_1$, the right side is in the graph $\mathscr{G}_2$, and the horizontal edges are in $\mathscr{H}$, we require the equation,

(*) The star triangle relation

$$\sum_{q_1',\xi_1,\xi_1',\eta_1} \begin{array}{ccccccc} & & p_1 & \longrightarrow & p_2 & & \\ & \swarrow & & \searrow^{\xi_1} & & \searrow & \\ q_1 & & & q_1' & \xrightarrow{\eta_1} & & q_2 \\ & \searrow & & \swarrow_{\xi_1'} & & \swarrow & \\ & & r_1 & \longrightarrow & r_2 & & \end{array} = \sum_{q_2',\xi_2,\xi_2',\eta_1} \begin{array}{ccccccc} & & p_1 & \longrightarrow & p_2 & & \\ & \swarrow & & \swarrow^{\xi_2} & & \searrow & \\ q_1 & & \xrightarrow{\eta_2} & q_2' & & & q_2 \\ & \searrow & & \searrow_{\xi_2'} & & \swarrow & \\ & & r_1 & \longrightarrow & r_2 & & \end{array} .$$

Each hexagon in the above equation is consisted from three cells. This means we multiply the weights of these three cells.

What we would like to do, is to get a connection on the graph $\mathcal{G}_2$ starting from that on the graph $\mathcal{G}_1$ by applying Roche's procedure [15, p. 405]. Here we rewrite his procedure in our terms.

1. Begin with a suitable connection on the graph $\mathcal{G}_1$.
2. Look at the symmetries of $\mathcal{G}_1$ and find the graph $\mathcal{G}_2$ and $\mathcal{H}$.
3. Look for a suitable cell system.
4. Define a connection on $\mathcal{G}_1$ so that the star triangle relations are satisfied.

For a while from now on, we use the graph $B(n, m, n, m)$ in [19, § 4] as the graph $\mathcal{G}_1$. This graph has the central square and legs of the same length diagonally across.

$\mathcal{G}_1$:

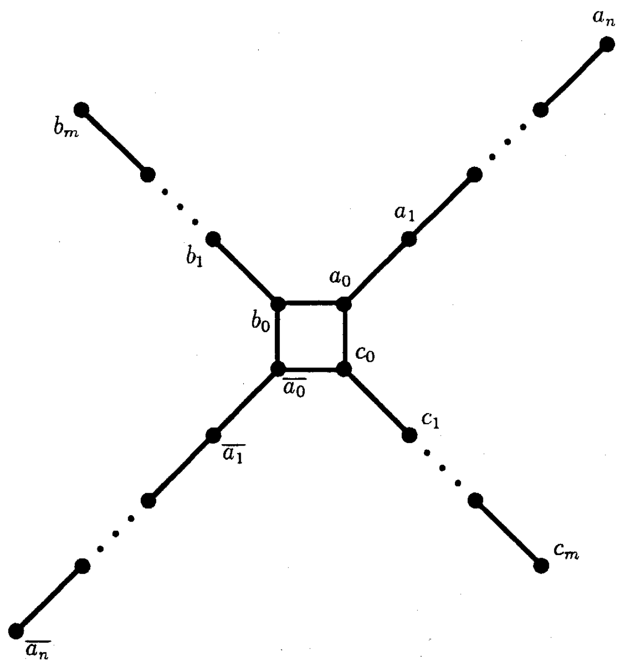


Figure 1.1.

We label the vertices of the graph $B(n, m, n, m)$ as above. Then there exists the symmetry from the graph $B(n, m, n, m)$ to the 5-star graph $S(n, m+1, m+1, m+1, m+1)$,

$\mathcal{G}_2$:

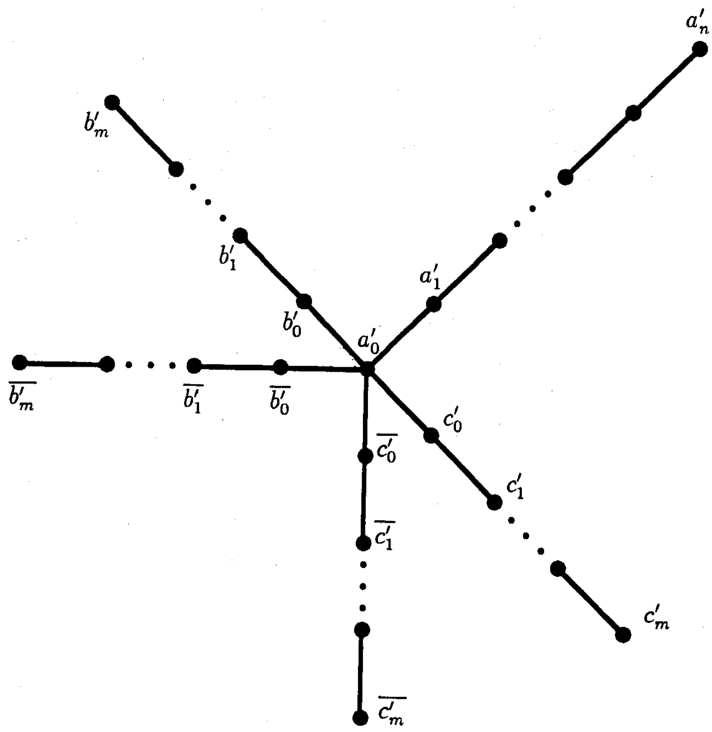


Figure 1.2.

Indeed, the matrix C is represented in the graph

$$\mathcal{H}: \quad \begin{array}{ccc} a_i & & \\ & \searrow & \\ & & a'_j \\ & \nearrow & \\ \overline{a_i} & & \end{array} \quad (i=0, 1, \cdots, n), \quad \begin{array}{ccc} & & b'_j \\ & \nearrow & \\ b_j & & \\ & \searrow & \\ & & \overline{b'_j} \end{array}, \quad \begin{array}{ccc} & & c'_j \\ & \nearrow & \\ c_j & & \\ & \searrow & \\ & & \overline{c'_j} \end{array} \quad (j=0, 1, \cdots, m)$$

Of course, the above symmetry is coming from the reflection for the 2nd-4th line, which is $\mathbf{Z}_2$-automorphism σ on the graph $B(n, m, n, m)$. And the Perron-Frobenius weights on the vertices of the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are as follows.

$\mathcal{G}_1$:

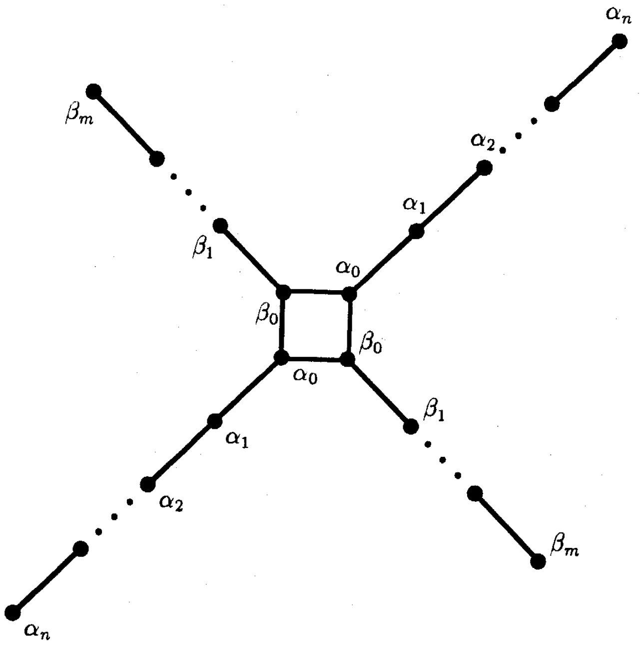


Figure 1.3.

$\mu(a_i)=\mu(\overline{a_i})=\alpha_i$ $(i=0, 1, \cdots, n)$ and $\mu(b_j)=\mu(\overline{b_j})=\beta_j$ $(j=0, 1, \cdots, m)$,

$\mathcal{G}_2$:

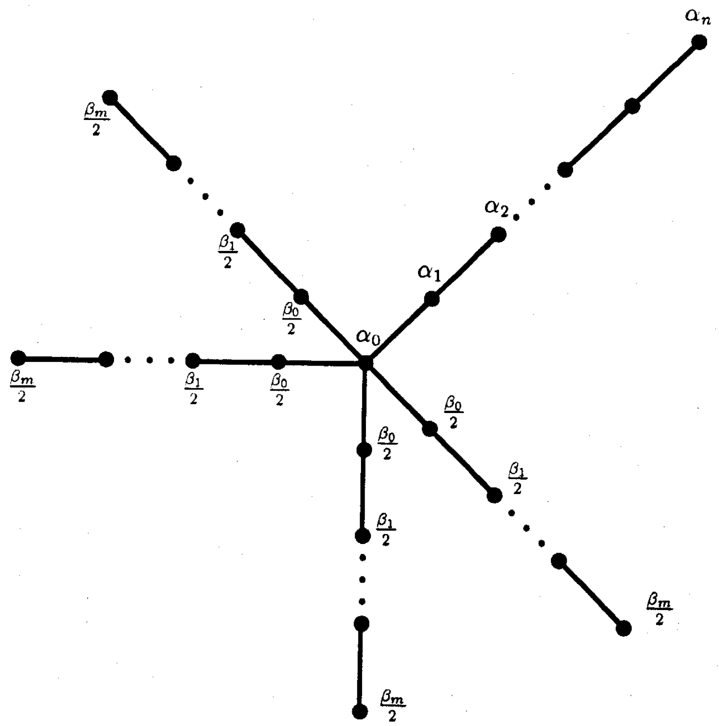


Figure 1.4.

$\mu(a'_i)=\alpha_i$ $(i=0, 1, \cdots, n)$ and $\mu(b'_j)=\mu(\overline{b'_j})=\mu(c'_j)=\mu(\overline{c'_j})=\beta_j/2$ $(j=0, 1, \cdots, m)$.

That is, the weghts on the fixed points under the $\mathbf{Z}_2$-automorphism σ

split equally. Here α_i $(i=0, 1, 2, \cdots, n)$ and β_j $(j=0, 1, 2, \cdots, m)$ satisfy the following relations.

LEMMA 1.4. ([19, Lemma 4.1])

(1) $\alpha_n^2=\alpha_i^2-\alpha_{i+1}\alpha_{i-1}$ *for* $i=1, 2, \cdots, n-1$,

(2) $\alpha_n^2=\alpha_0^2-2\alpha_1\beta_0$,

(3) $\beta_m^2=\beta_j^2-\beta_{j+1}\beta_{j-1}$ *for* $j=1, 2, \cdots, m-1$,

(4) $\beta_m^2=\beta_0^2-2\alpha_0\beta_1$,

(5) $2(\alpha_0^2-\beta_0^2)=\alpha_1\beta_0-\alpha_0\beta_1>0$.

Next we give a cell system between the graphs $B(n, m, n, m)$ and the 5-star graph $S(n, m+1, m+1, m+1, m+1)$. In this case, we can find the following unitary cell system.

$$\begin{array}{ccc} a_i & \longrightarrow & a_i' \\ \downarrow & & \downarrow \\ a_{i+1} & \longrightarrow & a_{i+1}' \end{array} = \begin{array}{ccc} \overline{a_i} & \longrightarrow & a_i' \\ \downarrow & & \downarrow \\ \overline{a_{i+1}} & \longrightarrow & a_{i+1}' \end{array} =1, \quad (i=0, 1, \cdots, n-1)$$

$$\begin{array}{ccc} a_i & \longrightarrow & a_i' \\ \downarrow & & \downarrow \\ a_{i-1} & \longrightarrow & a_{i-1}' \end{array} = \begin{array}{ccc} \overline{a_i} & \longrightarrow & a_i' \\ \downarrow & & \downarrow \\ \overline{a_{i-1}} & \longrightarrow & a_{i-1}' \end{array} =1, \quad (i=1, 2, \cdots, n)$$

$$\begin{pmatrix} \begin{array}{ccc} a_0 & \longrightarrow & a_0' \\ \downarrow & & \downarrow \\ b_0 & \longrightarrow & b_0' \end{array} & \begin{array}{ccc} a_0 & \longrightarrow & a_0' \\ \downarrow & & \downarrow \\ b_0 & \longrightarrow & \overline{b_0'} \end{array} \\ \begin{array}{ccc} \overline{a_0} & \longrightarrow & a_0' \\ \downarrow & & \downarrow \\ b_0 & \longrightarrow & b_0' \end{array} & \begin{array}{ccc} \overline{a_0} & \longrightarrow & a_0' \\ \downarrow & & \downarrow \\ b_0 & \longrightarrow & \overline{b_0'} \end{array} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \begin{array}{ccc} b_0 & \longrightarrow & b_0' \\ \downarrow & & \downarrow \\ a_0 & \longrightarrow & a_0' \end{array} & \begin{array}{ccc} b_0 & \longrightarrow & b_0' \\ \downarrow & & \downarrow \\ \overline{a_0} & \longrightarrow & a_0' \end{array} \\ \begin{array}{ccc} b_0 & \longrightarrow & \overline{b_0'} \\ \downarrow & & \downarrow \\ a_0 & \longrightarrow & a_0' \end{array} & \begin{array}{ccc} b_0 & \longrightarrow & \overline{b_0'} \\ \downarrow & & \downarrow \\ \overline{a_0} & \longrightarrow & a_0' \end{array} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cccc} a_0 \longrightarrow a_0' & & a_0 \longrightarrow a_0' \\ \downarrow \qquad\quad \downarrow & & \downarrow \qquad\quad \downarrow \\ c_0 \longrightarrow c_0' & & c_0 \longrightarrow \overline{c_0'} \\ \overline{a_0} \longrightarrow a_0' & & \overline{a_0} \longrightarrow a_0' \\ \downarrow \qquad\quad \downarrow & & \downarrow \qquad\quad \downarrow \\ c_0 \longrightarrow c_0' & & c_0 \longrightarrow \overline{c_0'} \end{array}\right) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cccc} c_0 \longrightarrow c_0' & & c_0 \longrightarrow c_0' \\ \downarrow \qquad\quad \downarrow & & \downarrow \qquad\quad \downarrow \\ a_0 \longrightarrow a_0' & & \overline{a_0} \longrightarrow a_0' \\ c_0 \longrightarrow \overline{c_0'} & & c_0 \longrightarrow \overline{c_0'} \\ \downarrow \qquad\quad \downarrow & & \downarrow \qquad\quad \downarrow \\ a_0 \longrightarrow a_0' & & \overline{a_0} \longrightarrow a_0' \end{array}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\begin{array}{c} b_j \longrightarrow b_j' \\ \downarrow \qquad\quad \downarrow \\ b_{j+1} \longrightarrow b_{j+1}' \end{array} = \begin{array}{c} \overline{b_j} \longrightarrow b_j' \\ \downarrow \qquad\quad \downarrow \\ \overline{b_{j+1}} \longrightarrow b_{j+1}' \end{array} = 1, \quad (j=0, 1, \cdots, m-1)$$

$$\begin{array}{c} b_j \longrightarrow b_j' \\ \downarrow \qquad\quad \downarrow \\ b_{j-1} \longrightarrow b_{j-1}' \end{array} = \begin{array}{c} \overline{b_j} \longrightarrow b_j' \\ \downarrow \qquad\quad \downarrow \\ \overline{b_{j-1}} \longrightarrow b_{j-1}' \end{array} = 1, \quad (j=1, 2, \cdots, m)$$

$$\begin{array}{c} c_j \longrightarrow c_j' \\ \downarrow \qquad\quad \downarrow \\ c_{j+1} \longrightarrow c_{j+1}' \end{array} = \begin{array}{c} \overline{c_j} \longrightarrow c_j' \\ \downarrow \qquad\quad \downarrow \\ \overline{c_{j+1}} \longrightarrow c_{j+1}' \end{array} = 1, \quad (j=0, 1 \cdots, m-1)$$

$$\begin{array}{c} c_j \longrightarrow c_j' \\ \downarrow \qquad\quad \downarrow \\ c_{j-1} \longrightarrow c_{j-1}' \end{array} = \begin{array}{c} \overline{c_j} \longrightarrow c_j' \\ \downarrow \qquad\quad \downarrow \\ \overline{c_{j-1}} \longrightarrow c_{j-1}' \end{array} = 1, \quad (j=1, 2, \cdots, m)$$

It is easy to check that the above cell system satisfies the conditions (1) and (2) of Definition 1.2.

Using this cell system we can construct a connection on the 5-star graph $S(n, m+1, m+1, m+1, m+1)$ according to Roche's procedure. In order to perform his procedure, we have to begin with the symmetry invariant connection of the graph $B(n, m, n, m)$ [15, p. 405]. Thus we shall construct the symmetry invariant connection on the graph $B(n, m, n, m)$ in the next section.

2. Z_2 symmetric connection and the star triangle relations

In this section, first we reconstruct the symmety invariant connection on $B(n, m, n, m)$. This is, we construct the connection W such that the conditions,

$$W\begin{pmatrix} & a & \\ \swarrow & & \searrow \\ b & & c \\ \searrow & & \swarrow \\ & d & \end{pmatrix} = W\begin{pmatrix} & \sigma(a) & \\ \swarrow & & \searrow \\ \sigma(b) & & \sigma(c) \\ \searrow & & \swarrow \\ & \sigma(d) & \end{pmatrix} \tag{2.1}$$

for all admissible cells, are satisfied.

We shall concentrate our interest upon the cell weights around the central square of $B(n, m, n, m)$. Because the determinations of the cell weight at the points left on the legs, are the routine argument.

We put cell weights on the central square as follows.

$$w_1 = W\begin{pmatrix} & a_0 & \\ \swarrow & & \searrow \\ b_0 & & b_0 \\ \searrow & & \swarrow \\ & a_0 & \end{pmatrix},\quad w_2 = W\begin{pmatrix} & a_0 & \\ \swarrow & & \searrow \\ b_0 & & c_0 \\ \searrow & & \swarrow \\ & a_0 & \end{pmatrix},\quad w_3 = W\begin{pmatrix} & a_0 & \\ \swarrow & & \searrow \\ c_0 & & c_0 \\ \searrow & & \swarrow \\ & a_0 & \end{pmatrix}, \tag{2.2}$$

$$w_4 = W\begin{pmatrix} & \overline{a_0} & \\ \swarrow & & \searrow \\ b_0 & & b_0 \\ \searrow & & \swarrow \\ & \overline{a_0} & \end{pmatrix},\quad w_5 = W\begin{pmatrix} & \overline{a_0} & \\ \swarrow & & \searrow \\ b_0 & & c_0 \\ \searrow & & \swarrow \\ & \overline{a_0} & \end{pmatrix},\quad w_6 = W\begin{pmatrix} & \overline{a_0} & \\ \swarrow & & \searrow \\ c_0 & & c_0 \\ \searrow & & \swarrow \\ & \overline{a_0} & \end{pmatrix},$$

$$w_4 = W\begin{pmatrix} & a_0 & \\ \swarrow & & \searrow \\ b_0 & & b_0 \\ \searrow & & \swarrow \\ & \overline{a_0} & \end{pmatrix},\quad w_5 = W\begin{pmatrix} & a_0 & \\ \swarrow & & \searrow \\ b_0 & & c_0 \\ \searrow & & \swarrow \\ & \overline{a_0} & \end{pmatrix},\quad w_6 = W\begin{pmatrix} & a_0 & \\ \swarrow & & \searrow \\ c_0 & & c_0 \\ \searrow & & \swarrow \\ & \overline{a_0} & \end{pmatrix},$$

Then we get the following matrices of the weights around the central square by applying renormalization rules.

The 3×3-matrix of the weight $W\begin{pmatrix} & a_0 & \\ \swarrow & & \searrow \\ * & & * \\ \searrow & & \swarrow \\ & a_0 & \end{pmatrix}$ becomes

$$\begin{pmatrix} \dfrac{\alpha_n}{\alpha_0}e^{i\theta_1} & \dfrac{\sqrt{\alpha_1\beta_0}}{\alpha_0}e^{i\theta_2} & \dfrac{\sqrt{\alpha_1\beta_0}}{\alpha_0}e^{i\theta_3} \\ \dfrac{\sqrt{\alpha_1\beta_0}}{\alpha_0}e^{i\theta_2} & w_1 & w_2 \\ \dfrac{\sqrt{\alpha_1\beta_0}}{\alpha_0}e^{i\theta_3} & w_2 & w_3 \end{pmatrix}, \tag{2.3}$$

and the 3×3-matrix of the weights $W\left(\begin{array}{ccc} & \overline{a_0} & \\ \swarrow & & \searrow \\ * & & * \\ \searrow & & \swarrow \\ & \overline{a_0} & \end{array}\right)$ becomes

$$(2.4)\qquad \begin{pmatrix} \frac{\alpha_n}{\alpha_0}e^{i\theta_4} & \frac{\sqrt{\alpha_1\beta_0}}{\alpha_0}e^{i\theta_5} & \frac{\sqrt{\alpha_1\beta_0}}{\alpha_0}e^{i\theta_6} \\ \frac{\sqrt{\alpha_1\beta_0}}{\alpha_0}e^{i\theta_5} & w_4 & w_5 \\ \frac{\sqrt{\alpha_1\beta_0}}{\alpha_0}e^{i\theta_6} & w_5 & w_6 \end{pmatrix}.$$

The 3×3-matrix of the weights $W\left(\begin{array}{ccc} & * & \\ \nearrow & & \searrow \\ b_0 & & b_0 \\ \searrow & & \nearrow \\ & * & \end{array}\right)$ is

$$(2.5)\qquad \begin{pmatrix} \frac{\beta_m}{\beta_0}e^{i\theta_7} & \frac{\sqrt{\alpha_0\beta_1}}{\beta_0}e^{i\theta_8} & \frac{\sqrt{\alpha_0\beta_1}}{\beta_0}e^{i\theta_9} \\ \frac{\sqrt{\alpha_0\beta_1}}{\beta_0}e^{i\theta_8} & \frac{\alpha_0}{\beta_0}w_1 & \frac{\alpha_0}{\beta_0}w_7 \\ \frac{\sqrt{\alpha_0\beta_1}}{\beta_0}e^{i\theta_9} & \frac{\alpha_0}{\beta_0}w_7 & \frac{\alpha_0}{\beta_0}w_4 \end{pmatrix},$$

and the 3×3-matrix of the weight $W\left(\begin{array}{ccc} & * & \\ \nearrow & & \searrow \\ c_0 & & c_0 \\ \searrow & & \nearrow \\ & * & \end{array}\right)$ is

$$(2.6)\qquad \begin{pmatrix} \frac{\beta_m}{\beta_0}e^{i\theta_{10}} & \frac{\sqrt{\alpha_0\beta_1}}{\beta_0}e^{i\theta_{11}} & \frac{\sqrt{\alpha_0\beta_1}}{\beta_0}e^{i\theta_{12}} \\ \frac{\sqrt{\alpha_0\beta_1}}{\beta_0}e^{i\theta_{11}} & \frac{\alpha_0}{\beta_0}w_3 & \frac{\alpha_0}{\beta_0}w_9 \\ \frac{\sqrt{\alpha_0\beta_1}}{\beta_0}e^{i\theta_{12}} & \frac{\alpha_0}{\beta_0}w_9 & \frac{\alpha_0}{\beta_0}w_6 \end{pmatrix}.$$

The 2×2-matrices of the weights $W\left(\begin{array}{ccc} & a_0 & \\ \swarrow & & \searrow \\ * & & * \\ \searrow & & \swarrow \\ & \overline{a_0} & \end{array}\right)$ and $W\left(\begin{array}{ccc} & \overline{a_0} & \\ \swarrow & & \searrow \\ * & & * \\ \searrow & & \swarrow \\ & a_0 & \end{array}\right)$ has the form

$$(2.7)\qquad \begin{pmatrix} w_7 & w_8 \\ w_8 & w_9 \end{pmatrix},$$

and the 2×2-matrices of the weights $W\left(\begin{array}{ccc} & * & \\ \nearrow & & \searrow \\ b_0 & & c_0 \\ \searrow & & \nearrow \\ & * & \end{array}\right)$ and $W\left(\begin{array}{ccc} & * & \\ \nearrow & & \searrow \\ c_0 & & b_0 \\ \searrow & & \swarrow \\ & * & \end{array}\right)$ has the form

$$\begin{pmatrix} \frac{\alpha_0}{\beta_0}w_2 & \frac{\alpha_0}{\beta_0}w_8 \\ \frac{\alpha_0}{\beta_0}w_8 & \frac{\alpha_0}{\beta_0}w_5 \end{pmatrix}. \tag{2.8}$$

The θ_k's in the above matrices can be taken any values in $[0, 2\pi)$. In order to determine the weights w_k, we need the next proposition.

PROPOSITION 2.1. *Let α_i and β_j be as in the previous section, where $n>m$. Then there exist real numbers s and t, a complex number u and $\theta\in[0, 2\pi)$ such that the following two matrices are both unitary.*

$$A=\begin{pmatrix} \frac{\alpha_n}{\alpha_0} & \frac{\sqrt{\alpha_1\beta_0}}{\alpha_0} & \frac{\sqrt{\alpha_1\beta_0}}{\alpha_0} \\ \frac{\sqrt{\alpha_1\beta_0}}{\alpha_0} & s & t \\ \frac{\sqrt{\alpha_1\beta_0}}{\alpha_0} & t & s \end{pmatrix},$$

$$B=\begin{pmatrix} \frac{\beta_m}{\beta_0}e^{i\theta} & \frac{\sqrt{\alpha_0\beta_1}}{\beta_0} & \frac{\sqrt{\alpha_0\beta_1}}{\beta_0} \\ \frac{\sqrt{\alpha_0\beta_0}}{\beta_0} & \frac{\alpha_0}{\beta_0}s & \frac{\alpha_0}{\beta_0}u \\ \frac{\sqrt{\alpha_0\beta_1}}{\beta_0} & \frac{\alpha_0}{\beta_0}u & \frac{\alpha_0}{\beta_0}s \end{pmatrix}.$$

PROOF. First we note that it follows that

$$\left(\frac{\alpha_n}{\alpha_0}\right)^2+2\frac{\alpha_1\beta_0}{\alpha_0^2}=1 \tag{2.9}$$

by the relation (2) in Lemma 1.4. The unitarity of the matrix A requires the equalities

$$
\text{(a)} \quad \begin{cases} \dfrac{\alpha_n}{\alpha_0}+s+t=0\,, \\[2ex] \dfrac{\alpha_1\beta_0}{\alpha_0^2}+s^2+t^2=1\,, \\[2ex] \dfrac{\alpha_1\beta_0}{\alpha_0^2}+2st=0\,. \end{cases}
$$

Using the relation (2.9), three equalities in (a) can be reduced to two equalites,

$$
\text{(a)}' \quad \begin{cases} \dfrac{\alpha_n}{\alpha_0}+s+t=0\,, \\[2ex] \dfrac{\alpha_1\beta_0}{\alpha_0^2}+s^2+t^2=1\,. \end{cases}
$$

Now by direct calculations and using the relation (2.9), we solve the equalities (a)′,

$$
(2.10) \qquad (s,\, t)=\left(\frac{1}{2}\left(-\frac{\alpha_n}{\alpha_0}+1\right),\, \frac{1}{2}\left(-\frac{\alpha_n}{\alpha_0}-1\right)\right) \quad \text{or}
$$

$$
(2.11) \qquad \left(\frac{1}{2}\left(-\frac{\alpha_n}{\alpha_0}-1\right),\, \frac{1}{2}\left(-\frac{\alpha_n}{\alpha_0}+1\right)\right).
$$

Next we note that it follows that

$$
(2.12) \qquad \left(\frac{\beta_m}{\beta_0}\right)^2+2\frac{\alpha_0\beta_1}{\beta_0^2}=1
$$

by the relation (4) in Lemma 1.4. And the unitarity of the matrix B requires the equalities

$$
\text{(b)} \quad \begin{cases} \dfrac{\beta_m}{\beta_0}e^{i\theta}+\dfrac{\alpha_0}{\beta_0}(s+u)=0\,, \\[2ex] \dfrac{\alpha_0\beta_1}{\beta_0^2}+\left(\dfrac{\alpha_0}{\beta_0}\right)^2(s^2+|u|^2)=1\,, \\[2ex] \dfrac{\alpha_0\beta_1}{\beta_0^2}+\left(\dfrac{\alpha_0}{\beta_0}\right)^2 s(u+\bar{u})=0\,. \end{cases}
$$

Similary, using the relation (2.12), three equalities in (b) can be reduced to two equalities,

$$
\text{(b)}' \quad \begin{cases} \dfrac{\alpha_0\beta_1}{\beta_0^2}+\left(\dfrac{\alpha_0}{\beta_0}\right)^2(s^2+|u|^2)=1\,, \\[2ex] \dfrac{\alpha_0\beta_1}{\beta_0^2}+\left(\dfrac{\alpha_0}{\beta_0}\right)^2 s(u+\bar{u})=0\,. \end{cases}
$$

From these equalities, we have

$$\begin{cases} |u|^2 = \left(\dfrac{\beta_0}{\alpha_0}\right)^2 - \dfrac{\beta_1}{\alpha_0} - s^2, \\ \Re(u) = -\dfrac{1}{2s}\dfrac{\beta_1}{\alpha_0}, \end{cases}$$

where $\Re(u)$ is the real part of u.

In order to show the existence of a complex number u, it is enough to see the inequality

$$|\Re(u)|^2 < |u|^2$$

that is

$$\frac{1}{4s^2}\left(\frac{\beta_1}{\alpha_0}\right)^2 < \left(\frac{\beta_0}{\alpha_0}\right)^2 - \frac{\beta_1}{\alpha_0} - s^2. \tag{2.13}$$

The inequality (2.13) can be reformed to

$$\left(2s^2 + \frac{\beta_1}{\alpha_0}\right)^2 < 4s^2\left(\frac{\beta_0}{\alpha_0}\right)^2$$

Hence the inequality that we have to show, is

$$2s^2 + \frac{\beta_1}{\alpha_0} < 2|s|\frac{\beta_0}{\alpha_0}. \tag{2.14}$$

From (a)′, we obtain

$$2s^2 = \frac{\alpha_1\beta_0}{\alpha_0^2} - 2s\frac{\alpha_n}{\alpha_0},$$

so the inequality (2.14) can be changed to

$$\frac{\alpha_1\beta_0}{\alpha_0^2} - 2s\frac{\alpha_n}{\alpha_0} + \frac{\beta_1}{\alpha_0} < 2|s|\frac{\beta_0}{\alpha_0},$$

$$\frac{\alpha_1\beta_0 + \alpha_0\beta_1}{\alpha_0^2} < 2\left(s\frac{\alpha_n}{\alpha_0} + |s|\frac{\beta_0}{\alpha_0}\right). \tag{2.15}$$

Now we shall show the inequality (2.15) for each case of (2.10) and (2.11).

Case I) $(s, t) = \left(\dfrac{1}{2}\left(-\dfrac{\alpha_n}{\alpha_0} + 1\right), \dfrac{1}{2}\left(-\dfrac{\alpha_n}{\alpha_0} - 1\right)\right)$.

Substitute s in (2.15) by $\dfrac{1}{2}\left(-\dfrac{\alpha_n}{\alpha_0} + 1\right) > 0$, we have

$$\frac{\alpha_1\beta_0 + \alpha_0\beta_1}{\alpha_0^2} < \left(-\frac{\alpha_n}{\alpha_0} + 1\right)\frac{\alpha_n}{\alpha_0} + \left(-\frac{\alpha_n}{\alpha_0} + 1\right)\frac{\beta_0}{\alpha_0},$$

$$\alpha_1\beta_0 + \alpha_0\beta_1 < (\alpha_0 - \alpha_n)(\beta_0 + \alpha_n). \tag{2.16}$$

The inequality (2.16) is our desired one in this case. We consider the difference,

$$D_1=(\text{The right hand of (2.16)})-(\text{The left hand of (2.16)}).$$

Using the relations (2) and (5) in Lemma 1.4, we obtain the following equalities,

$$\begin{aligned}D_1&=(\alpha_0-\alpha_n)(\beta_0+\alpha_n)-(\alpha_1\beta_0+\alpha_0\beta_1)\\&=(\alpha_0-\beta_0)\alpha_n+\alpha_0\beta_0-\alpha_n^2-\alpha_1\beta_0-\alpha_0\beta_1\\&=(\alpha_0-\beta_0)\alpha_n+\alpha_0\beta_0-\alpha_0^2+\alpha_1\beta_0-\alpha_0\beta_1\\&=(\alpha_0-\beta_0)\alpha_n+\alpha_0\beta_0+\alpha_0^2-2\beta_0^2\\&=(\alpha_0-\beta_0)(\alpha_0+2\beta_0+\alpha_n).\end{aligned}$$

Since we know that $\alpha_0>\beta_0$, it is clear the last expression is in positive. Thus we get the desired inequality for this case.

Case II) $(s,t)=\left(\frac{1}{2}\left(-\frac{\alpha_n}{\alpha_0}-1\right),\frac{1}{2}\left(-\frac{\alpha_n}{\alpha_0}+1\right)\right)$.

Similary, subsitute s in (2.15) by $\frac{1}{2}\left(-\frac{\alpha_n}{\alpha_0}-1\right)<0$, we have

$$\alpha_1\beta_0+\alpha_0\beta_1<(\alpha_0+\alpha_n)(\beta_0-\alpha_n). \tag{2.17}$$

The inequality (2.17) is our desired one in this case. We also consider the difference,

$$D_2=(\text{The right hand of (2.17)})-(\text{The left hand of (2.17)}).$$

Using the relations (2) and (5) in Lemma 1.4, we obtain the following equalities,

$$\begin{aligned}D_2&=(\alpha_0+\alpha_n)(\beta_0-\alpha_n)-(\alpha_1\beta_0+\alpha_0\beta_1)\\&=-(\alpha_0-\beta_0)\alpha_n+\alpha_0\beta_0-\alpha_n^2-\alpha_1\beta_0-\alpha_0\beta_1\\&=(\alpha_0-\beta_0)(\alpha_0+2\beta_0-\alpha_n).\end{aligned}$$

And as we know that $\alpha_0>\beta_0$ and $\alpha_0>\alpha_n$, the last expression is in positive. Hence we get the desired inequality (2.17).

Now there exist real numbers s and t, and a complex number u satisfying the demanded conditions. Of course, we have the relation

$$\left|\frac{\alpha_0}{\beta_0}(s+u)\right|=\frac{\beta_m}{\beta_0}, \tag{2.18}$$

and θ can be determined by the first equality in (b). □

REMARK 2.2. From the equalities (a) and (b) in the proof of Proposition 2.1, it can be proved that

$$\left(\frac{\beta_0}{\alpha_0}\right)^2 - t^2 = 1 - |u|^2 > 0\,, \tag{2.19}$$

by using the relations in Lemma 1.4.

From these observations, let us define w_k and θ_k as follows.

$$\begin{cases} w_1 = w_3 = w_4 = w_6 = s, \quad w_2 = w_5 = t, \\ w_7 = u, \quad w_8 = i\sqrt{1-|u|^2}, \quad w_9 = \bar{u}, \end{cases} \tag{2.20}$$

and

$$\begin{cases} \theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 = \theta_6 = 0\,, \\ \theta_8 = \theta_9 = \theta_{11} = \theta_{12} = 0\,, \\ \theta_7 = \theta_{10} = \theta\,. \end{cases} \tag{2.21}$$

Then it is easy to see that all the matrices from (2.3) to (2.8) are unitary, and that w_k's satisfy the invariant condition. Hence we conclude that the graph $B(n, m, n, m)$ has the $\mathbf{Z}_2$-symmetric connection, of which symmetry is arised from the reflection of the 2nd-4th line.

We construct the connection on the 5-star graph $S(n, m+1, m+1, m+1, m+1)$ from the above symmetry. Fortunately, we can use the results in [2] that if the connection on the graph is invariant under the $\mathbf{Z}_2$-symmetry, like the above, we can move this connection to one on the new graph which is obtained from orbifold of this symmetry.

For convenience, we list the calculation for the parametrizations of the weights at the central point in the matrix form below.

$$
\begin{pmatrix}
W\left(\begin{smallmatrix} & a_0 & \\ \swarrow & & \searrow \\ a_1 & & a_1 \\ \searrow & & \swarrow \\ & a_0 & \end{smallmatrix}\right) & \frac{1}{\sqrt{2}}W\left(\begin{smallmatrix} & a_0 & \\ \swarrow & & \searrow \\ a_1 & & b_0 \\ \searrow & & \swarrow \\ & a_0 & \end{smallmatrix}\right) & \frac{1}{\sqrt{2}}W\left(\begin{smallmatrix} & a_0 & \\ \swarrow & & \searrow \\ a_1 & & b_0 \\ \searrow & & \swarrow \\ & a_0 & \end{smallmatrix}\right) & \frac{1}{\sqrt{2}}W\left(\begin{smallmatrix} & a_0 & \\ \swarrow & & \searrow \\ a_1 & & c_0 \\ \searrow & & \swarrow \\ & a_0 & \end{smallmatrix}\right) & \frac{1}{\sqrt{2}}W\left(\begin{smallmatrix} & a_0 & \\ \swarrow & & \searrow \\ a_1 & & c_0 \\ \searrow & & \swarrow \\ & a_0 & \end{smallmatrix}\right) \\
\frac{1}{\sqrt{2}}W\left(\begin{smallmatrix} & a_0 & \\ \swarrow & & \searrow \\ b_0 & & a_1 \\ \searrow & & \swarrow \\ & a_0 & \end{smallmatrix}\right) & \frac{w_1+w_7}{2} & \frac{w_1-w_7}{2} & \frac{w_2+w_8}{2} & \frac{w_2-w_8}{2} \\
\frac{1}{\sqrt{2}}W\left(\begin{smallmatrix} & a_0 & \\ \swarrow & & \searrow \\ b_0 & & a_1 \\ \searrow & & \swarrow \\ & a_1 & \end{smallmatrix}\right) & \frac{w_1-w_7}{2} & \frac{w_1+w_7}{2} & \frac{w_2-w_8}{2} & \frac{w_2+w_8}{2} \\
\frac{1}{\sqrt{2}}W\left(\begin{smallmatrix} & a_0 & \\ \swarrow & & \searrow \\ c_0 & & a_1 \\ \searrow & & \swarrow \\ & a_0 & \end{smallmatrix}\right) & \frac{w_2+w_8}{2} & \frac{w_2-w_8}{2} & \frac{w_3+w_9}{2} & \frac{w_3-w_9}{2} \\
\frac{1}{\sqrt{2}}W\left(\begin{smallmatrix} & a_0 & \\ \swarrow & & \searrow \\ c_0 & & a_1 \\ \searrow & & \swarrow \\ & a_0 & \end{smallmatrix}\right) & \frac{w_2-w_8}{2} & \frac{w_2+w_8}{2} & \frac{w_3-w_9}{2} & \frac{w_3+w_9}{2}
\end{pmatrix}
$$

The rows and the columns in the above matrix, are indexfied in the order, $a_1', b_0', \overline{b_0'}, c_0'$ and $\overline{c_0'}$.

Substitute each weight in the above matrix by (2.20), we have the 5×5 matrix of the weights at the central point of the 5-star graph $S(n, m+1, m+1, m+1, m+1)$,

$$
U=\begin{pmatrix}
\frac{\alpha_n}{\alpha_0} & \frac{1}{\sqrt{2}}\frac{\sqrt{\alpha_1\beta_0}}{\alpha_0} & \frac{1}{\sqrt{2}}\frac{\sqrt{\alpha_1\beta_0}}{\alpha_0} & \frac{1}{\sqrt{2}}\frac{\sqrt{\alpha_1\beta_0}}{\alpha_0} & \frac{1}{\sqrt{2}}\frac{\sqrt{\alpha_1\beta_0}}{\alpha_0} \\
\frac{1}{\sqrt{2}}\frac{\sqrt{\alpha_1\beta_0}}{\alpha_0} & \frac{s+u}{2} & \frac{s-u}{2} & \frac{t+i\sqrt{1-|u|^2}}{2} & \frac{t-i\sqrt{1-|u|^2}}{2} \\
\frac{1}{\sqrt{2}}\frac{\sqrt{\alpha_1\beta_0}}{\alpha_1} & \frac{s-u}{2} & \frac{s+u}{2} & \frac{t-i\sqrt{1-|u|^2}}{2} & \frac{t+i\sqrt{1-|u|^2}}{2} \\
\frac{1}{\sqrt{2}}\frac{\sqrt{\alpha_1\beta_0}}{\alpha_0} & \frac{t+i\sqrt{1-|u|^2}}{2} & \frac{t-i\sqrt{1-|u|^2}}{2} & \frac{s+\bar{u}}{2} & \frac{s-\bar{u}}{2} \\
\frac{1}{\sqrt{2}}\frac{\sqrt{\alpha_1\beta_0}}{\alpha_0} & \frac{t-i\sqrt{1-|u|^2}}{2} & \frac{t+i\sqrt{1-|u|^2}}{2} & \frac{s-\bar{u}}{2} & \frac{s+\bar{u}}{2}
\end{pmatrix}
$$

and, of course, the above matrix $U=(u_{ij})$ is a unitay matrix and the entries u_{ij} satisfy the following norm condition,

$$(|u_{ij}|)=\begin{pmatrix} \frac{\alpha_n}{\alpha_0} & \frac{1}{\sqrt{2}}\frac{\sqrt{\alpha_1\beta_0}}{\alpha_0} & \frac{1}{\sqrt{2}}\frac{\sqrt{\alpha_1\beta_0}}{\alpha_0} & \frac{1}{\sqrt{2}}\frac{\sqrt{\alpha_1\beta_0}}{\alpha_0} & \frac{1}{\sqrt{2}}\frac{\sqrt{\alpha_1\beta_0}}{\alpha_0} \\ \frac{1}{\sqrt{2}}\frac{\sqrt{\alpha_1\beta_0}}{\alpha_0} & \frac{\beta_m}{2\alpha_0} & \frac{\beta_0}{2\alpha_0} & \frac{\beta_0}{2\alpha_0} & \frac{\beta_0}{2\alpha_0} \\ \frac{1}{\sqrt{2}}\frac{\sqrt{\alpha_1\beta_0}}{\alpha_0} & \frac{\beta_0}{2\alpha_0} & \frac{\beta_m}{2\alpha_0} & \frac{\beta_0}{2\alpha_0} & \frac{\beta_0}{2\alpha_0} \\ \frac{1}{\sqrt{2}}\frac{\sqrt{\alpha_1\beta_0}}{\alpha_0} & \frac{\beta_0}{2\alpha_0} & \frac{\beta_0}{2\alpha_0} & \frac{\beta_m}{2\alpha_0} & \frac{\beta_0}{2\alpha_0} \\ \frac{1}{\sqrt{2}}\frac{\sqrt{\alpha_1\beta_0}}{\alpha_0} & \frac{\beta_0}{2\alpha_0} & \frac{\beta_0}{2\alpha_0} & \frac{\beta_0}{2\alpha_0} & \frac{\beta_m}{2\alpha_0} \end{pmatrix}$$

which corresponds the key Lemma in [5]. Here we reach the theorem,

THEOREM 2.4. *The 5-star graph $S(n, m+1, m+1, m+1, m+1)$ $(n>m)$ has a connection.*

In the case when $m=0$, the graph $\mathcal{G}_1$ is the graph $B(n, 0, n, 0)$,

$\mathcal{G}_1$:

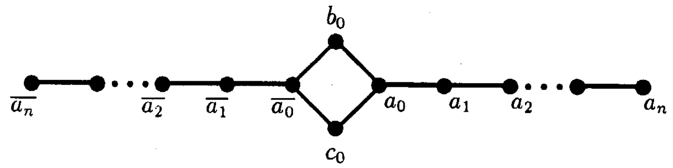


Figure 2.1.

it is a little easier to find the $\mathbf{Z}_2$-symmetric connection. Because the matrix B in the Proposition 2.1 should be changed to the 2×2 matrix of the form

$$B'=\begin{pmatrix} \frac{\alpha_0}{\beta_0}s & \frac{\alpha_0}{\beta_0}u \\ \frac{\alpha_0}{\beta_0}u & \frac{\alpha_0}{\beta_0}s \end{pmatrix}.$$

Hence we are able to put the purely imaginal number $i\sqrt{\left(\frac{\beta_0}{\alpha_0}\right)^2-s^2}$ as the complex number u.

3. Remark on the symmetries of Sunder's graph

In this section, we give some remarks on the symmetries of the graphs in [17]. First, we consider the graph $B(n, n, n, n)$ as the graph $\mathcal{G}_1$, that is the graph has the central square and four legs of the same length.

$\mathcal{G}_1$:

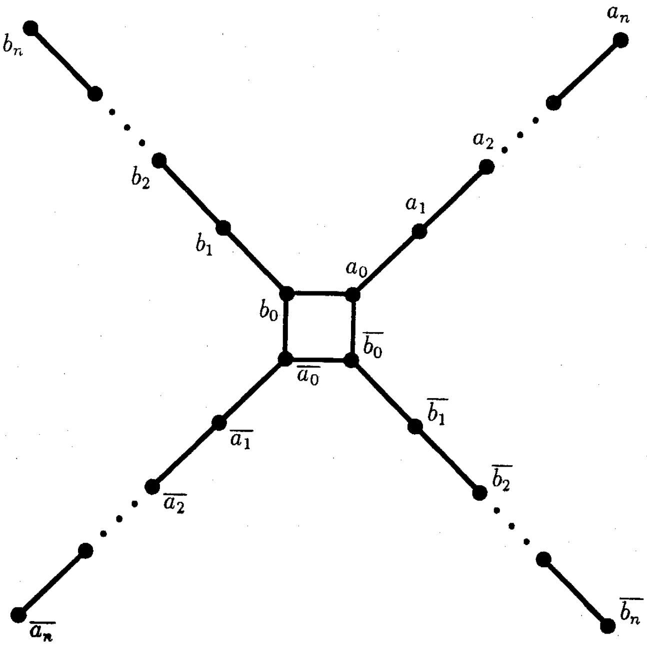


Figure 3.1.

This graph has the following two different symmetries.

1. Symmetry to 5-star $S(n, n+1, n+1, n+1, n+1)$.
2. Symmetry to the Sunder's graph of $N=2$.

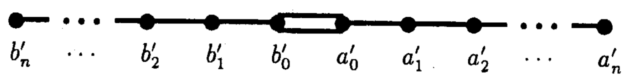
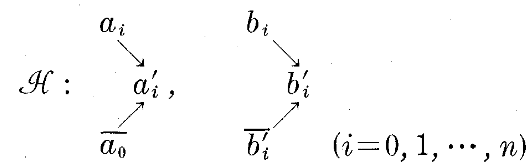


Figure 3.2.

The first symmetry is arised from the reflection for the 2nd-4th line of the graph $\mathcal{G}_1$, as we have seen the previous section. And the second symmetry is coming from the π-rotation at the center of the graph $\mathcal{G}_1$, which is also $\mathbf{Z}_2$-symmetry but there is no fixed point in this case.

Here we write the matrix C which gives us the second symmetry, in the graph $\mathcal{H}$.



We constructed the connection on the graph $B(n, n, n, n)$ in [19, § 2]. And this connection is invariant under both the first and the second symmetries in above. For convenient, we rewrite the weights around the central square

$$w_1 = w_3 = w_4 = w_6 = u ,$$

$$w_2 = w_5 = w_7 = w_9 = \bar{u} \quad \text{and}$$

$$w_8 = i\sqrt{1-|u|^2} e^{i(\theta - \pi/2)}$$

where $u = -(\alpha_n/2\alpha_0) + i(1/2)$ or $u = -(\alpha_n/2\alpha_0) - i(1/2)$ and $\theta = \arg(\bar{u})$. Of course, the w_k's mean the cell weights as in the previous section and we set $\theta_k = 0$ for all θ_k.

Using this connection, we obtain the connections both on the 5-star graph $S(n, n+1, n+1, n+1, n+1)$ and on the Sunder's graph of $N=2$ by Roche's procedure. As for on the 5-star graph $S(n, n+1, n+1, n+1, n+1)$, we have the same parametrization and it is not so difficult to find a cell system for the second symmetry, so we omite the details. And Sunder's construction of the connection on the graph of $N=2$ can be regarded as a sort of the above procedure [17, Lemma 7 and Lemma 9].

On the other hand, there exiets a symmetry from the 5-star graph $S(n, n+1, n+1, n+1, n+1)$ to the Sunder's graph of $N=2$ directly. And this symmetry can be extended to the general case. That is, there exists the symmetry from the graph $\mathcal{G}_1$, the (N^2+1)-star graph $S(n, \underbrace{n+1, n+1, \cdots, n+1}_{N^2})$, to the graph $\mathcal{G}_2$, Sunder's graph of N.

We lable the vertices of each graphs as follows.

$\mathcal{G}_1$:

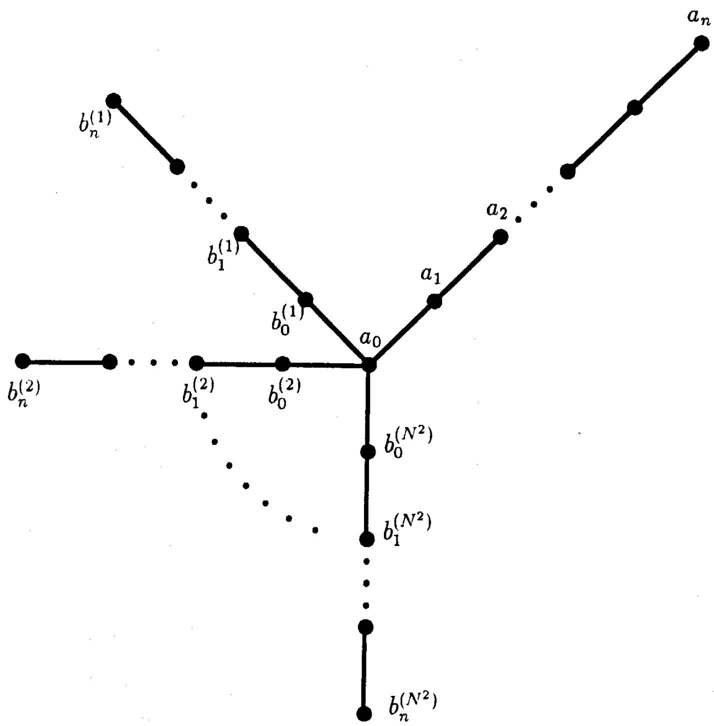


Figure 3.3.

$\mathcal{G}_2$:

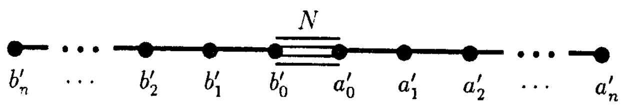
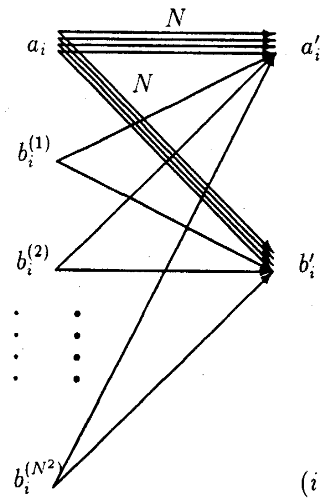


Figure 3.4.

Then the matrix C which gives us the symmetry, is represented in the graph

$\mathscr{H}$:



$(i = 0, 1, 2, \cdots, n)$.

Figure 3.5.

The Perron-Frobenius weights on each graph are as follows.

$$\mathscr{G}_1 : \begin{cases} \mu(a_i) = \alpha_i, & (i=0, 1, \cdots, n), \\ \mu(b_i^{(l)}) = \dfrac{\alpha_i}{N} & (i=0, 1, \cdots, n) \quad \text{for all } l, \end{cases}$$

$$\mathscr{G}_2 : \mu(a'_i) = \mu(b'_i) = \alpha_i \quad (i=0, 1, \cdots, n).$$

where α_i's satisfies the relations:

(1) $\alpha_n^2 = \alpha_i^2 - \alpha_{i+1}\alpha_{i+1}$ for $i=1, 2, \cdots, n-1$,

(2) $\alpha_n^2 = \alpha_0^2 - N\alpha_0\alpha_1$,

Furthermore, we shall give the remark on the symmetry for Sunder's graph. The symmetry from the graph $B(n, n, n, n)$ to Sunder's graph of $N=2$, can be extended to the following symmetry.

We consider the graph $P_{2N}(n)$. The graph $P_{2N}(n)$ has the central $2N$-polygon with $N(N-2)$ many inner edges and leges of length n at each corner vertex of the polygon. Here the inner edges of the polygon are connected as following manner. We label the vertices of the $2N$-polygon $p_1, p_2, \cdots, p_{2N-1}, p_{2N}$. And for each odd number labeled vertex, we connect this point to the all even number labeled vertices. Conversely, for each even number labeled vertex, we connect this point to the all odd number labeled vertices. In the above connecting, we should not make the edges multi-count. That is all the edges should be single.

Now we have the symmetry from the graph $P_{2N}(n)$ to Sunder's graph of N, which is arised from the $(2\pi/N)$-rotation at the center of the $2N$-

polygon in graphical. Of course, this symmetry is $\boldsymbol{Z}_n$-symmetry and the matrix C, which gives us this symmetry, is represented in the graph

$$\mathcal{H}: p_i^{(\text{odd})} \longrightarrow a_i', \quad p_i^{(\text{even})} \longrightarrow b_i', \qquad (i=0, 1, \cdots, n)$$

where $p_i^{(l)}$ is the i-th vertex on the l-th leg of the graph $P_{2N}(n)$, and a_i' and b_i' as in Figure 3.4.

At the finally, we would like to say as a remark that, for the case when $N=3$, we can get a $\boldsymbol{Z}_3$-symmetry invariant connection on the following graph $P_6(n)$. And U. Haagerup has also obtained a connection on this graph [4].

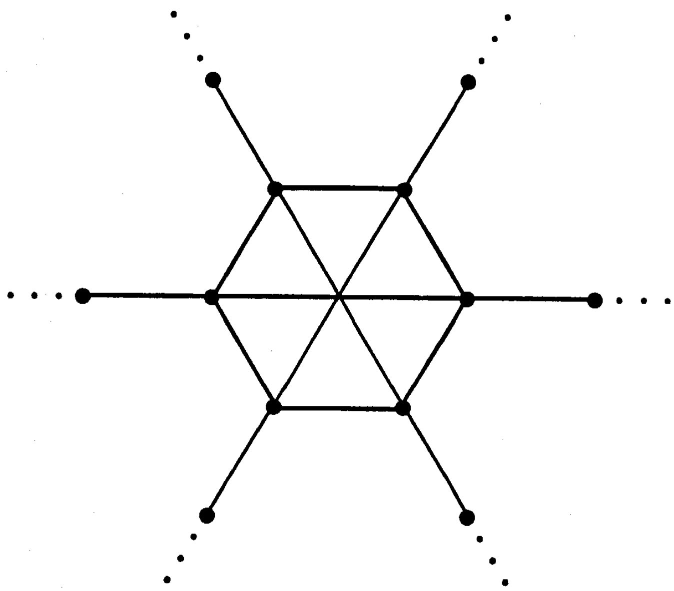


Figure 3.6.

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