Natural Science Report, Ochanomizu University, Vol. 44, No. 1, 1993
お茶の水女子大学 自然科学報告 第44巻 第1号

Supersingular K3 Surfaces with Various Values of the Artin Invariant in Characteristic 2

Mari Ohhira

Department of Mathematics, Faculty of science,
Ochanomizu University, Tokyo

(Received April 8, 1993)

§ 0. Introduction

Let C be a non-singular complete curve defined over an algebraically closed field of positive characteristic p, and let $f: S \to C$ be a relatively minimal elliptic surface defined over k.

DEFINITION 0.1. $f: S \to C$ is called a unirational elliptic surface of base change type if there exist a curve C' and a morphism $g: C' \to C$ such that the fiber product $S \times_C C'$ is rational.

Let $f: S \to C$ be a unirational elliptic surface of base change type. Naturally, S gives an example of unirational algebraic surfaces (i.e. there exists a generically surjective rational map from the projective surface $\boldsymbol{P}^2$ to S), which are the focus of many mathematicians' attention. T. Katsura ([3]) has completely determined irrational unirational elliptic surfaces of base change type in characteristic more than two, and the author make it up in characteristic 2 ([8]). Incidentally, such surfaces are all K3 surfaces in characteristic 2, so, we get certain numbers of unirational K3 surfaces.

It is known a unirational algebraic surface is supersingular, i.e. its second Betti mumber is equal to its Picard number ([12]). In conclusion, it seems that our previous paper gives many examples of supersingular elliptic K3 surfaces in characteristic 2, in appearance.

Now, in this paper, we show that our set of K3 surfaces is indeed, rather varied, i.e. the members of the set give various values of Artin's σ-invariant, which is the most important characteristic value of a supersingular K3 surface.

DEFINITION 0.2. ([1]) Let X be a supersingular K3 surface defined over an algebraically closed field k of positive characteristic p. then the determinant of its Néron-Severi group disc $NS(X)$ is given as follows:

$$\operatorname{disc} NS(X) = -p^{2\sigma}, \qquad 1 \leqq \sigma \leqq 10$$

σ is called Artin's σ-invariant of X.

THEOREM 0.3. *Let $f: X\to \boldsymbol{P}^1$ be an irrational unirational elliptic surface of base change type over an algebraically closed field k of characteristic 2. Let $K=k(t)$ be the function field of $\boldsymbol{P}^1$, and let $a, b, \cdots, f\in k$. then the minimal Weierstrass normal form of $f: X\to \boldsymbol{P}^1$ is given by one of the following, and its σ-invariant σ is determined as follows:*

(1) $$y^2+t^6y=x^3+t(at^3+bt^2+ct+d)x^2+et^6x+ft^{11}, \qquad (d, e\neq 0).$$

$$\begin{cases} \sigma=4 & \text{if } f=0, \\ \sigma=5 & \text{if } f\neq 0. \end{cases}$$

(2) $$y^2+t^6y=x^3+t(at^3+bt^2+ct+d)x^2, \qquad (d\neq 0).$$

$$\sigma=3.$$

(3) $$y^2+t^2y=x^3+t(at+b)x^2+t^3(ct^4+dt^2+e), \qquad (c, e\neq 0).$$

$$\sigma=5.$$

(4) $$y^2+t^2y=x^3+t(at^2+bt+c)x^2+dt^4x+t^3(et^2+f), \qquad (a, d, f\neq 0).$$

Let $F(T)=T^3+cT^2+g$, $G(T)=aT^2+dT+e$ be polynomials of one variable T.

* *If F and G have two common roots, then $\sigma=4$.*

* *If F and G have only one common root, and if another root of F is $(ac+d)/a$, and if the other root of G is c, then $\sigma=4$.*

* *Excepting above cases, if F and G have only one common root, or if $(ac+d)/a$ is a root of F, or if c is a root of G, then $\sigma=5$.*

* *If f conforms to no above cases, then $\sigma=6$.*

(5) $$y^2+t^2y=x^3+t(at^2+bt+c)x^2+dt^2x+et^3, \qquad (a, cd+e\neq 0).$$

$$\begin{cases} \sigma=4 & \text{if } ec=0, \\ \sigma=5 & \text{if } ec\neq 0. \end{cases}$$

(6) $$y^2+t^2y=x^3+at^2x^2+t^5(bt^2+c), \qquad (b\neq 0).$$

$$\sigma=3.$$

(7) $$y^2+t^2y=x^3+t^2(at+b)x^2+ct^4x+dt^5, \qquad (a, c\neq 0).$$

$$\begin{cases} \sigma=3 & \text{if } d=0, \\ \sigma=4 & \text{if } d\neq 0. \end{cases}$$

(8) $$y^2+t^2y=x^3+t^2(at+b)x^2, \qquad (a\neq 0).$$

$$\sigma=2.$$

(9) $$y^2+t^2y=x^3+t(at+b)x^2+t^6(ct+d), \qquad (b, c\neq 0).$$

$$\sigma=4\,.$$

(10) $$y^2+t^2y=x^3+t(at^2+bt+c)x^2+dt^4x+et^5\,,\qquad (a,c,d\neq 0)\,.$$

* *If* $e=0$ *and* $d=ac$, *then* $\sigma=3$.

* *Excepting the above case, if* $e=0$, *or if* $d=ac$, *or if* $ac^2+dc+e=0$, *then* $\sigma=4$.

* *If* f *conforms to no above cases, then* $\sigma=5$.

(11) $$y^2+t^2y=x^3+t(at^2+bt+c)x^2\,,\qquad (a,c\neq 0)\,.$$

$$\sigma=3\,.$$

(12) $$y^2+t^2(t+1)^2y=x^3+t^2(t+1)^2(at+b)x+t^3(t+1)^3(ct+d)\,,\qquad (a,c\neq 0)\,.$$

Let $F(T)=T^3+bT+c$, $G(T)=T^3+(a+b)T+c+d$ *be polynomials of one variable* T.

* *If* F *and* G *have three common roots, then* $\sigma=3$.

* *If* F *and* G *have only one common root, then* $\sigma=4$.

* *If* F *and* G *have no common roots, then* $\sigma=5$.

(13) $$y^2+t^2(t+1)^2y=x^3+at^3(t+1)^3x+bt^5(t+1)^4\,,\qquad (b\neq 0)\,.$$

$$\sigma=3\,.$$

(14) $$y^2+t^2(t+1)^2y=x^3+at^3(t+1)^3x\,,\qquad (a\neq 0)\,.$$

$$\sigma=2\,.$$

(15) $$y^2+t^2(t+1)^2y=x^3+at(t+1)x^2\,,\qquad (a\neq 0)\,.$$

$$\sigma=2\,.$$

(16) $$y^2+t^2(t+1)^2y=x^3+at^2(t+1)x^2+t^2(t+1)^2(bt+c)x+t^3(t+1)^3(dt+e)\,,$$

$$(a,e,a(b+c)+d+e\neq 0)\,.$$

Let $F(T)=T^3+cT+e$, $G(T)=T^3+aT^2+(b+c)T+d+e$ *be polynomials of one vaxiable* T.

* *If* F *and* G *have two common roots, then* $\sigma=4$.

* *If either* $F(T)$ *and* $G(T)$ *or* $F(T+a)$ *and* $G(T)$ *have only one common root, then* $\sigma=5$.

* *If* f *conforms to no above cases, then* $\sigma=6$.

(17) $$y^2+t^2(t+1)^2y=x^3+t(t+1)(at+b)x^2+ct^3(t+1)^3x+dt^5(t+1)^4\,,$$

$$(ac+d,b,a+\sigma\neq 0)\,.$$

$$\begin{cases} \sigma=3 & \textit{if } d=0 \textit{ and } c=(a+b)b\,, \\ \sigma=3 & \textit{if } a=0 \textit{ and } d=b^3+bc\,, \\ \sigma=4 & \textit{if } d=0 \textit{ and } c=(a+b)b\,, \\ \sigma=4 & \textit{if } d=0 \textit{ and } c\neq(a+b)b\,, \\ \sigma=4 & \textit{if } d=0 \textit{ and } c=(a+b)b\,, \\ \sigma=5 & \textit{if } ec\neq 0\,. \end{cases}$$

(18) $$y^2+t^2(t+1)^2y=x^3+t^2(t+1)^2(at^2+bt+c)x+t^3(t+1)^3(dt^3+et+f)\,,$$
$(d, f, d+e+f\neq 0)$.

Let $F_1(T)=T^3+cT+f=(T+\lambda_{11})(T+\lambda_{12})(T+\lambda_{13})$, $F_2(T)=T^3+(a+b+c)T+(d+e+f)=(T+\lambda_{21})(T+\lambda_{22})(T+\lambda_{23})$, *and* $F_3(T)=T^2+aT+d=(T+\lambda_{31})(T+\lambda_{32})(T+\lambda_{33})$ *be polynomials of one variable* T. *then*

$$3\geqq\sigma=7-\log_2(I+J+1)\geqq 7\,.$$

Here,

$$I=\mathrm{Card.}\{(i, j, k)\,|\,\lambda_{1i}+\lambda_{2j}+\lambda_{3k}=0\}\,,$$

$$\begin{aligned} J=&(\textit{number of common roots of } F_1 \textit{ and } F_2)\\ &+(\textit{number of common roots of } F_1 \textit{ and } F_3)\\ &+(\textit{number of common roots of } F_2 \textit{ and } F_3)\,. \end{aligned}$$

(19) $$y^2+t^2(t+1)^2y=x^3\,,$$

$$\sigma=1\,.$$

(20) $$y^2+t^2(t+1)^2y=x^3+t(t+1)(at+b)x^2\,, \qquad (a, b, a+b\neq 0)\,.$$

$$\sigma=3\,.$$

(21) $$y^2+t^2(t+1)^2y=x^3+at^2(t+1)x^2+bt^3(t+1)^2x+ct^4(t+1)^3\,,$$
$(a, ab+c\neq 0)$.

$$\begin{cases} \sigma=3 & \textit{if } c=0\,, \\ \sigma=4 & \textit{if } c\neq 0\,. \end{cases}$$

The proof of the above theorem is an application of the Mordell-Weil lattice theory constructed by T. Shioda. We summarize some part of this theory for our preliminaries in Section 1, then we study the structure of some fundamental Mordell-Weil lattices in detail (Section 2). The proof of Theorem 0.3 is completed in Section 3.

The author thanks Professor T. Shioda who introduced her to the meaningful Mordell-Weil lattice theory, Professor K. Oguiso who taught her many aspects of the Mordell-Weil lattice of rational surfaces, and Professor T. Urabe who gave her a lot of information on root lattices. The author also thanks Professor T. Katsura who gave her many useful suggestions.

§ 1. Preliminaries on Mordell-Weil lattices

In this section, we summarize some part of the theory of Mordell-Weil lattices, which is the most important tool for our study. The fundamental papers of T. Shioda ([13, 14]) describe the theory entirely, so, the author requests anyone who wants to know more, to refer to these papers.

We use the following notations in this section.

k: an algebraically closed field of non-negative characteristic p.

$K=k(C)$: the function field of a smooth projective curve C/k,

E/K: an elliptic curve,

$O\in E(K)$: the origin of E,

$f: X\to C$: the associated relatively minimal elliptic surface, (we assume f is not smooth),

(P): the section of f who corresponds to $P\in E(K)$,

$E(K)^0 := \{P\in E(K) | (P)$ and (O) passes the same part of any fibers of $f\}$,

$R := \{v\in C |$ the fiber $f^{-1}(v)$ is reducible$\}$.

For $v\in R$, we decompose the singular fiber to irreducible components as follows:

$$f^{-1}(v)=\Theta_{v,0}+\sum_{i=1}^{m_v-1}\mu_{v,i}\Theta_{v,i},$$

here, $\Theta_{v,0}$ is the irreducible component which passes the zero-section (O).

The intersection matrix of the above components is denoted as follows:

$$A_v=(\Theta_{v,i}\cdot\Theta_{v,j})_{1\leq i,j\leq m_v-1}.$$

At first, Mordell-Weil lattice theory gives a bilinear form on the finitely-generated Abelian group $E(K)$ (the Mordell-Weil group).

DEFINITION 1.1. We set the following bilinear form $\langle P, P'\rangle$ $(P, P'\in E(K))$ on $E(K)$, and call $\langle\ ,\ \rangle$ the height pairing on $E(K)$.

$$\langle P, P'\rangle := -((O)-(P))\cdot((O)-(P'))-\sum_{v\in R}\mathrm{Contr}_v(P, P'),$$

here,

$$\mathrm{Contr}_v(P, P')=\begin{cases} 0, & \text{if } P \text{ or } P' \text{ passes } \Theta_{v,0}, \\ (-A_v^{-1})_{i,j} & \text{if } \Theta_{v,i}\cdot(P)=\Theta_{v,j}\cdot(P')=1. \end{cases}$$

The height pairing relates $E(K)$ to the Néron-Severi group of X $NS(X)$

as follows.

LEMMA 1.1. *Let r be the rank of the free part of Mordell-Weil group $S:=E(K)/E(K)_{\mathrm{tor}}$, and let $P_1, \cdots, P_r \in S$. Then, the followings are equivalent.*

(#) $\langle P_1, \cdots, P_r \rangle \subset S$ *is the subgroup of index* ν,

(##) $$|\det NS(X)| = \frac{|\det(\langle P_i, P_j \rangle)_{i,j}| \prod_{v \in R} m_v^{(1)}}{(\# E(K)_{\mathrm{tor}})^2 \cdot \nu^2}.$$

Since our goal is to calculate $\det NS(X)$, Lemma 1.1 is very useful for us.

§2. Lemmas on fundamental root lattices

Here, we recall from our previous paper ([8]) on unirational elliptic surfaces of base change type in characteristic 2, and prepare the lemmas on lattices concerning our elliptic surfaces.

From now on, k is fixed in characteristic 2.

Let $f: X \to \boldsymbol{P}^1$ be an irrational unirational elliptic K3 surfaces of base change type in characteristic 2, let $K' = k(s) = k(t^{1/2})$ be the purely inseparable extension of degree 2 of the funtion field $K = k(\boldsymbol{P}^1) = k(t)$, and let $g: \boldsymbol{P}^1 \to \boldsymbol{P}^1$ be the morphism corresponding to the extension K'/K. Then, we know that $X' \underset{\sim}{\cdot} X \times_{\boldsymbol{P}^1} \boldsymbol{P}^1$ is a rational elliptic surface (cf. the proof of Proposition 3.3 of [8]). So, we can relate the σ-invariant of X, say, σ, to the Mordell-Weil lattice S' of the rational elliptic surface $f': X' \to \boldsymbol{P}^1$, whose structure is already clarified by K. Oguiso and T. Shioda ([7]).

LEMMA 2.1. *Under the above notations,*

$$[S' : \tilde{S}] = \left(\frac{2^{2\sigma + r}}{\det S' \prod_{v \in R} m_v^{(1)}} \right)^{1/2} \cdot \# S_{\mathrm{tor}}$$

PROOF. Apply the fundamental theorem Lemma 1.2 and the functoriality of height pairing (cf. [14]) to X and X'. □

By the above lemma, the calculus of σ is reduced to the one of the order of the finite group $\tilde{S}/2S'$. So, in the rest of this section, we study the quotient group of modulo 2 of some fundamental root lattices.

LEMMA 2.2. *Let S be the root lattice E_6^* (resp. E_7^*) (resp. E_8) (cf. [2]). Then, the representative elements of the quotient group $S/2S$ can be selected out of the elements of S whose length is smaller than or equal to 2 (resp. 7/2) (resp. 4).*

PROOF. Here we use the data on root lattices from [2].

Let $S=E_6^*$. Since the length of the minimal vectors of E_6^* is 4/3, the length of the minimal vectors of $2E_6^*$ is 16/3. Let $x, x'\in S$ be two minimal vectors such that $x'\neq\pm x$. Then, it follows that $|x+x'|^2$ or $|x-x|^2$ is smaller than 16/3, so, $x-x'$ cannot be an element of $2S$. Since the number of the minimal vectors of S (denoted by $N(4/3)$) is 54, we can select 27 representative elements from minimal vectors.

Let $y, y'\in S$ be elements of length 2, and assume that $y'\neq\pm y$. Then, we see that x, y, y' represent different elements of $S/2S$ each other.

Now, the number of non-zero elements of $S/2S$ is $2^6-1=63$, while we see that $N(4/3)/2+N(2)/2=27+36=63$, so we can select all the representative elements such that their length is 4/3 or 2.

The case where $S=E_7^*$ or $S=E_8$ can be proved similarly. □

§3. The proof of the theorem

Here, we prove our theorem Theorem 0.3. Since the method is essentially same for 21 cases ((1), ⋯, (21) in statement of the theorem), here, we treat only the most complicated case (18).

So we fix the notations as the following.

$f: X\to\boldsymbol{P}^1$: the irrational unirational elliptic surface of base change type in characteristic 2.

$y^2+t^2(t+1)^2y=x^3+t^2(t+1)^2(at^2+bt+c)x+t^3(t+1)^3(dt^3+et+f)$, $(d, f, d+e+f\neq 0)$: the mimimal Weierstrass normal form of f.

$g: \boldsymbol{P}^1\to\boldsymbol{P}^1$: the purely inseparable morphism of degree 2.

$f': X'\simeq\to X\times_{P^1}\boldsymbol{P}^1$: the rational elliptic surface obtained from f by the base change g.

$(S, \langle\,,\rangle_S)$: the Mordell Weil lattice of f.

$(S'', \langle\,,\rangle_{S'})$: the Mordell Weil lattice of f'. (S' is isomorphic to E_8.)

$\iota: S\to S'$: the natural injection.

$\tilde{S}:=\mathrm{im}(\iota)$.

σ: the σ-invariant of X.

Let $F_1(T)=T^3+cT+f=(T+\lambda_{11})(T+\lambda_{12})(T+\lambda_{13})$, $F_2(T)=T^3+(a+b+c)T+(d+e+f)=(T+\lambda_{21})(T+\lambda_{22})(T+\lambda_{23})$ and $F_3(T)=T^3+aT+d=(T+\lambda_{31})(T+\lambda_{32})(T+\lambda_{33})$ be polynomials of one variable T.

At first, recall the representative elements of the quotient group $S'/2S'$ can be selected out of the elements of S whose length is smaller than or equal to 4.

So from Lemma 1.1, we see that

(18.1) $$2^{7-\sigma}=\#\tilde{S}/2S'=(\#\{P\in S|\langle P, P\rangle_S=1\}/2+\#\{P\in S|\langle P, P\rangle_S=2\}/16+1)$$

Recall also that the singular fibers of f are over $t=0, 1, \infty$ and all of type C_4 (cf. Néron ([6]).

Now, let us calculate the number of the minimal vectors of S.

At first, let $P\in S$ be a section such that $(P)\cdot(O)=0$. This is equivalent to $\langle P, P\rangle_S\leqq 4$. Then, we can write the coordinates of P as the following:

$$P:\begin{cases} x=\alpha_0+\cdots+\alpha_4t^4\,, \\ y=\beta_0+\cdots+\beta_6t^6\,. \end{cases}$$

If P is a minimal vector, P and O must pass different components of all the singular fibers each other. For example, if P and O pass different components of the singular fiber over $t=0$, we see, applying Néron's quadratic transformation ([6]) to f, it holds that $\alpha_0=\beta_0=\beta_1=0$ and $\alpha_1=\zeta$, where ζ is a root of $F_1(T)$. After analyzing similaly over $t=1, \infty$, we see that if $\langle P, P\rangle_S=1$, it must hold that

$$P:\begin{cases} x=\zeta t+\tau t^2+\rho t^3\,, \\ y=\beta t^2(t+1)^2\,, \end{cases}$$

where τ(resp. ρ) is a root of $F_2(T)$(resp. $F_3(T)$) and it also holds that $\zeta+\tau+\rho=0$.

In conclusion, there is 2 : 1-correspondence between the set of minimal vectors of S and the set $\{(i, j, k)\,|\,\lambda_{1i}+\lambda_{2j}+\lambda_{3k}=0\}$. Similarly, we see that there is 16 : 1-correspondence between the set of vectors of S with length 2 and the union of the set of common roots of F_1 and F_2, the set of common roots of F_1 and F_3, and the set of common roots of F_2 and F_3.

So, from the equation (18.1), we can calculate σ as in the statement of Theorem 0.3.

References

[1] M. Artin: *Supersingular K3 surfaces*, Ann. Sci. Ecole Norm. Sup 4^e serie, **7** (1974), 543-568.

[2] J. Conway and N. J. A. Sloane: *Sphere Packings, Lattices and Groups*, Grund. Math. Wiss. 290, Springer Verlag, Berlin-Heidelberg-New York, 1988.

[3] T. Katsura: *Unirational elliptic surfaces in characteristic p*, Tôhoku. Math. J. **33** (1981), 521-553.

[4] T. Katsura: *The unirationality of certain elliptic surfaces in characteristic p*, Tôhoku. Math. J. J. **36** (1984), 217-231.

[5] T. Katsura: *Generalized Kummer Surfaces and their unirationality in characteristic* p, J. Fac. Sci. Univ. Tokyo, Sec. IA **34** (1987), 1-41.

[6] A. Néron: *Modèles minimaux des variétés abéliennes sur les corps locaux et globaux*, Publ. Math. I. H. E. S. **21** (1964).

[7] K. Oguiso and T. Shioda: *The Mordell-Weil lattice of a rational elliptic surface*, preprint.

[8] M. Ohhira: *Unirational elliptic surfaces in characteristic 2*, to appear in J.

Math. Soc. Japan.

[9] M. Ohhira: *Supersingular K3 surfaces with various values of the Artin invariant in characteristic 3,* in preparation.

[10] A. N. Rudakov and I. R. Shafarevich: *Supersingular K3 surfaces over fields of characteristic* 2, Math. USSR-Izv. **13** (1979), 147-165.

[11] T. Shioda: *On elliptic modular surfaces,* J. Math. Soc. Japan **24** (1972), 20-59.

[12] T. Shioda: *Some results on unirationality of algebraic surfaces,* Math. Ann. **230** (1977), 153-168.

[13] T. Shioda: *Mordell-Weil lattices and Galois representation, I, II, III,* Proc. Japan. Acad. (1989), 268-271, 296-299, 300-303.

[14] T. Shioda: *On the Mordell-Weil Lattices,* Comment. Math. Univ. St. Pauli. **39** (1990), 211-240.

[15] J. H. Silverman: *The arithmetic of elliptic curves,* Grad. Texts in Math. 106, Springer-Verlag, Berlin-Heidelberg-New York, 1986.

[16] J. Tate: *Algorithm for determining the type of singular fibre in an elliptic pencil, in Modular Funtion of One Variable IV,* Lect. Notes in Math. **476** (1975), Springer-Verlag, Berlin-Heidelberg-New York, 33-52.