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A Remark on Pluri-Genera of Algebraic Surfaces

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§ 0. Introduction.

Let X be a non-singular complete algebraic surface defined over the field $\boldsymbol{C}$ of complex numbers. We denote by $p_g(X)$ the geometric genus of X, and by $q(X)$ the irregularity of X. E. Stagnaro investigated algebraic surfaces X with $p_g(X)=q(X)=0$ (cf. [4]) and he gave the following question.

QUESTION. *For algebraic surfaces X with $p_g(X)=q(X)=0$, are the 2-genera $P_2(X)$ bounded?*

If the Kodaira dimension $\kappa(X)$ of X is equal to $-\infty$ or 0, then we have $P_2(X)\leqq 1$. If $\kappa(X)=2$, that is, X is of general type, then from Noether's formula and Yau-Miyaoka's inequality, it follows that $P_2(X)\leqq 10$ under the condition $p_g(X)=q(X)=0$. This class of surfaces of general type really contains many interesting surfaces such as Godeaux surfaces, Campedelli surfaces, Mumford's fake projective plane, etc.. For algebraic surfaces X with $\kappa(X)=1$, however, the situation is different. In fact, we show in this note that $P_2(X)$'s are not bounded for these surfaces X even if $p_g(X)=q(X)=0$. We also determine the structure of such algebraic surfaces. These results are consequences of Kodaira's theory on analytic elliptic surfaces. We impose multiple fibres on elliptic surfaces by means of Kodaira's logarithmic transformations, and see later that the resulting surfaces X are algebraic since $p_g(X)=q(X)=0$.

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NOTATION.

For a compact complex analytic manifold X of dimension n, we use the following notation:

$\mathcal{O}_X$: the sheaf of germs of holomorphic functions on X,

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Ω_X^1: the sheaf of germs of holomorphic 1-forms on X,
K_X: a canonical divisor on X,
$p_g(X)=\dim_C H^n(X, \mathcal{O}_X)$: the geometric genus of X,
$q(X)=\dim_C H^1(X, \mathcal{O}_X)$: the irregularity of X,
$h(X)=\dim_C H^0(X, \Omega_X^1)$,
$P_m(X)=\dim_C H^0(X, \mathcal{O}_X(mK_X))$: the m-genus of X,
$\chi(\mathcal{O}_X)=\sum_{i=0}^n(-1)^i \dim_C H^i(X, \mathcal{O}_X)$,
$c_n(X)$: the n-th Chern number (i.e., the Euler number) of X,
$b_1(X)$: the first Betti number of X.

§1. Preliminaries.

In this section, we summarize the facts on elliptic surfaces which we will use in the next section. For details, see Kodaira [1] and [3].

Let $f: X \rightarrow \boldsymbol{C}$ be a (not necessarily algebraic) elliptic surface defined over $\boldsymbol{C}$ with a non-singular complete curve $\boldsymbol{C}$. Throughout this paper, we assume that no fibers of f contain exceptional curves of the first kind. We denote by

$$\{m_1 D_1, m_2 D_2, \cdots, m_\nu D_\nu\}$$

the multiple fibers of f with multiplicities m_i $(i=1, \cdots, \nu)$. We set

$$\boldsymbol{f}=R^1 f_* \mathcal{O}_X .$$

Then, we know that in characteristic $0, \boldsymbol{f}$ is torsion-free and is an invertible sheaf on C. Hence, we have a divisor $\mathfrak{f}$ on C such that $\boldsymbol{f}=\mathcal{O}_C(\mathfrak{f})$. It is well-known that

$$\deg \mathfrak{f}=-\chi(\mathcal{O}_X) . \tag{1.1}$$

A canonical divisor K_X on X is given by

$$K_X=f^*(K_C-\mathfrak{f})+\textstyle\sum_{i=1}^{\nu}(m_i-1) D_i . \tag{1.2}$$

By this formula, we have

$$K_X{}^2=0 , \tag{1.3}$$

hence, by Noether's formula, we have

$$12 \chi(\mathcal{O}_X)=c_2(X) . \tag{1.4}$$

Moreover, we have

$$H^0(X, \mathcal{O}_X(K_X)) \cong H^0(C, \mathcal{O}_C(K_C-\mathfrak{f})) . \tag{1.5}$$

§ 2. Elliptic surfaces with $p_g(X)=q(X)=0$.

In this section, we give a negative answer to the question by E. Stagnaro.

LEMMA 2.1. *Let X be a compact complex analytic surface with $p_g(X)=q(X)=0$. Then, X is an algebraic surface.*

PROOF. By Kodaira [3, Theorem 10] and [2], for a compact complex analytic surface X with $p_g(X)=0$, X is algebraic if and only if $b_1(X)$ is even. Moreover, we have $b_1(X)=q(X)+h(X)$ and $q(X)\geqq h(X)$ (cf. Kodaira [2]). Since $q(X)=0$, we have $h(X)=0$, therefore, we conclude $b_1(X)=0$. Hence, X is algebraic. q.e.d.

Now, we construct elliptic surfaces $f: X\to \boldsymbol{P}^1$ with $\kappa(X)=1$ and $p_g(X)=q(X)=0$, using the method by Kodaira [3]. First, we take a rational elliptic surface

$$\pi: S \longrightarrow \boldsymbol{P}^1 \tag{2.1}$$

free from multiple fibers. Let $E_1, \cdots, E_\nu$ ($\nu\geqq 2$) be fibers of π which are regular or of type I_b ($b\geqq 1$). (For the notation, see Kodaira [1], [3, Section 4].) We set $a_i=\pi(E_i)$ ($i=1, 2, \cdots, \nu$). Let $m_1, m_2, \cdots, m_\nu$ be integers such that $m_i\geqq 2$ ($i=1, 2, \cdots, \nu$). Then, by means of logarithmic transformations $L_{a_1}, \cdots, L_{a_\nu}$ (cf. Kodaira [3, Section 4]) we get an elliptic surface

$$f: X \longrightarrow \boldsymbol{P}^1 \tag{2.2}$$

which has multiple fibers m_iD_i ($i=1, 2, \cdots, \nu$) of multiplicities m_i ($i=1, \cdots, \nu$) such that $f(m_iD_i)=a_i$ ($i=1, \cdots, \nu$). Since to impose a multiple fibres is a local problem, we can impose ν of them with ν arbitrary integer.

Now, we calculate numerical invariants of the elliptic surface in (2.2). Since the Euler number of an elliptic surface is equal to the sum of Euler numbers of the fibers, we have $c_2(X)=c_2(S)=12\chi(\mathcal{O}_S)=12$. Since $K_X{}^2=0$, by Noether's formula we have

$$\chi(\mathcal{O}_X)=(K_X{}^2+c_2(X))/12=1\,. \tag{2.3}$$

Therefore, by (1.1), we have

$$\deg(K_{P^1}-f)=-2+\chi(\mathcal{O}_X)=-1\,.$$

By (1.5), we have $p_g(X)=\dim H^2(X, \mathcal{O}_X)=\dim H^0(X, \mathcal{O}_X(K_X))=0$. Therefore, by (2.3), we have $q(X)=0$. Hence, by Lemma 2.1, we conclude that X is an algebraic surface. By (1.2), we have

$$\begin{aligned} mK_X &= mf^*(K_{P^1}-f)+\textstyle\sum_{i=1}^{\nu} m(m_i-1)D_i \\ &= mf^*(K_{P^1}-f)+\textstyle\sum_{i=1}^{\nu}[m(m_i-1)/m_i]f^*(a_i) \\ &\quad +\textstyle\sum_{i=1}^{\nu}\{m(m_i-1)-[m(m_i-1)/m_i]m_i\}D_i\,, \end{aligned}$$

where $[x]$ means the integral part of a rational number x. Therefore, we have

$$\begin{aligned} (2.4)\qquad P_m(X) &= \dim_C H^0(X, \mathcal{O}_X(mK_X)) \\ &= \dim_C H^0(\boldsymbol{P}^1, \mathcal{O}_{P^1}(mK_{P^1}-mf+\textstyle\sum_{i=1}^{\nu}[m(m_i-1)/m_i]a_i)) \\ &= -m+\textstyle\sum_{i=1}^{\nu}[m(m_i-1)/m_i]+1 \qquad (\nu\geqq 2,\ m>0)\,. \end{aligned}$$

In particular, we have

$$(2.5)\qquad P_2(X)=-2+\nu+1=\nu-1 \qquad (\nu\geqq 2)\,.$$

Summarizing these facts, we have the following results.

PROPOSITION 2.2. *The m-genus or the elliptic surface X in (2.2) is given by*

$$P_m(X)=(-m+1)+\textstyle\sum_{i=1}^{\nu}[m(m_i-1)/m_i] \qquad (\nu\geqq 2)\,.$$

THEOREM 2.3. *Any elliptic surface X with $\kappa(X)=1$ and $p_g(X)=q(X)=0$ is algebraic. Moreover, for algebraic elliptic surfaces X with $\kappa(X)=1$ and $p_g(X)=q(X)=0$, the 2-genera $P_2(X)$ are not bounded.*

As for the structure of elliptic surfaces X with $\kappa(X)=1$ and $p_g(X)=q(X)=0$, we have the following.

THEOREM 2.4. *Any elliptic surface X with $\kappa(X)=1$ and $p_g(X)=q(X)=0$ is constructed as in the above way from a suitable rational elliptic surface $\pi: S\to \boldsymbol{P}^1$ free from multiple fibers.*

PROOF. Let $f: X\to C$ be an elliptic surface with $\kappa(X)=1$ and $p_g(X)=q(X)=0$. Since $q(X)\geqq h(X)$, we have $h(X)=0$. Therefore, we have $h(C)=0$, hence, C is isomorphic to $\boldsymbol{P}^1$. By (1.2) and $\kappa(X)=1$, we see that the number of multiple fibers of f is greater than or equal to 2. By Kodaira [3, p. 771], there exists an elliptic surface $\pi: S\to\boldsymbol{P}^1$ free from multiple fibers from which $f: X\to\boldsymbol{P}^1$ is obtained by means of a finite number of suitable logarithmic transformations. By the same reason as above, we have $\chi(\mathcal{O}_S)=\chi(\mathcal{O}_X)=1$. A canonical divisor on S is given by $K_S=\pi^*(K_{P^1}-f)$ with $\mathcal{O}_{P^1}(f)=R^1\pi_*\mathcal{O}_S$. Since $\deg f=-\chi(\mathcal{O}_S)=-1$, we have $\deg(K_{P^1}-f)=-1$. Therefore, we have

$$p_g(S)=0 \quad\text{and}\quad P_2(S)=0\,.$$

Since $1=\chi(\mathcal{O}_S)=p_g(S)-q(S)+1$, we have $q(X)=0$. Hence, by Castelnuovo's criterion of rationality, we conclude that $\pi: S\to \boldsymbol{P}^1$ is a rational elliptic surface.
q.e.d.

REMARK 2.5. Let $\pi: S\to\boldsymbol{P}^1$ be a rational elliptic surface as in (2.1) with fibers $E_1, \cdots, E_\nu$ ($\nu\geqq 2$) which are regular or of type I_b ($b\geqq 1$). Let $f: X\to\boldsymbol{P}^1$ be the elliptic surface with $\kappa(X)=1$ and multiple fibers $m_1D_1, \cdots, m_\nu D_\nu$ which is constructed as above from $\pi: S\to\boldsymbol{P}^1$. Then, by construction, $S\backslash\{E_1\cup\cdots\cup E_\nu\}$ is analytically isomorphic to $X\backslash\{m_1D_1\cup\cdots\cup m_\nu D_\nu\}$. However, by Proposition 2.2 and the fact $P_m(S)=0$ ($m\geqq 1$) we see that they are not isomorphic to each other algebraically.

References

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