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Several Topologies Related to the Sazonov Topology

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§1. Introduction.

The Sazonov topology is a vector topology τ defined on a Hilbert space H. It was constructed by V. V. Sazonov for the purpose of generalizing the Bochner's Theorem. This topology τ is defined as follows; a function f from H into $\boldsymbol{C}$ is a characteristic function of a Radon probability measure on H iff it possesses the following three properties: 1.) $f(0)=1$, 2.) f is positive definite and 3.) f is τ-continuous at the origin.

Later this word is used in more general senses (*e.g.* [7]). There are many investigations related to the Sazonov topology. In this paper we deal with several topologies which have relation to the Sazonov topology.

§2. Notations and terminologies.

Let E be a real separable Banach space with norm $\|\cdot\|_E$, E' its topological dual, $(\cdot, \cdot)$ the natural inner product on $E'\times E$ and $\mathscr{B}(E)$ the Borel σ-algebra on E. Let F be a finite-dimensional linear subspace of E'. A subset C of E is a cylindrical set based on F if it is of the form

$$C=\{x\in E;\ ((\xi_1, x)\cdots(\xi_n, x))\in D\}$$

where $n\in N$, $\xi_1, \cdots, \xi_n\in F$ and $D\in\mathscr{B}(\boldsymbol{R}^n)$. Let $\mathcal{C}_F$ denote the set of all cylindrical sets based on F, and $\mathcal{C}=\cup\mathcal{C}_F$ where F runs over all finite dimensional subspaces of E'. It is clear that each $\mathcal{C}_F$ is a σ-algebra and $\mathcal{C}$ is an algebra. It is the important result that the σ-algebra generated by $\mathcal{C}$ is $\mathscr{B}(E)$.

A non-negative set function $\mu: \mathcal{C}\to[0, 1]$ is called a cylindrical measure if for each finite dimensional subspace F of E', $\mu|_{\mathcal{C}_F}$ is a probability measure.

Suppose μ is a cylindrical measure on E. The characteristic function of μ is the function $\hat{\mu}: E'\to\boldsymbol{C}$ defined by

$$\hat{\mu}(\xi)=\int_E \exp\{i(\xi, x)\}d\mu(x)\,.$$

DEFINITION. (i) A Borel probability measure λ on $\boldsymbol{R}$ is *Gaussian* if either

(1) $\lambda=\delta_0$ (=unit mass at $0\in\boldsymbol{R}$).

(2) There exists $a>0$ such that for $B\in\mathscr{B}(\boldsymbol{R})$

$$\lambda(B)=(2\pi a)^{-1/2}\int_B \exp\{-t^2/2a\}dt\,.$$

(ii) A cylindrical measure μ is *Gaussian* if every one-dimensional distribution $\xi\mu$ of μ is Gaussian on R.

(iii) A *Gaussian measure* on E is a Borel probability measure on E which restricts to a Gaussian cylindrical measure.

Let E and F be Banach spaces. We denote $L(E, F)$ the collection of continuous linear operators of E into F.

DEFINITION. Let μ be a Gaussian cylindrical measure on E. We say μ has *variance* A if there is a self-adjoint $A\in L(E', E)$ such that $\hat{\mu}(\xi)=\exp-(\xi, A\xi)/2$ for all $\xi\in E'$. The word "self-adjoint" means that $(\xi, A\eta)=(\eta, A\xi)$ for all ξ and η in E'. Such an operator A is positive, *i.e.* $(\xi, A\xi)\geqq 0$ for all $\xi\in E'$.

Let H be a real separable Hilbert space with norm $|\cdot|_H=\sqrt{\langle\cdot,\cdot\rangle_H}$, where $\langle\cdot,\cdot\rangle_H$ means the inner product. We identify H' and H and denote by I_H the identity operator of H. Let γ_H be the Gaussian cylindrical measure which has variance I_H. We call it the *canonical Gauss cylindrical measure.*

Let $\mathscr{P}(E)=\{A\in L(E', E)\,;\ A$ is self-adjoint and positive$\}$. P. Baxendale showed that every $A\in\mathscr{P}(E)$ is the variance of a Gaussian cylindrical measure on E and also every Gaussian measure does have a variance. The next propositions are important ([1]).

PROPOSITION 1 ([1]). *Suppose $A\in\mathscr{P}(E)$. Then there is a separable real Hilbert space H and an injection $j\in L(H, E)$ such that $A=jj^*$, where j^* is a dual operator of j.*

REMARK. Let $j^*=T$, then we get $j^{**}=j=T^*$ and $A=T^*T$. We call T satisfying the above conditions the *square root* of A.

PROPOSITION 2 ([1]). *Every Gaussian measure on E is of the form $j\gamma_H$ for an injection $j\in L(H, E)$.*

DEFINITION. Let $\mathscr{G}(E)=\{A\in\mathscr{P}(E)\,;\ A$ is a variance of a Gaussian measure$\}$.

DEFINITION. *A norm* $\|\cdot\|$ *on* H *is a measurable norm if for all* $\varepsilon>0$ there exists a finite-dimensional orthogonal projection P_0 such that for any other such projection P perpendicular to P_0, we have

$$\gamma_H\{x\in H;\ \|Px\|>\varepsilon\}<\varepsilon .$$

DEFINITION. Let $\|\cdot\|$ be a measurable norm on H, B the completion of H with respect to $\|\cdot\|$ and i the inclusion map of H into B. The triple (i, H, B) is called an *abstract Wiener space.*

We continue to explain some definitions of operators defined on H and E. $L_{(1)}(H)$ denotes the collection of trace class operators of H, *i.e.*, if $u\in L_{(1)}(H)$, then u is a compact operator of H satisfying $\sum_{n=1}^{\infty}\lambda_n<+\infty$, where λ_n's are the eigenvalues of $(u^*u)^{1/2}$. An operator is called an *S-operator* of H if it is in $L_{(1)}(H)$, positive definite and self-adjoint.

Let H and K be separable Hilbert spaces and u be a continuous linear operator of H into K satisfying $\sum_{n=1}^{\infty}|ue_n|_K^2<+\infty$ for some orthonormal basis $\{e_n\}_{n\in N}$ of H. Then we call u to be of Hilbert-Schmidt type.

Let E and F be Banach spaces. For any $p\in(0,\infty)$ an operator $T\in L(E,F)$ is called p-absolutely summing if there is a constant $c\geqq 0$ such that

$$\sum_{i=1}^{n}\|Tx_i\|_F^p\leqq c^p\sup_{\|a\|_{F'}\leqq 1}\sum_{i=1}^{n}|(a,x_i)|^p$$

for any finite family of elements $x_1,\cdots,x_n\in E$.

Now we close this section by the following definition and Proposition 3 ([4]).

DEFINITION. A Banach space E is said to be of *type* 2 if for every sequence $\{x_i\}_{i=1}^{\infty}\subset E$ satisfying $\sum_{i=1}^{\infty}\|x_i\|_E^2<+\infty$, we have that $\sum x_i\varepsilon_i$ converges *a.e.* where ε_i are independent identically distributed symmetric Bernoulli random variables, *i.e.* $P(\varepsilon_i=+1)=P(\varepsilon_i=-1)=1/2$.

PROPOSITION 3 ([4]). *The following are equivalent.*

(1) *The space E is of type* 2.

(2) *For every positive operator* $A: E'\to E$, $\exp[-1/2(y, Ay)]$ $(y\in E')$ *is the characteristic function of a Gaussian measure on E iff the square root of A is 2-absolutely summing on* $E'\to H$ (*see Proposition* 1).

REMARK. The "only if" part of (2) is true in any Banach space.

§3. Several topologies.

1. First we introduce the topology $\mathcal{T}$ which was defined by V. V. Sazonov.

Let $\mathcal{S}$ denote the family of all S-operators on H. The class of sets $\{[x\,;\, \langle Sx, x\rangle < 1], S \in \mathcal{S}\}$ defines a system of neighborhoods at the origin. $\mathcal{T}$ is the topology defined as above.

Next we show the topology $\mathcal{T}_m$ which was defined by H. H. Kuo.

The topology $\mathcal{T}_m$ denotes the weakest topology on H satisfying that all measurable seminorms are continuous.

The topology $\mathcal{T}_m$ is really stronger than $\mathcal{T}$.

2. Here we explain a topology defined on a general linear space X.

If p is a seminorm on X, then we denote by X_p the factor space $X/p^{-1}(0)$, equipped with the norm $\bar{p}$ defined by the equation $\bar{p}(\tilde{x}) = p(Q_P^{-1}\tilde{x})$, where $\tilde{x} \in X/p^{-1}(0)$ and Q_p is the canonical mapping of X onto $X/p^{-1}(0)$. We denote by $\bar{X}_p$ the Banach space obtained by completing X_p. The map $\bar{Q}_p = i_p Q_p$, where i_p is the natural embedding of X_p in $\bar{X}_p$, is easily to be continuous.

If p and q are two seminorms on X and if for each $x \in X$

$$(*) \qquad p(x) \leqq Cq(x)\,,$$

then we can define a canonical continuous linear transformation $\phi_{qp} : \bar{X}_q \to \bar{X}_p$, as follows. If $\tilde{x} \in i_q(X_q)$, then $\phi_{qp}(\tilde{x}) = \bar{Q}_p(\bar{Q}_q^{-1}\tilde{x})$ $(\in \bar{X}_p)$ (that this is well-defined follows from (*)). If $\tilde{x} \notin i_q(X_q)$ and $\tilde{x} \in \bar{X}_q$, then there exists a sequence $\{\tilde{x}_n\}$ of elements in $i_q(X_q)$ which converges to $\tilde{x}$. The sequence $\{\phi_{qp}(\tilde{x}_n)\}$ converges in $\bar{X}_p$ (this, too, follows from the inequality connecting the seminorms p and q). In this case we set $\phi_{qp}(\tilde{x}) = \lim \phi_{qp}(\tilde{x}_n)$. In what follows we usually do not distinguish between X_p and $i_p(X_p)$.

A seminorm p on X is a Hilbert seminorm if there exists on $X \times X$ a positive definite bilinear form b such that $p^2(x) = b(x, x)$ for all $x \in X$, and if p is a Hilbert seminorm, then $\bar{X}_p$ is a Hilbert space.

Let (X, τ) be a locally convex space, where τ is a topology on X. O. G. Smolyanov and S. V. Fomin defined a topology $\mathcal{F}$ associated with τ as follows. Let $\boldsymbol{P}$ be the family of seminorms on X that are determined in the following way: $p \in \boldsymbol{P}$ if and only if p is a continuous Hilbert seminorm on X for which there is a continuous Hilbert seminorm q on X such that $p(x) \leqq q(x)$ for all x and the mapping $\phi_{qp} : \bar{X}_q \to \bar{X}_p$ is a Hilbert-Schmidt transformation. $\mathcal{F}$ is the topology on X given by the family $\boldsymbol{P}$.

All topologies described as above are Sazonov topologies. Moreover, in 1975 D. Mouchtari introduced a Sazonov topology $\mathcal{M}$ on dual Banach spaces satisfying some conditions ([5]).

3. Now we shall construct three topologies defined on dual Banach spaces analogous to the above topologies.

First we define the topology $\mathcal{F}_a$ which is a generalization of $\mathcal{F}$. Let E be a real Banach space and $\boldsymbol{P}_a$ be the family of continuous Hilbert seminorms on E'. $\boldsymbol{P}_a$ is determined in the following way: $p\in\boldsymbol{P}_a$ if and only if p is a continuous Hilbert seminorm on E' for which there is a continuous seminorm q on E' such that $p(x)\leqq q(x)$ for all x, and $\bar{E}'_q$ is the dual space of some Banach space B and triple pair $(\phi^*_{qp}, \bar{E}'_p, B)$ is an abstract Wiener space. $\mathscr{F}_a$ is the topology on E' given by the family $\boldsymbol{P}_a$.

Next we introduce the topologies τ_g and τ_2. Let $\boldsymbol{G}$ and $\boldsymbol{T}$ be the families of subsets of E' defined as follows:

$\boldsymbol{G}=\{[x'\in E';\ |j^*x'|_H<1];$ where H is a real separable Hilbert space embedded in E and j is the inclusion map of H into E satisfying $jj^*\in\mathscr{G}(E)\}$,

$\boldsymbol{T}=\{[x'\in E';\ |j^*x'|_H<1];$ where H is a real separable Hilbert space embedded in E and j is the inclusion map of H into E satisfying that j^* is a 2-absolutely summing operator of E' into $H\}$.

Let τ_g be the topology that has the family $\boldsymbol{G}$ as the fundamental system of neighborhood at the origin in E'. Also let τ_2 be the topology that has $\boldsymbol{T}$ as the fundamental system of neighborhood at the origin in E'.

Clearly we have

THEOREM 1. *$\mathscr{F}_a, \tau_g$ and τ_2 are vector topologies defined on E'.*

We have many interesting results between these topologies and the Sazonov topology (see [3]).

§ 4. Relations among these topologies.

Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be two topologies defined on a space X. We denote by $\mathscr{T}_1\prec\mathscr{T}_2$ if $\mathscr{T}_1\subset\mathscr{T}_2$ and say that $\mathscr{T}_2$ is stronger than $\mathscr{T}_1$. We know that $\mathscr{T}\precneqq\mathscr{T}_m$.

THEOREM 2. *Let E be a real separable reflexive Banach space and E' be the dual of E. If $\mathscr{F}, \mathscr{F}_a, \tau_g$ and τ_2 are defined on E', then $\mathscr{F}\prec\mathscr{F}_a\prec\tau_g\prec\tau_2$. Moreover, there exist examples of E' such that $\mathscr{F}\neq\mathscr{F}_a$, $\mathscr{F}_a\neq\tau_g$ and $\tau_g\neq\tau_2$ respectively.*

PROOF. It is clear that if $p\in\boldsymbol{P}$, then there exists q such that $(\phi^*_{qp}, \bar{E}'_p, \bar{E}'_q)$ is an abstract Wiener space. Therefore we have $\mathscr{F}\prec\mathscr{F}_a$.

Next we show that $\mathscr{F}_a\prec\tau_g$. If $p\in\boldsymbol{P}_a$, then we have a seminorm q and an abstract Wiener space $(\phi^*_{qp}, \bar{E}'_p, B)$. Put $j=\pi\phi^*_{qp}$, where π is the canonical map of B into E. Since j is an inclusion map of $\bar{E}'_p$ into E and $(\phi^*_{qp}, \bar{E}'_p, B)$ is an abstract Wiener space, we have $jj^*\in\mathscr{G}(E)$ and so $\mathscr{F}_a\prec\tau_g$.

The relation $\tau_g \prec \tau_2$ is induced by Proposition 1.

Now we shall show the latter part. If E is a real separable Hilbert space, then we have the following results: $\mathscr{F}=\mathscr{T}$ and $\mathscr{F}_a=\mathscr{T}_m$ and also $\mathscr{T} \neq \mathscr{T}_m$.

The example of E' which satisfys the relation $\mathscr{F}_a \neq \tau_g$, is shown in the main theorem of [3].

Let E be a real separable Banach space which is not of type 2. It follows from Proposition 3 that $\tau_g \neq \tau_2$. □

REMARK. The assumption of reflexivity is not necessary for the proof of $\mathscr{F} \prec \mathscr{F}_a$ and $\tau_g \prec \tau_2$.

COROLLARY. *If E is a real separable Banach space and of type 2, then $\tau_g=\tau_2$.*

The relation between these topologies and $\mathscr{M}$ will be treated in elsewhere.

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