Natural Science Report, Ochanomizu University, Vol. 44, No. 1, 1993 31
お茶の水女子大学 自然科学報告 第44巻 第1号

The Image Formation Derived from the Propagation of Wave

Giiti Iwata

Department of Physics, Faculty of Science
Ochanomizu University, Tokyo
(Received April 1, 1993)

§1. Introduction

Light rays issuing from a source are observed to form an image of the source after reflections and refractions at spherical surfaces. Geometrical optics traces light rays and explains the formation of image, further allows to construct many precise optical instruments. Light is a wave, however, so it is diffracted. Image formation is analyzed by Abbe[1], taking into consideration the wave property of light. But he has grafted wave optics upon geometrical optics, borrowing notions of the latter such as focus. The position of the image there has not been derived by wave optics. We intend to derive the formation of image from the propagation of waves. §2–§5 are intended to prepare basic relations for wave propagation when a wave is reflected at a curved surface or refracted at a curved surface separating two different media. §6, 7, 8 are intended to analyze the formation of image through an optical system composed of different media separated by curved surfaces, each surface being generally perpendicular to an axis.

§2. Gauss's integral theorem and Green's formula

Gauss's integral theorem states that an integral of the divergence $\nabla\cdot\boldsymbol{A}=\operatorname{div}\boldsymbol{A}$ of a 3-dimensional vector $A=(F, G, H)$ over a finite domain D is equal to the integral of the outward normal component of the vector $\boldsymbol{A}$ over the whole surface S of the domain D, or explicitly

$$\int\nabla\cdot\boldsymbol{A}d\boldsymbol{x}=\int_D\left(\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}+\frac{\partial H}{\partial z}\right)dxdydz=\int_S\boldsymbol{A}\cdot\boldsymbol{n}dS \qquad (1)$$

where dS stands for the surface element. We assume here the existence of a function $\phi(\boldsymbol{x})$, which is positive inside the domain D, negative outside the domain and vanishes at the surface S. For example, if the domain D is a sphere of radius r, centered at the origin, we may put $\phi=r^2-x^2-y^2-z^2$.

Then we may write (1) in the following form

$$\int_D\left(\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}+\frac{\partial H}{\partial z}\right)dxdydz=\int\left(\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}+\frac{\partial H}{\partial z}\right)\theta(\phi)dxdydz$$

where the function $\theta(x)$ is the Heaviside function, $\theta(x)$ being equal to 1 for $x>0$, 0 for $x<0$, and 1/2 for $x=0$. The integration on the right side extends to the whole space. By partial integration we change the term related to F as follows,

$$\int_{-\infty}^{\infty}\frac{\partial F}{\partial x}\theta(\phi)dx=F\theta(\phi)\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}F\delta(\phi)\frac{\partial\phi}{\partial x}dx$$

$$=-\int_{-\infty}^{\infty}F\delta(\phi)\frac{\partial\phi}{\partial x}dx$$

since $\theta(\phi)$ vanishes outside the domain D. Accordingly we may write Gauss's integral theorem as follows

$$\int_D\nabla\cdot\boldsymbol{A}d\boldsymbol{x}=-\int\left(F\frac{\partial\phi}{\partial x}+G\frac{\partial\phi}{\partial y}+H\frac{\partial\phi}{\partial z}\right)\delta(\phi)d\boldsymbol{x}$$

$$=-\int\boldsymbol{A}\cdot\nabla\phi\,\delta(\phi)d\boldsymbol{x}\,,\qquad d\boldsymbol{x}=dxdydz\,. \tag{2}$$

So we have an easier representation of Gauss's theorem in place of (1), where the surface element dS is not so easy to treat. The representation (2) holds irrespective of the dimensionality of the space.

Referring to a Cartesian coordinate system $x_1, x_2, \cdots, x_n$, a function $u(\boldsymbol{x})$ is assumed to satisfy the following differential equation

$$\sum a_{kl}\frac{\partial^2 u}{\partial x_k\partial x_l}+\sum b_k\frac{\partial u}{\partial x_k}+cu=-f \tag{3}$$

within a domain and a given boundary condition at the surface of the domain. Summation $\sum$ means to make the sum with respect to suffices appearing twice. Introducing an operator L defined by

$$L=\sum a_{kl}\frac{\partial^2}{\partial x_k\partial x_l}+\sum b_k\frac{\partial}{\partial x_k}+c \tag{4}$$

$a_{kl}=a_{lk}$, b_k, c, being all constants, we may replace (3) by

$$Lu=-f\,. \tag{5}$$

For any two functions u, v an operator M defined by

$$M(v)=\sum\frac{\partial^2 v}{\partial x_k\partial x_l}a_{kl}-\sum\frac{\partial v}{\partial x_k}b_k+vc$$

satisfies an equality

$$vLu-M(v)u=\sum\frac{\partial F_k}{\partial x_k} \tag{6}$$

the right member representing the divergence of a vector $\boldsymbol{F}=(F_1, F_2, \cdots, F_n)$ defined by

$$F_k=\sum\left(va_{kl}\frac{\partial u}{\partial x_l}-\frac{\partial v}{\partial x_l}a_{kl}u\right)+vb_k u\,. \tag{7}$$

Using the function $\phi(x)$ introduced earlier, we get Green's formula

$$\int(vLu-M(v)u)\theta(\phi(\boldsymbol{x}))d\boldsymbol{x}=\int\sum\frac{\partial F_k}{\partial x_k}\theta(\phi(\boldsymbol{x}))d\boldsymbol{x}=-\int\boldsymbol{F}\cdot\nabla\phi\,\delta(\phi(\boldsymbol{x}))d\boldsymbol{x} \tag{8}$$

by virtue of (2). If we take a solution $v(\boldsymbol{x},\boldsymbol{y})$ to the equation

$$M(v)=-\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{9}$$

we have then, from (8), (9),

$$\int(v(\boldsymbol{x},\boldsymbol{y})Lu-M(v(\boldsymbol{x},\boldsymbol{y})u)\theta(\phi(\boldsymbol{x}))d\boldsymbol{x}=\int(-v(\boldsymbol{x},\boldsymbol{y})f(\boldsymbol{x})+\delta(\boldsymbol{x}-\boldsymbol{y})u)\theta(\phi(\boldsymbol{x}))d\boldsymbol{x}$$

$$=-\int v(\boldsymbol{x},\boldsymbol{y})f(\boldsymbol{x})\theta(\phi(\boldsymbol{x}))d\boldsymbol{x}+u(\boldsymbol{y})\theta(\phi(\boldsymbol{y}))$$

consequently we get

$$u(\boldsymbol{y})\theta(\phi(\boldsymbol{y}))=\int v(\boldsymbol{x},\boldsymbol{y})f(\boldsymbol{x})\theta(\phi(\boldsymbol{x}))d\boldsymbol{x}-\int\boldsymbol{F}\cdot\nabla\phi\,\delta(\phi(\boldsymbol{x}))d\boldsymbol{x}\,. \tag{10}$$

Therefore, if the value of F is given at the boundary, we can determine the value of $u(\boldsymbol{y})$ inside the domain. If L is equal to $\Delta+c$, Δ being the Laplacian operator, M is equal to L, and vector $\boldsymbol{F}$ is equal to $v\nabla u-(\nabla v)u$. So we have

$$u(\boldsymbol{y})\theta(\phi(\boldsymbol{y}))=\int v(\boldsymbol{x},\boldsymbol{y})f(\boldsymbol{x})\theta(\phi(\boldsymbol{x}))d\boldsymbol{x}$$

$$+\int\{(\nabla v(\boldsymbol{x},\boldsymbol{y})\cdot\nabla\phi)u(\boldsymbol{x})-v(\boldsymbol{x},\boldsymbol{y})\nabla u\cdot\nabla\phi\}\delta(\phi(\boldsymbol{x}))d\boldsymbol{x}\,. \tag{11}$$

Most boundary conditions encountered in physics specify u at the boundary or the normal derivative of u, that is, $\partial u/\partial n$, or a linear combination of u and $\partial u/\partial n$, that is, $\alpha u+\beta\partial u/\partial n$. The vector $\nabla\phi$ is normal to the boundary surface $\phi=0$, so $\partial u/\partial n$ is equal to $\nabla\phi\cdot\nabla u/|\nabla\phi|$. We denote $\nabla u\cdot\nabla\phi$ by $\partial_\phi u$ and replace the normal derivative of u by $\partial_\phi u$ under the assumption $|\nabla\phi|=1$ on the boundary. The assumption is readily realized by taking $\phi/|\nabla\phi|$ as a new ϕ, because we see that $\nabla(\phi/|\nabla\phi|)=\nabla\phi/|\nabla\phi|+$

$\phi\nabla(1/|\nabla\phi|)$ and ϕ vanishes en the boundary.

Further we set the condition $\alpha^2+\beta^2=1$. Then we have the following relation

$$
\begin{aligned}
&(\nabla\phi\cdot\nabla v)u-v(\nabla\phi\cdot\nabla u)=(\partial_\phi v)u-v\partial_\phi u\\
&=\begin{vmatrix} u & \partial_\phi u \\ v & \partial_\phi v \end{vmatrix}\cdot\begin{vmatrix} \alpha & -\beta \\ \beta & \alpha \end{vmatrix}=\begin{vmatrix} \alpha u+\beta\partial_\phi u & -\beta u+\alpha\partial_\phi u \\ \alpha v+\beta\partial_\phi v & -\beta v+\alpha\partial_\phi v \end{vmatrix}\\
&=(\alpha u+\beta\partial_\phi u)(-\beta v+\alpha\partial_\phi v)-(\alpha v+\beta\partial_\phi v)(-\beta u+\alpha\partial_\phi u)\,. \qquad (12)
\end{aligned}
$$

Accordingly we get a relation equivalent to (11)

$$
\begin{aligned}
u(\boldsymbol{y})=\int v(\boldsymbol{x},\boldsymbol{y})f(\boldsymbol{x})\theta(\phi(\boldsymbol{x}))d\boldsymbol{x}+\int\{&(\alpha u+\beta\partial_\phi u)(-\beta v+\alpha\partial_\phi v)\\
&-(\alpha v+\beta\partial_\phi v)(-\beta u+\alpha\partial_\phi u)\}\delta(\phi(\boldsymbol{x}))d\boldsymbol{x}\,.
\end{aligned}
$$

If v is chosen so as to satisfy the condition $\alpha v+\beta\partial_\phi v=0$, (11) may be written as

$$
u(\boldsymbol{y})=\int v(\boldsymbol{x},\boldsymbol{y})f(\boldsymbol{x})\theta(\phi(\boldsymbol{x}))d\boldsymbol{x}+\int(\alpha u+\beta\partial_\phi u)(-\beta v+\alpha\partial_\phi v)\delta(\phi(\boldsymbol{x}))d\boldsymbol{x}\,. \quad (13)
$$

This gives the required solution that takes the specified value of $\alpha u+\beta\partial_\phi u$ at the boundary surface. So the boundary value problem reduces to the determination of a solution v to the differential equation (9), that satisfies the boundary condition $\alpha v+\beta\partial_\phi v=0$.

§ 3. Green's function for a Helmholtz equation

The wave equation

$$
\left(\Delta-\frac{\partial^2}{\partial t^2}\right)u=-\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t') \qquad (1)
$$

may be changed into a Helmholtz equation

$$
(\Delta-q^2)v=-\delta(\boldsymbol{x}-\boldsymbol{x}') \qquad (2)
$$

by means of a Laplace transform

$$
v=\int_0^\infty e^{-q(t-t')}u\,dt\,, \qquad \operatorname{Re} q>0 \qquad (3)
$$

with an initial condition $u=\partial u/\partial t=0$ at $t=0$.

The Green's function for the Helmholtz equation (2) is given by

$$
G(\boldsymbol{x},\boldsymbol{x}')=\frac{1}{(2\pi)^n}\int\frac{e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')}}{q^2+\boldsymbol{p}^2}d\boldsymbol{p} \qquad (4)
$$

the integration extending to the whole space. A remark that $\mathrm{Re}(q+p^2/q)$ is positive since $\mathrm{Re}\, q>0$ together with $\mathrm{Re}(1/q)>0$, leads to the representation

$$\frac{1}{q^2+p^2}=\frac{1}{q+p^2/q}\frac{1}{q}=\frac{1}{q}\int_0^\infty e^{-s(q+p^2/q)}ds\,.$$

Hence we have

$$v=\frac{1}{(2\pi)^n}\int d\boldsymbol{p}\int_0^\infty e^{-s(q+p^2/q)+i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\frac{ds}{q}$$

$$=\frac{1}{(2\pi)^n}\int_0^\infty e^{-sq-q(\boldsymbol{x}-\boldsymbol{x}')^2/4s}\left(\frac{\pi q}{s}\right)^{n/2}\frac{ds}{q}\,.$$

An integral representation of modified Bessl function

$$K_\nu(z)=\frac{1}{2}\int_0^\infty e^{-z(s+1/s)/2}s^{-\nu-1}ds$$

gives

$$G(\boldsymbol{x},\boldsymbol{x}')=\frac{1}{(2\pi)^{n/2}}\left(\frac{q}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)^\nu K_\nu(q|\boldsymbol{x}-\boldsymbol{x}'|)\,,\qquad \nu=\frac{n}{2}-1\,. \tag{5}$$

Green's function has the following properties

1) $$G(\boldsymbol{x},\boldsymbol{x}')=G(\boldsymbol{x}',\boldsymbol{x})$$

as is easily seen from (5).

2) For $$|\boldsymbol{x}-\boldsymbol{x}'|\longrightarrow\infty\,,$$

$$G(\boldsymbol{x},\boldsymbol{x}')\sim\frac{q^{(n-2)/2}}{2(2\pi|\boldsymbol{x}-\boldsymbol{x}'|)^{(n-1)/2}}e^{-q|\boldsymbol{x}-\boldsymbol{x}'|} \tag{6}$$

from an asymptotic expansion of modified Bessel function.

3) For $$|\boldsymbol{x}-\boldsymbol{x}'|\longrightarrow 0$$

$$G(x,x')\sim\frac{\Gamma(\nu)}{4\pi^{n/2}}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|^{n-2}} \tag{7}$$

4) $$\lim_{x_1\to x_1'}\frac{\partial}{\partial x_1}G(\boldsymbol{x},\boldsymbol{x}')=-\frac{1}{2}\delta(x_2-x_2')\cdots\delta(x_n-x_n')\cdot\lim_{x_1\to x_1'}\mathrm{sgn}\,(x_1-x_1')\,. \tag{8}$$

In the integral representation of Green's function (4), an integration with respect to p_1 gives

$$\frac{1}{2\pi}\int_{-\infty}^\infty\frac{e^{ip_1(x_1-x_1')}}{p_1^2+\lambda^2}dp_1=\frac{e^{-\lambda|x_1-x_1'|}}{2\lambda},\quad \lambda=\sqrt{(p_2^2+\cdot+p_n^2+q^2)},\quad \mathrm{Re}\,\lambda>0\,. \tag{9}$$

A diffrentiation with respect to x_1 leads to

$$-\frac{1}{2}e^{-\lambda|x_1-x_1'|}\,\mathrm{sgn}\,(x_1-x_1')$$

which tends to

$$-\frac{1}{2}\lim_{x_1\to x_1'}\mathrm{sgn}\,(x_1-x_1')$$

as $x_1\to x_1'$. Remaining integrations in (4) give $\delta(x_2-x_2')\cdots\delta(x_n-x_n')$

5)
$$\int G(\boldsymbol{y}, \boldsymbol{x})\frac{\partial}{\partial x_1}G(\boldsymbol{x}, \boldsymbol{z})\delta(x_1)d\boldsymbol{x}=\frac{1}{2}G(\boldsymbol{y}^\dagger, \boldsymbol{z}^{\dagger *})\,\mathrm{sgn}\,z_1 \tag{10}$$

$$\boldsymbol{y}^\dagger=(|y_1|, y_2, \cdots, y_n)\,,\quad \boldsymbol{z}^{\dagger *}=(-|z_1|, z_2, \cdots, z_n)\,.$$

The suffix † here means an operation where the first component of vector is changed into its absolute value. The suffix * means another operation where the first component of a vector changes its sign. $z^{\dagger *}$ means $(z^\dagger)^*$, so $z^{\dagger *}$ differs from $z^{*\dagger}$ in general. The integration in (10) extends to the whole space. To prove the relation (10), we use two integral representations (4) for two Green's functions and have the left member of (10)

$$\text{l.m. of }(10)=\frac{1}{(2\pi)^{2n}}\iiint\frac{e^{i\boldsymbol{p}\cdot(\boldsymbol{y}-\boldsymbol{x})}}{\boldsymbol{p}^2+q^2}d\boldsymbol{p}\frac{\partial}{\partial x_1}\frac{e^{i\boldsymbol{r}\cdot(\boldsymbol{x}-\boldsymbol{z})}}{\boldsymbol{r}^2+q^2}d\boldsymbol{r}\delta(x_1)d\boldsymbol{x}\,.$$

Factors involving p_1 and r_1 are integrated to yield

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ip_1(y_1-x_1)}}{p_1^2+\lambda^2}dp_1=\frac{e^{-\lambda|y_1-x_1|}}{2\lambda}\,,\quad \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ir_1(x_1-z_1)}}{r_1^2+\mu^2}dr_1=\frac{e^{-\mu|x_1-z_1|}}{2\mu}$$

$$\lambda=\sqrt{(q^2+p_2^2+\cdots+p_n^2)}\,,\ \mathrm{Re}\,\lambda>0\,,\quad \mu=\sqrt{(q^2+r_2^2+\cdots+r_n^2)}\,,\ \mathrm{Re}\,\mu>0\,.$$

Subsequent integrations with respect to $x_2, \cdots, x_n$ yield the product of delta functions $\delta(p_2-r_2)\cdots\delta(p_n-r_n)$. The product makes μ equal to λ. Then we get

$$\frac{\partial}{\partial x_1}\frac{e^{-\lambda|x_1-z_1|}}{2\lambda}=\frac{1}{2}e^{-\lambda|x_1-z_1|}\,\mathrm{sgn}\,(z_1-x_1)\,.$$

After integrating with respect to $r_2, \cdots, r_n$, we have

$$\text{l.m. of }(10)=\frac{1}{2}\frac{1}{(2\pi)^{n-1}}\int e^{ip_2(y_2-z_2)+\cdots+ip_n(y_n-z_n)}\frac{e^{-\lambda(|y_1|+|z_1|)}}{2\lambda}dp_2\cdots dp_n\,\mathrm{sgn}\,z_1\,,$$

and utilizing (9)

$$=\frac{1}{2}\frac{1}{(2\pi)^n}\int e^{ip_1(|y_1|+|z_1|)+ip_2(y_2-z_2)+\cdots+ip_n(y_n-z_n)}\cdot\frac{d\boldsymbol{p}\,\mathrm{sgn}\,z_1}{p^2+q^2}$$

therefore we get (10).

§ 4. Setting up of a unifying integral equation in one domain

From the preceding section § 2, we see that determination of a solution to the wave equation (1) satisfying the given boundary condition $\alpha u+\beta\partial_\phi u=h$ reduces to determination of a solution $v(\boldsymbol{x})$ to the Helmholtz equation (2) satisfying the boundary condition $\alpha v+\beta\partial_\phi v=0$. Remembering the relation (13) in § 2, we see that the solution $v(\boldsymbol{x})$ should satisfy the following integral equation

$$v(\boldsymbol{y})=G(\boldsymbol{y},\boldsymbol{x})+\int\{(\alpha v(\boldsymbol{x})+\beta\partial_\phi v(\boldsymbol{x}))(-\beta G(\boldsymbol{y},\boldsymbol{x})+\alpha\partial\phi \mathrm{G}(\boldsymbol{y},\boldsymbol{x}))$$
$$-(-\beta v(\boldsymbol{x})+\alpha\partial_\phi v(\boldsymbol{x}))(\alpha G(\boldsymbol{y},\boldsymbol{x})+\beta\partial_\phi G(\boldsymbol{y},\boldsymbol{x}))\}\delta(\phi(\boldsymbol{x}))d\boldsymbol{x}\,. \tag{1}$$

We remark here that ϕ depends on $\boldsymbol{x}$, not on $\boldsymbol{y}$.

If we impose the boundary condition $\alpha v+\beta\partial_\phi v=0$, this equation becomes

$$v(\boldsymbol{y})=G(\boldsymbol{y},\boldsymbol{x}')-\int(-\beta v(\boldsymbol{x})+\alpha\partial_\phi v(\boldsymbol{x}))(\alpha G(\boldsymbol{y},\boldsymbol{x})+\beta\partial_\phi G(\boldsymbol{y},\boldsymbol{x}))\delta(\phi(\boldsymbol{x}))d\boldsymbol{x}\,. \tag{2}$$

We want to state that a solution v to this integral equation satisfies both the equation (2) in § 3 in the domain D and the boundary condition $\alpha v+\beta\partial_\phi v=0$ on the boundary. In short, the equation (2) in § 3 and the boundary condition are unified into the single integral equation (2).

Operating $\Delta_y-q^2=\sum(\partial^2/\partial y_k^2)-q^2$ on both sides of (2), we have

$$(\Delta_y-q^2)v(\boldsymbol{y})=-\delta(\boldsymbol{y}-\boldsymbol{x}')-\int(-\beta v(\boldsymbol{x})+\alpha\partial_\phi v(\boldsymbol{x}))$$
$$\cdot(-\alpha\delta(\boldsymbol{y}-\boldsymbol{x})+\beta\partial_\phi\delta(\boldsymbol{y}-\boldsymbol{x}))\delta(\phi(\boldsymbol{x}))d\boldsymbol{x}\,.$$

Since $\boldsymbol{y}$ is inside the domain and the integration variable $\boldsymbol{x}$ is on the boundary, $\delta(\boldsymbol{y}-\boldsymbol{x})$ vanishes. Therefore we see that

$$(\Delta_y-q^2)v(\boldsymbol{y})=-\delta(\boldsymbol{y}-\boldsymbol{x}') \tag{3}$$

whence the equation (1) may be derived. Then the difference of (1) and (2) should vanish, so we have

$$\int(\alpha v(\boldsymbol{x})+\beta\partial_\phi v(\boldsymbol{x}))(-\beta G(\boldsymbol{y},\boldsymbol{x})+\alpha\partial_\phi G(\boldsymbol{y},\boldsymbol{x}))\delta(\phi(\boldsymbol{x}))d\boldsymbol{x}=0\,. \tag{4}$$

We set here

$$\alpha v(\boldsymbol{x})+\beta\partial_\phi v(\boldsymbol{x})=w(\boldsymbol{x}) \tag{5}$$

then (4) becomes

$$\int w(\boldsymbol{x})(\beta G(\boldsymbol{y},\boldsymbol{x})-\alpha\partial_\phi G(\boldsymbol{y},\boldsymbol{x}))\delta(\phi(\boldsymbol{x}))d\boldsymbol{x}=0\,. \tag{6}$$

Here we employ an identity

$$\int \nabla \cdot (G(\boldsymbol{y}, \boldsymbol{x}) w(\boldsymbol{x}) \delta(\phi(\boldsymbol{x})) \nabla \phi) d\boldsymbol{x} = 0 \tag{7}$$

where $\nabla\phi$ is a vector. An integral of the divergence of a vector may be turned into an integral at the boundary of domain. In this case, the integral extends to the whole space. The Green's function $G(\boldsymbol{y}, \boldsymbol{x})$ decreases exponentially at a great distance, as shown in (6) in § 3. Under the assumption that $w(\boldsymbol{x})$ remains finite at infinity we attain to (7). The integrand of (7) turns out to be

$$\begin{aligned} &\nabla G(\boldsymbol{y}, \boldsymbol{x}) \cdot \nabla \phi w(\boldsymbol{x}) \delta(\phi(\boldsymbol{x})) + G(\boldsymbol{y}, \boldsymbol{x}) \nabla \cdot (w(\boldsymbol{x}) \delta(\phi(\boldsymbol{x})) \nabla \phi) \\ &= \partial_\phi G(\boldsymbol{y}, \boldsymbol{x}) w(\boldsymbol{x}) \delta(\phi(\boldsymbol{x})) + G(\boldsymbol{y}, \boldsymbol{x}) \nabla \cdot (w(\boldsymbol{x}) \delta(\phi(\boldsymbol{x})) \nabla \phi) . \end{aligned} \tag{8}$$

Consequently we may write (6) as

$$\int G(\boldsymbol{y}, \boldsymbol{x}) P(\boldsymbol{x}) d\boldsymbol{x} = 0 \tag{9}$$

$$P(\boldsymbol{x}) = \beta w(\boldsymbol{x}) \delta(\phi(\boldsymbol{x})) + \alpha \nabla \cdot (w(\boldsymbol{x}) \delta(\phi(\boldsymbol{x})) \nabla \phi) . \tag{10}$$

Operating $\Delta_y - q^2$ on (9), we conclude that

$$\begin{aligned} P(\boldsymbol{x}) &= \beta w(\boldsymbol{x}) \delta(\phi(\boldsymbol{x})) + \alpha w(\boldsymbol{x}) \delta(\phi(\boldsymbol{x})) \Delta \phi + \alpha w(\boldsymbol{x}) \delta'(\phi(\boldsymbol{x})) (\nabla \phi)^2 + \alpha \partial_\phi w \delta(\phi(\boldsymbol{x})) \\ &= (\beta w(\boldsymbol{x}) + \alpha w(\boldsymbol{x}) \Delta \phi + \alpha \partial_\phi w) \delta(\phi(\boldsymbol{x})) + \alpha w(\boldsymbol{x}) \delta'(\phi(\boldsymbol{x})) = 0 . \end{aligned} \tag{11}$$

We multiply $P(\boldsymbol{x})$ by $\phi(\boldsymbol{x})$ and get

$$\alpha w(\boldsymbol{x}) \delta(\phi(\boldsymbol{x})) = 0$$

because of the properties of the delta function $x\delta(x) = 0$ and $x\delta'(x) = -\delta(x)$. So, if $\alpha \neq 0$, we conclude that the boundary condition $w(\boldsymbol{x})\delta(\phi(\boldsymbol{x})) = 0$ is satisfied by the solution to (4). If $\alpha = 0$, then $\beta = 1$ and the same conclusion holds more readily from (11).

EXAMPLE 1. Reflection at the plane $x_1 = 0$. A boundary condition $v = 0$ is set. The domain is defined by $\phi = x_1 > 0$. Equation (2) becomes

$$v(\boldsymbol{y}) = G(\boldsymbol{y}, \boldsymbol{x}') - \int \frac{\partial v}{\partial x_1} G(\boldsymbol{y}, \boldsymbol{x}) \delta(x_1) d\boldsymbol{x} . \tag{12}$$

A solution to the equation (2) in § 3 and the boundary condition $v = 0$ at the plane $x_1 = 0$ is, as is well known,

$$v(\boldsymbol{x}) = G(\boldsymbol{x}, \boldsymbol{x}') - G(\boldsymbol{x}, \boldsymbol{x}'^*) \tag{13}$$

the point $\boldsymbol{x}'^*$ being a point of reflection of $\boldsymbol{x}$ with respect to the plane $x_1 = 0$. If we use (13), we see that

$$\int \frac{\partial}{\partial x_1}(G(\boldsymbol{x}, \boldsymbol{x}') - G(\boldsymbol{x}, \boldsymbol{x}'^*))G\boldsymbol{y}, \boldsymbol{x})\delta(x_1)d\boldsymbol{x}$$

$$= \frac{1}{2}G(\boldsymbol{y}^\dagger, \boldsymbol{x}'^*)\operatorname{sgn} x_1' - \frac{1}{2}G(\boldsymbol{y}^\dagger, \boldsymbol{x}'^{*\dagger *})\operatorname{sgn}(-x_1')$$

$$= \frac{1}{2}G(\boldsymbol{y}, \boldsymbol{x}'^*) + \frac{1}{2}G(\boldsymbol{y}, \boldsymbol{x}'^*) = G(\boldsymbol{y}, \boldsymbol{x}'^*)$$

hence the solution (13) satisfies the equation (2) in § 3 and the boundary condition. When the boundary condition is set $\partial v/\partial x_1 = 0$, we have the equation

$$v(\boldsymbol{y}) = G(\boldsymbol{y}, \boldsymbol{x}') + \int \frac{\partial G(\boldsymbol{y}, \boldsymbol{x})}{\partial x_1} v(\boldsymbol{x})\delta(x_1)d\boldsymbol{x} \tag{14}$$

instead of (12), and the solution

$$v(\boldsymbol{x}) = G(\boldsymbol{x}, \boldsymbol{x}') + G(\boldsymbol{x}, \boldsymbol{x}'^*)\,. \tag{15}$$

It is readily shown, by virtue of (10) in § 3, and $x_1 > 0$, $x_1' > 0$, that

$$\int v(\boldsymbol{x}) \frac{\partial G(\boldsymbol{y}, \boldsymbol{x})}{\partial x_1}\delta(x_1)d\boldsymbol{x} = \int (G(\boldsymbol{x}, \boldsymbol{x}') + G(\boldsymbol{x}, \boldsymbol{x}'^*)) \frac{\partial G(\boldsymbol{y}, \boldsymbol{x})}{\partial x_1}\delta(x_1)d\boldsymbol{x}$$

$$= \frac{1}{2}G(\boldsymbol{x}'^\dagger, \boldsymbol{y}^{\dagger *})\operatorname{sgn} y_1 + \frac{1}{2}G(\boldsymbol{x}'^{*\dagger}, \boldsymbol{y}^{\dagger *})\operatorname{sgn} y_1$$

$$= \frac{1}{2}G(\boldsymbol{x}', \boldsymbol{y}^*) + \frac{1}{2}G(\boldsymbol{x}', \boldsymbol{y}^*) = G(\boldsymbol{x}', \boldsymbol{y}^*) = G(\boldsymbol{y}, \boldsymbol{x}'^*)\,.$$

The solution to the equation (2) may be constructed as a series

$$v = v_0 + v_1 + v_2 + \cdots + v_k + \cdots \tag{16}$$

where the successive terms are formed as

$$v_{k+1} = -\int \frac{\partial v_k}{\partial x_1} G(\boldsymbol{y}, \boldsymbol{x})\delta(x_1)d\boldsymbol{x}\,, \qquad k = 0, 1, 2, \cdots$$

and the first term is set

$$v_0 = G(\boldsymbol{y}, \boldsymbol{x}')\,.$$

If the series (16) converges, it certainly represents a solution to (12). We see that

$$v_1(\boldsymbol{y}) = -\int G(\boldsymbol{y}, \boldsymbol{x}) \frac{\partial G(\boldsymbol{x}, \boldsymbol{x}')}{\partial x_1}\delta(x_1)d\boldsymbol{x}$$

$$= -\frac{1}{2}G(\boldsymbol{y}^\dagger, \boldsymbol{x}'^{\dagger *})\operatorname{sgn} x_1' = -\frac{1}{2}G(\boldsymbol{y}, \boldsymbol{x}'^*)$$

$$v_2(\boldsymbol{y})=\frac{1}{2}\int G(\boldsymbol{y},\boldsymbol{x})\frac{\partial G(\boldsymbol{x},\boldsymbol{x}'^*)}{\partial x_1}\delta(x_1)d\boldsymbol{x}$$

$$=\frac{1}{4}G(\boldsymbol{y}^\dagger,\boldsymbol{x}'^{*\dagger*})\,\mathrm{sgn}\,(-x_1')=-\frac{1}{4}G(\boldsymbol{y},\boldsymbol{x}'^*)$$

and so forth, consequently the series (16) converges to the solution (13)

$$v_0(\boldsymbol{y})+v_1(\boldsymbol{y})+v_2(\boldsymbol{y})+\cdots=G(\boldsymbol{y},\boldsymbol{x}')-\frac{1}{2}G(\boldsymbol{y},\boldsymbol{x}'^*)-\frac{1}{4}G(\boldsymbol{y},\boldsymbol{x}'^*)-\cdots$$

$$=G(\boldsymbol{y},\boldsymbol{x}')-G(\boldsymbol{y},\boldsymbol{x}'^*)\,.$$

If the boundary condition is set $\partial v/\partial n=0$ instead of $v=0$, the successive approximation series gives the solution (15).

§ 5. Setting up of unifying integral equations in two domains separated by a boundary surface

When two domains D and D' are separated by a boundary $\phi=0$, a solution v is required to satisfy an equation

$$(\Delta-q^2)v=-\delta(\boldsymbol{x}-\boldsymbol{x}') \tag{1}$$

in the domain D ($\phi>0$), and a solution w is required to satisfy an equation

$$(\Delta-r^2)w=0 \tag{2}$$

in the domain D' ($\phi<0$). Two solutions v and w are further required to satisfy on the boundary $\phi=0$ two conditions

$$\alpha v=w \tag{3}$$

$$\beta\partial_\phi v=\partial_\phi w\,. \tag{4}$$

The equation (2) lacks the term representing the source since a wave is considered to expand from the source at $\boldsymbol{x}'$ and cross the boundary into the doman $\phi<0$, to be refracted. We put the Green's function in the domain $\phi>0$ as $G(\boldsymbol{x},\boldsymbol{y})$ satisfying

$$(\Delta-q^2)G(\boldsymbol{x},\boldsymbol{y})=-\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{5}$$

and the Green's function in the domain $\phi<0$ as $H(\boldsymbol{x},\boldsymbol{y})$ satisfying

$$(\Delta-r^2)H(\boldsymbol{x},\boldsymbol{z})=-\delta(\boldsymbol{x}-\boldsymbol{z})\,. \tag{6}$$

Since v satisfies (1), it satisfies the following relation

$$v(\boldsymbol{y})=G(\boldsymbol{y},\boldsymbol{x}')+\int\{v(\boldsymbol{x})\partial_\phi G(\boldsymbol{y},\boldsymbol{x})-\partial_\phi v(\boldsymbol{x})G(\boldsymbol{y},\boldsymbol{x})\}\delta(\phi(\boldsymbol{x}))d\boldsymbol{x} \tag{7}$$

which becomes, by virtue of the boundary conditions (3) and (4),

$$v(\boldsymbol{y})=G(\boldsymbol{y},\boldsymbol{x}')+\int\left\{\frac{1}{\alpha}w(\boldsymbol{x})\partial_\phi G(\boldsymbol{y},\boldsymbol{x})-\frac{1}{\beta}\partial_\phi w(\boldsymbol{x})G(\boldsymbol{y},\boldsymbol{x})\right\}\delta(\phi(\boldsymbol{x}))d\boldsymbol{x}. \quad (8)$$

Similarly $w(\boldsymbol{z})$ satisfies

$$w(\boldsymbol{z})=-\int\{w(\boldsymbol{x})\partial_\phi H(\boldsymbol{z},\boldsymbol{x})-\partial_\phi w(\boldsymbol{x})H(\boldsymbol{z},\boldsymbol{x})\}\delta(\phi(\boldsymbol{x}))d\boldsymbol{x}. \quad (9)$$

It is to be noted here that, because of the relation $\phi<0$ in the domain D' the sign of ∂_ϕ is changed. The boundary conditions (3) and (4) change (9) into

$$w(\boldsymbol{z})=-\int\{\alpha v(\boldsymbol{x})\partial_\phi H(\boldsymbol{z},\boldsymbol{x})-\beta\partial_\phi w(\boldsymbol{x})H(\boldsymbol{z},\boldsymbol{x})\}\delta(\phi(\boldsymbol{x}))d\boldsymbol{x}. \quad (10)$$

The relation (8) and (10) are satisfied rightly when v and w satisfy the equations (1) and (2) and the boundary conditions (3) and (4).

We state, similarly to § 4, that if v and w satisfy (8) and (10) simultaneously, then v and w satisfy the equations (1) and (2) and the boundary conditions (3) and (4).

It is readily shown as in § 4 that v and w satisfy (1) and (2) repectively by operating Δ_y-q^2 and Δ_z-r^2 on (8) and (10) respectively. If v satisfies (8), then v satisfies (1) so that (7) follows. Consequently from (7) and (8) follows the relation

$$\int\left\{\left(v(\boldsymbol{x})-\frac{1}{\alpha}w(\boldsymbol{x})\right)\partial_\phi G(\boldsymbol{y},\boldsymbol{x})-\left(\partial_\phi v(\boldsymbol{x})-\frac{1}{\beta}\partial_\phi w(\boldsymbol{x})\right)G(\boldsymbol{y},\boldsymbol{x})\right\}\delta(\phi(\boldsymbol{x}))d\boldsymbol{x}=0. \quad (11)$$

Similarly, from (9) and (10) follows the relation

$$\int\{(\alpha v(\boldsymbol{x})-w(\boldsymbol{x}))\partial_\phi H(\boldsymbol{z},\boldsymbol{x})-(\beta\partial_\phi v(\boldsymbol{x})-\partial_\phi w(\boldsymbol{x}))H(\boldsymbol{z},\boldsymbol{x})\}\delta(\phi(\boldsymbol{x}))d\boldsymbol{x}=0. \quad (12)$$

Here we put

$$v(\boldsymbol{x})-\frac{1}{\alpha}w(\boldsymbol{x})=A(\boldsymbol{x}), \quad \partial_\phi v(\boldsymbol{x})-\partial_\phi w(\boldsymbol{x})/\beta=B(\boldsymbol{x}) \quad (13)$$

and use the identity

$$A(\boldsymbol{x})\partial_\phi G(\boldsymbol{y},\boldsymbol{x})\delta(\phi(\boldsymbol{x}))=A(\boldsymbol{x})\sum\frac{\partial G(\boldsymbol{y},\boldsymbol{x})}{\partial x_k}\frac{\partial\phi}{\partial x_k}\delta(\phi(\boldsymbol{x}))$$

$$=\sum\frac{\partial}{\partial x_k}\left(A(\boldsymbol{x})\frac{\partial\phi}{\partial x_k}\delta(\phi(\boldsymbol{x}))G(\boldsymbol{y},\boldsymbol{x})\right)-G(\boldsymbol{y},\boldsymbol{x})\sum\frac{\partial}{\partial x_k}\left(A(\boldsymbol{x})\frac{\partial\phi}{\partial x_k}\delta(\phi(\boldsymbol{x}))\right)$$

to get

$$\int A(\boldsymbol{x})\partial_\phi G(\boldsymbol{y},\boldsymbol{x})\delta(\phi(\boldsymbol{x}))d\boldsymbol{x}=-\int G(\boldsymbol{y},\boldsymbol{x})\nabla\cdot(A(\boldsymbol{x})\delta(\phi(\boldsymbol{x}))\nabla\phi)d\boldsymbol{x}$$

assuming the finiteness of $A(x)$. Referring to (11), we get

$$\int G(\boldsymbol{y}, \boldsymbol{x})(\nabla\cdot(A(\boldsymbol{x})\delta(\phi(\boldsymbol{x}))\nabla\phi)+B(\boldsymbol{x})\delta(\phi(\boldsymbol{x})))d\boldsymbol{x}=0\,.$$

Operating $\Delta_y - q^2$ on this relation we have

$$B(\boldsymbol{x})\delta(\phi(\boldsymbol{x}))+\nabla\cdot(A(\boldsymbol{x})\delta(\phi(\boldsymbol{x}))\nabla\phi)=0$$

or

$$(B(\boldsymbol{x})+\partial_\phi A(\boldsymbol{x})+A\Delta\phi)\delta(\phi(\boldsymbol{x}))+A(\boldsymbol{x})\delta'(\phi(\boldsymbol{x}))(\nabla\phi)^2=0 \tag{14}$$

Multiplying by $\phi(\boldsymbol{x})$ we have

$$A(\boldsymbol{x})\delta(\phi(\boldsymbol{x}))(\nabla\phi)^2=0$$

Since $(\nabla\phi)^2=1$, we get

$$A(\boldsymbol{x})\delta(\phi(\boldsymbol{x}))=0$$

or $A(\boldsymbol{x})=0$ on the boundary, so that we have the relation

$$B(x)\delta(\phi(\boldsymbol{x}))=0\,. \tag{16}$$

Hence two boundary conditions are satisfied. Therefore two equations (1) and (2), and two boundary conditions (3), (4) are unified into two integral equations (8) and (10).

EXAMPLE. The plane $x_1=0$ is taken to be a boundary separating two media. In the domain $\phi>0$ we take the equation for v

$$(\Delta-aq^2)v=-\delta(\boldsymbol{x}-\boldsymbol{x}')$$

and, in the domain $\phi<0$, the equation for w

$$(\Delta-bq^2)w=0\,.$$

Boundary conditions are

$$\alpha v=w$$

$$\beta\frac{\partial v}{\partial x_1}=\frac{\partial w}{\partial x_1}\,.$$

If we write equations and boundary conditions for Fourier transforms of v and w

$$\tilde{v}=\int e^{-ip_2(x_2-x_2')-\cdots-ip_n(x_n-x_n')}v\,dx_2\cdots dx_n$$

$$\tilde{w}=\int e^{-ip_2(x_2-x_2')-\cdots-ip_n(x_n-x_n')}w\,dx_2\cdots dx_n$$

we get

$$\frac{\partial^2\tilde{v}}{\partial x_1^2}-\lambda^2\tilde{v}=-\delta(x_1-x_1')$$

$$\frac{\partial^2 \tilde{w}}{\partial x_1^2} - \mu^2 \tilde{w} = 0$$

$$\lambda = \sqrt{(aq^2 + p_2^2 + \cdots + p_n^2)}\,,\ \mathrm{Re}\,\lambda > 0\,,\quad \mu = \sqrt{(bq^2 + p_2^2 + \cdots + p_n^2)}\,,\ \mathrm{Re}\,\mu > 0$$

together with

$$\alpha \tilde{v} = \tilde{w}\,,\quad \beta \frac{\partial \tilde{v}}{\partial x_1} = \frac{\partial \tilde{w}}{\partial x_1} \quad \text{at } x_1 = 0\,.$$

In this case we have

$$G(y_1, x_1) = \frac{e^{-\lambda |y_1 - x_1|}}{2\lambda}\,,\quad H(z_1, x_1) = \frac{e^{-\mu |z_1 - x_1|}}{2\mu}$$

and solutions

$$\tilde{v} = \frac{e^{-\lambda |x_1 - x_1'|}}{2\lambda} + \frac{\beta\lambda - \alpha\mu}{2\lambda(\beta\lambda + \alpha\mu)} e^{-\lambda x_1' - \lambda x_1} \tag{17}$$

$$\tilde{w} = \frac{\alpha\beta}{\beta\lambda + \alpha\mu} e^{-\lambda x_1' + \mu x_1}\,. \tag{18}$$

Equations (8) and (10) become

$$\tilde{v}(x_1) = \frac{e^{-\lambda |x_1 - x_1'|}}{2\lambda} - \frac{e^{-\lambda x_1}}{2\lambda} \frac{1}{\beta} \tilde{w}'(0) + \frac{e^{-\lambda x_1}}{2} \frac{1}{\alpha} \tilde{w}(0) \tag{19}$$

$$\tilde{w}(x_1) = \frac{e^{\mu x_1}}{2\mu} \beta \tilde{v}'(0) + \frac{e^{\mu x_1}}{2} \alpha \tilde{v}(0)\,. \tag{20}$$

Solutions (17) and (18) satisfy certainly (19) and (20). Inversely we can get solutions (17) and (18) from (19) and (20) by the method of successive approximation, relying on the condition

$$\left| 2 - \frac{\alpha\mu}{\beta\lambda} - \frac{\beta\lambda}{\alpha\mu} \right| < 4\,.$$

§ 6. Reflection at a curved surface (Paraxial approximation)

Referring to a Cartesian coordinate system (x, y, z), a wave propagates generally in the direction of z-axis from a source $\boldsymbol{x}' = (x', y', z')$ and is reflected at a curved surface

$$\phi = z - (x^2 + y^2)/2a = 0 \tag{1}$$

with the boundary condition $v = 0$ there. Then the equation (2) in § 4 becomes

$$v(\boldsymbol{x}'') = G(\boldsymbol{x}'', \boldsymbol{x}') - \int G(\boldsymbol{x}'', \boldsymbol{x}) \partial_\phi v(\boldsymbol{x}) \delta(\phi(\boldsymbol{x})) d\boldsymbol{x} \tag{2}$$

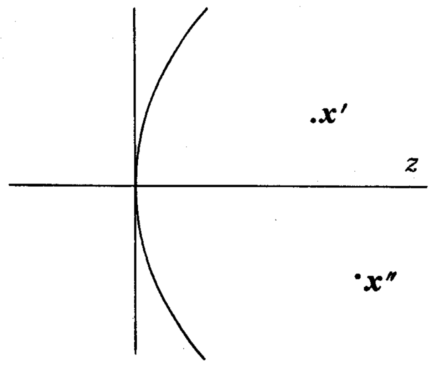

Fig. 1

the Green's function here being given by

$$G(\boldsymbol{x}, \boldsymbol{x}') = \frac{e^{-q|\boldsymbol{x}-\boldsymbol{x}'|}}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}. \tag{3}$$

We assume the solution v may be expanded as

$$v = v_0 + v_1 + v_2 + \cdots \tag{4}$$

while

$$v_0(\boldsymbol{x}'') = G(\boldsymbol{x}'', \boldsymbol{x}')$$

$$v_{k+1}(\boldsymbol{y}) = -\int G(\boldsymbol{y}, \boldsymbol{x}) \partial_\phi v_k(\boldsymbol{x}) \delta(\phi(\boldsymbol{x})) d\boldsymbol{x}, \qquad k = 0, 1, 2, \cdots.$$

The paraxial approximation assumes that coordinates x and y are small compared with the axial distance z. Then we see that

$$|\boldsymbol{x}-\boldsymbol{x}'| = |z-z'| + ((x-x')^2 + (y-y')^2)/2|z-z'|$$

$$\partial_\phi G(\boldsymbol{x}, \boldsymbol{x}') = qG(\boldsymbol{x}, \boldsymbol{x}') + \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} G(\boldsymbol{x}, \boldsymbol{x}')$$

If we assume that $|q(\boldsymbol{x}-\boldsymbol{x}')| \gg 1$, we may put

$$\partial_\phi G(\boldsymbol{x}, \boldsymbol{x}') = \frac{q}{4\pi} \frac{1}{|\boldsymbol{z}-\boldsymbol{z}'|} \exp\left[-q\left\{|\boldsymbol{z}-z'| + \frac{(x-x')^2 + (y-y')^2}{2|z-z'|}\right\}\right]$$

consequently we have

$$v_1(\boldsymbol{x}'') = -\frac{q}{(4\pi)^2} \int \exp\left[-q\left(z''-z+\frac{(x-x'')^2+(y-y'')^2}{2(z''-z)} + z'-z\right.\right.$$

$$\left.\left. + \frac{(x-x')^2+(y-y')^2}{2(z'-z)}\right)\right] \cdot \frac{\delta(x-(y^2+a^2)/2z)}{(z''-z)(z'-z)} dxdydz.$$

An integration with respect to z replaces z by $(x^2+y^2)/2a$ in the integrand except z's in the denominators. These z's are replaced by 0 according to the paraxial approximation, in other words, $z'-z$ and $z''-z$ are replaced by z' and z'' respectively. Then we have

$$v_1(\boldsymbol{x}'')=-\frac{q}{(4\pi)^2}\frac{e^{-qz'-qz''}}{z'z''}\int \exp\left[-\frac{q}{2}\left\{A(x^2+y^2)-2Xx-2Yy\right.\right.$$

$$\left.\left.+\frac{x''^2+y''^2}{z''}+\frac{x'^2+y'^2}{z'}\right\}\right]dxdy \tag{6}$$

$$=-\frac{q}{(4\pi)^2}\frac{1}{z'z''}\int \exp\left[-q|\boldsymbol{x}'|-q|\boldsymbol{x}''|-\frac{q}{2}A(x^2+y^2)+qXx+qYy\right]dxdy$$

where the following notations are used

$$A=\frac{1}{z'}+\frac{1}{z''}-\frac{2}{a}, \quad X=\frac{x''}{z''}+\frac{x'}{z'}, \quad Y=\frac{y''}{z''}+\frac{y'}{z'}. \tag{7}$$

When A is positive, integrating with respect to x and y, we get

$$v_1(\boldsymbol{x}'')=-\frac{1}{8\pi Az'z''}\exp\left[\frac{q}{2A}(X^2+Y^2)-q|\boldsymbol{x}'|-q|\boldsymbol{x}''|\right] \tag{8}$$

and

$$u_1(\boldsymbol{x}'')=\frac{1}{2\pi i}\int_L e^{(t-t')q}v_1(\boldsymbol{x}'')dq$$

$$=-\frac{1}{8\pi Az'z''}\delta[t-t'-|\boldsymbol{x}'|-|\boldsymbol{x}''|+(X^2+Y^2)/2A]. \tag{9}$$

The solution $u_1(\boldsymbol{x}'')$ determines the wave front

$$(t-t'-|\boldsymbol{x}'|-|\boldsymbol{x}''|)A+\frac{1}{2}(X^2+Y^2)=0.$$

Since the quantity A decreases as z'' increases, A may vanish at a certain z''. At the plane perpendicular to the z-axis at z'', the wave front is to satisfy $X^2+Y^2=0$ since $A=0$. The condition $X^2+Y^2=0$ entails two conditions $X=0$, $Y=0$, which determine a unique point $x''=-x'z''/z'$, $y''=-y'z''/z'$. So the wave front is to converge at the point $(-x'z''/z', -y'z''/z', z'')$, z'' being determined by $A=0$.

The condition $A=0$ gives the well-known relation of the z-coordinate of the source and that of its image in geometrical optics. If we consider a purely monochromatic wave and replace q by $i\omega$ (ω: frequency), then the integral (6) with $A=0$ reduces to

$$v_1(\boldsymbol{x}'')=\frac{-i}{4\omega}\frac{e^{-i\omega|x'|-i\omega|x''|}}{z'z''}\delta\left(\frac{x''}{z''}+\frac{x'}{z'}\right)\delta\left(\frac{y''}{z''}+\frac{y'}{z'}\right) \tag{10}$$

showing that a wave starts from the source (x', y', z') and converges to the image of the source (x'', y'', z'') very sharply.

When $A<0$, the integral in (6) diverges, since the range of integration extends to infinity. In this section, we used the paraxial approximation assuming that $|x|$, $|y|$ are small compared with the axial distance, so that the range of integration with respect to x and y should be confined to a finite region, for example, a circle of radius R, centered at the z-axis $x^2+y^2<R^2$ and the integral (6) should be evaluated more carefully. Therefore we should replace (6) by

$$v_1(\boldsymbol{x}'')=\frac{-q}{(4\pi)^2}\frac{1}{z'z''}\int\exp\left[-q|\boldsymbol{x}'|-q|\boldsymbol{x}''|-\frac{1}{2}qA(x^2+y^2)+qXx+qYy\right]$$
$$\cdot\theta(R^2-x^2-y^2)dxdy \tag{11}$$

multiplying the integrand by $\theta(R^2-x^2-y^2)$. $\theta(x)$ has an integral representation

$$\theta(x)=\frac{1}{2\pi i}\int_L e^{xs}\frac{ds}{s}, \qquad \mathrm{Re}\, s>0 \tag{12}$$

where the path of integration L is a straight line parallel to the imaginary axis. Replacing s by $s/2$, we have

$$\int\exp\left[-\frac{q}{2}A(x^2+y^2)+qXx+qYy\right]\theta(R^2-x^2-y^2)dxdy$$
$$=\frac{1}{2\pi i}\iint_L \exp\left[\frac{1}{2}R^2s-\frac{1}{2}(s+qA)(x^2+y^2)+qXx+qYy\right]\frac{ds}{s}dxdy$$
$$=\frac{2\pi}{2\pi i}\int_L \exp\left[\frac{1}{2}R^2s+\frac{q^2(X^2+Y^2)}{2(s+qA)}\right]\frac{ds}{s(s+qA)}, \qquad \mathrm{Re}\,(s+qA)>0$$
$$=\frac{2\pi e^{-qAR^2/2}}{q}\frac{1}{2\pi i}\int_C \exp\left[\frac{1}{2}R^2qs'+\frac{q(X^2+Y^2)}{2s'}\right]\frac{ds'}{s'(s'-A)}, \quad s+qA=s'q. \tag{13}$$

We change here the path of integration into a circle C encircling the origin and A. The path of integration is now changed into a circle of radius $R\sqrt{(X^2+Y^2)}$ centered at the origin. If the path of integration crosses the point A during its deformation, the integral should be added with the residue at A. So we have

$$(13)=\frac{2\pi}{q}e^{-qAR^2/2}\left[\frac{1}{2\pi i}\int\exp\left[\frac{1}{2}qR\sqrt{(X^2+Y^2)}(\zeta+1/\zeta)\right]\frac{d\zeta}{\zeta(\zeta-AR/\sqrt{(X^2+Y^2)})}\right.$$
$$\left.\cdot\frac{R}{\sqrt{(X^2+Y^2)}}+\exp\left[\frac{1}{2}qAR^2+\frac{1}{2}\frac{q(X^2+Y^2)}{A}\right]\frac{1}{A}\theta(AR/\sqrt{(X^2+Y^2)}-1)\right] \tag{14}$$

where the variable s' is changed into ζ by $s'=\zeta R\sqrt{(X^2+Y^2)}$. We change again the path of integration into the unit circle and put $\zeta=e^{i\theta}$. Then $\zeta+1/\zeta$ is real. The inverse Laplace transform of $v(x)$ is found to be

$$u_1(\boldsymbol{x}'')=\frac{1}{8\pi z'z''}\frac{R}{\sqrt{(X^2+Y^2)}}\frac{1}{2\pi}\int_{-\pi}^{\pi}$$

$$\cdot\delta\left[t-t'-|\boldsymbol{x}'|-|\boldsymbol{x}''|-\frac{1}{2}AR^2+\frac{1}{2}R\sqrt{(X^2+Y^2)}\cos\theta\right]$$

$$\cdot\frac{d\theta}{e^{i\theta}-AR/\sqrt{(X^2+Y^2)}}-\frac{1}{8\pi z'z''}\delta\left[t-t'-|\boldsymbol{x}'|-|\boldsymbol{x}''|+\frac{X^2+Y^2}{2A}\right]$$

$$\cdot\frac{1}{A}\theta\left(\frac{AR}{\sqrt{(X^2+Y^2)}}-1\right) \tag{15}$$

The integral in (15) may be evaluated by a formula

$$\int_{-\pi}^{\pi}\delta(a+b\cos\theta)g(\cos\theta,\sin\theta)d\theta$$

$$=\frac{1}{\sqrt{(b^2-a^2)}}\left\{g\left(-\frac{a}{b},\frac{\sqrt{(b^2-a^2)}}{b}\right)+g\left(-\frac{a}{b},-\frac{\sqrt{(b^2-a^2)}}{b}\right)\right\}\theta(b^2-a^2)\,. \tag{16}$$

§ 7. Refraction at a curved boundary (Paraxial approximation)

A curved surface $\phi=z-(x^2+y^2)/2a$ separates two different media. We assume that the medium in the domain $\phi<0$ has the refractive index n and the medium in the domain $\phi>0$ has the refractive index m.

A wave starting from a source $\boldsymbol{x}'=(x', y', z')$ situated in the left medium propagates generally from left to right. The Laplace transform of u is denoted by v in the left medium and by w in the right medium. Equations are put

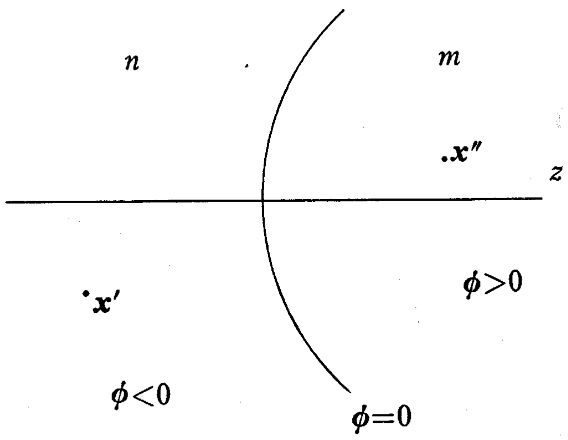

Fig. 2

$$\Delta v - n^2 q^2 v = -\delta(\boldsymbol{x} - \boldsymbol{x}') \tag{1}$$

$$\Delta w - m^2 q^2 w = 0 \tag{2}$$

respectively. Boundary conditions are set

$$\alpha v = w \tag{3}$$

$$\beta \partial_\phi v = \partial_\phi w \,. \tag{4}$$

Respective Green's functions are

$$G(\boldsymbol{x}, \boldsymbol{x}') = \frac{e^{-nq|\boldsymbol{x}-\boldsymbol{x}'|}}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}, \quad H(\boldsymbol{x}, \boldsymbol{x}'') = \frac{e^{-mq|\boldsymbol{x}-\boldsymbol{x}''|}}{4\pi|\boldsymbol{x}-\boldsymbol{x}''|}.$$

In this case, integral equations (8), (10) in § 5 governing v and w become

$$v(\boldsymbol{y}) = G(\boldsymbol{y}, \boldsymbol{x}') - \int \left\{ \frac{1}{\alpha} w(\boldsymbol{x}) \partial_\phi G(\boldsymbol{y}, \boldsymbol{x}) - \frac{1}{\beta} \partial_\phi w(\boldsymbol{x}) G(\boldsymbol{y}, \boldsymbol{x}) \right\} \delta(\phi(\boldsymbol{x})) d\boldsymbol{x} \tag{5}$$

and

$$w(\boldsymbol{z}) = \int \{\alpha v(\boldsymbol{x}) \partial_\phi H(\boldsymbol{z}, \boldsymbol{x}) - \beta \partial_\phi v(\boldsymbol{x}) H(\boldsymbol{z}, \boldsymbol{x})\} \delta(\phi(\boldsymbol{x})) d\boldsymbol{x} \,. \tag{6}$$

We take $G(\boldsymbol{y}, \boldsymbol{x})$ for the first approximation to v, and we have, as the first approximation to w,

$$w(\boldsymbol{x}'') = \int \{\alpha G(\boldsymbol{x}, \boldsymbol{x}') \partial_\phi H(\boldsymbol{x}'', \boldsymbol{x}) - \beta \partial_\phi G(\boldsymbol{x}, \boldsymbol{x}') \cdot H(\boldsymbol{x}'', \boldsymbol{x})\} \delta(\phi(\boldsymbol{x})) d\boldsymbol{x} \tag{7}$$

replacing $\boldsymbol{z}$ by $\boldsymbol{x}''$. According to the paraxial approximation and the assumption $|qr| \gg 1$, we use the following approximation

$$\partial_\phi G(\boldsymbol{x}, \boldsymbol{x}') = \frac{\partial}{\partial z} G(\boldsymbol{x}, \boldsymbol{x}') = -nqG(\boldsymbol{x}, \boldsymbol{x}') \tag{8}$$

$$\partial_\phi H(\boldsymbol{x}, \boldsymbol{x}'') = \frac{\partial}{\partial z} H(\boldsymbol{x}, \boldsymbol{x}'') = mqH(\boldsymbol{x}, \boldsymbol{x}'') \tag{9}$$

and get

$$\begin{aligned} w(\boldsymbol{x}'') &= (\alpha m + \beta n) q \int G(\boldsymbol{x}, \boldsymbol{x}') H(\boldsymbol{x}, \boldsymbol{x}'') \delta(z - (x^2 + y^2)/2a) dx dy dz \\ &= \frac{(\alpha m + \beta n) q}{(4\pi)^2 |z' z''|} \int \exp \left[q \left\{ nz' - mz'' - (n-m) z + n \frac{(x-x')^2 + (y-y')^2}{2z'} \right. \right. \\ &\qquad \left. \left. - m \frac{(x-x'')^2 + (y-y'')^2}{2z''} \right\} \right] \delta\left(z - \frac{x^2 + y^2}{2a} \right) dx dy dz \,. \end{aligned} \tag{10}$$

When z is replaced by $(x^2 + y^2)/2a$, the expression inside the braces may be rewritten

$$\{\ \}=n\left(z'+\frac{x'^2+y'^2}{2z'}\right)-m\left(z''+\frac{x''^2+y''^2}{2z''}\right)-\frac{1}{2}A(x^2+y^2)$$

$$+\left(\frac{mx''}{z''}-\frac{nx'}{z'}\right)x+\left(\frac{my''}{z''}-\frac{ny'}{z'}\right)y$$

$$=-n|\boldsymbol{x}'|-m|\boldsymbol{x}''|-\frac{1}{2}A(x^2+y^2)+Xx+Yy$$

where

$$A=\frac{m}{z''}-\frac{n}{z'}+\frac{n-m}{a}$$

$$X=\frac{mx''}{z''}-\frac{nx'}{z'},\quad Y=\frac{my''}{z''}-\frac{ny'}{z'}. \tag{11}$$

At the plane where A is positive, integration with respect to x and y gives

$$w(\boldsymbol{x}'')=\frac{\alpha m+\beta n}{8\pi A|z'z''|}\exp\left[\frac{q}{2A}(X^2+Y^2)-qn|\boldsymbol{x}'|-qm|\boldsymbol{x}''|\right]. \tag{12}$$

The inverse Laplace transform of $w(\boldsymbol{x}'')$ becomes

$$u(\boldsymbol{x}'')=\frac{\alpha m+\beta n}{8\pi A|z'z''|}\delta\left[t-t'-n|\boldsymbol{x}'|-m|\boldsymbol{x}''|+\frac{1}{2A}(X^2+Y^2)\right]. \tag{13}$$

As in the reflection in §6, we see the wave front converging to the point $(nx'z''/mz',\ ny'z''/mz',\ z'')$, z'' being determined by $A=0$. The condition $A=0$ is the well known relation of the z coordinate of the source and that of its image in geometrical optics.

As in the reflection in §6, for a monochromatic wave, $q=i\omega$, we get

$$w(\boldsymbol{x}'')=\frac{(\alpha m+\beta n)i}{4\omega|z'z''|}e^{-i\omega n|\boldsymbol{x}'|-i\omega m|\boldsymbol{x}''|}\delta\left(\frac{mx''}{z''}-\frac{nx'}{z'}\right)\delta\left(\frac{my''}{z''}-\frac{ny'}{z'}\right) \tag{14}$$

after refraction when A is equal to 0.

§8. A system of lenses

It is assumed that m boundary surfaces $\phi_k=z-d_k-(x^2+y^2)/2a_k=0$, $k=1,2,\cdots,m$ separate $m+1$ media, among whom medium 1 is defined by $\phi_1<0$ and has the refraction index $n_0=n'$, medium k is defined by $\phi_k>0$ and $\phi_{k+1}<0$, and has the refractive index n_k, $k=1,2,\cdots,m-1$, medium m is defined by $\phi_m>0$ and has the refractive index $n_m=n$, as shown in Fig, 3. A point in the neighbourhood of surface $\phi_k=0$ is denoted by $\boldsymbol{x}_k=(x_k, y_k, z_k)$. A wave function in medium k with the refractive index n_k is denoted by u_k, its Laplace transform by v_k, the Green's function there by

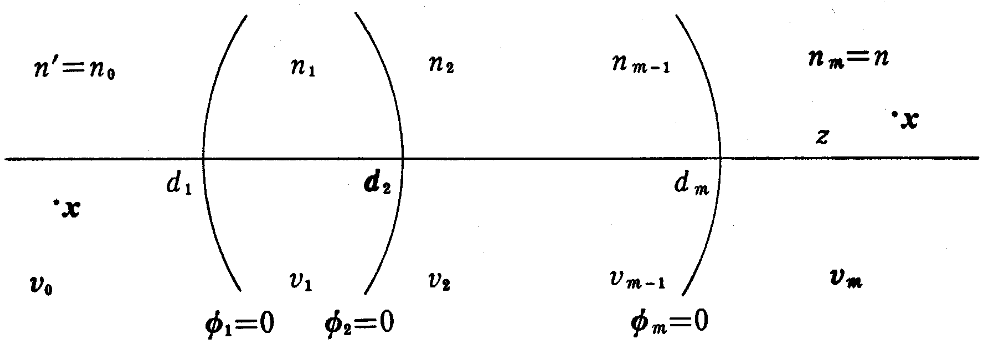

Fig. 3

$$G_k(\boldsymbol{x}, \boldsymbol{y}) = \frac{e^{-n_k q|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}. \tag{1}$$

Boundary conditions at the boundary $\phi_k=0$ are set as

$$\sigma_k v_{k-1} = v_k, \quad \tau_k \partial v_{k-1}/\partial n = \partial v_k/\partial n. \tag{2}$$

The symbol $\partial/\partial n$ means the normal derivative generally directed to the right direction, approximating $\partial_\phi v$. A wave starts from the source $\boldsymbol{x}'=\boldsymbol{x}_0$ situated in the medium 0 with the refractive index $n_0=n'$, propagates generally to the right, is refracted and reflected at each boundary, finally attains to medium m.

Referring to (7), (8), (9) in §7, the first approximation $v_m(\boldsymbol{x})$ may be given by

$$v_m(\boldsymbol{x}) = q^m \prod_{k=1}^{m} (\sigma_k n_k + \tau_k n_{k-1}) \cdot \int G_m(\boldsymbol{x}, \boldsymbol{x}_m) \cdots G_1(\boldsymbol{x}_2, \boldsymbol{x}_1) G_0(\boldsymbol{x}_1, \boldsymbol{x}') \cdot$$

$$\delta(\phi_m(\boldsymbol{x}_m)) \cdots \delta(\phi_1(\boldsymbol{x}_1)) d\boldsymbol{x}_1 \cdots d\boldsymbol{x}_m. \tag{3}$$

If we replace the denominator of Green's function by its value at the z-axis and put

$$|\boldsymbol{x}_k - \boldsymbol{x}_{k-1}| = |z_k - z_{k-1}| = d_k - d_{k-1}$$

then the integral in (3) reduces to the integral

$$w = \int \exp\left[-q \prod_{k=0}^{m} n_k |\boldsymbol{x}_{k+1} - \boldsymbol{x}_k|\right] \prod_{k=1}^{m} \delta(\phi_k(\boldsymbol{x}_k)) d\boldsymbol{x}_k \tag{4}$$

except for the denominators. We put

$$\prod_{k=0}^{m} n_k |\boldsymbol{x}_{k+1} - \boldsymbol{x}_k| = L. \tag{5}$$

Because of the factor $\delta(\phi_k(\boldsymbol{x}_k))$ in (4), the point $\boldsymbol{x}_k$ lies on the boundary $\phi_k=0$, Therefore L means the optical distance of a path connecting $\boldsymbol{x}'=\boldsymbol{x}_0, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_m, \boldsymbol{x}_{m+1}=\boldsymbol{x}$ with straight lines. Geometrical optics is deducible from Fermat's principle stating that the optical distance L of the path of light connecting $\boldsymbol{x}_0$ to $\boldsymbol{x}_{m+1}$ is stationary for the infinitesimal variations of $\boldsymbol{x}_k$ on the boundary $\phi_k=0$.

Employing the paraxial approximation and replacing z_k by $d_k+(x_k^2+y_k^2)/2a_k$ we rewrite L as

$$
\begin{aligned}
L=&\sum_{k=0}^{m} n_k\{(z_{k+1}-z_k)^2+(x_{k+1}-x_k)^2+(y_{k+1}-y_k)^2\}^{1/2}\\
=&\sum_{k=0}^{m} n_k\{z_{k+1}-z_k+(x_k^2+y_k^2+x_{k+1}^2+y_{k+1}^2-2x_kx_{k+1}-2y_ky_{k+1})/2(z_{k+1}-z_k)\}\\
=&\sum_{k=0}^{m} n_k\left\{d_{k+1}+\frac{x_{k+1}^2+y_{k+1}^2}{2a_{k+1}}-d_k-\frac{x_k^2+y_k^2}{2a_k}\right.\\
&\left.+\frac{x_k^2+y_k^2+x_{k+1}^2+y_{k+1}^2}{2(d_{k+1}-d_k)}-\frac{x_kx_{k+1}+y_ky_{k+1}}{d_{k+1}-d_k}\right\}\\
=&n_0\sqrt{(d_1-d_0)^2+x'^2+y'^2)}+\sum_{k=1}^{m-1} n_k(d_{k+1}-d_k)+n_{m+1}\sqrt{((d_{m+1}-d_m)^2+x^2+y^2)}\\
&+\frac{1}{2}\sum_{k=1}^{m}\alpha_k(x_k^2+y_k^2)-\sum_{k=1}^{m-1}\beta_k(x_kx_{k+1}+y_ky_{k+1})-\beta'(x'x_1+y'y_1)-\beta(xx_m+yy_m)
\end{aligned}
\tag{6}
$$

where the following notations are used

$$
\begin{aligned}
\alpha_k&=\frac{n_{k+1}-n_k}{a_k}+\frac{n_k}{d_{k+1}-d_k}+\frac{n_{k-1}}{d_k-d_{k-1}}\\
\beta_k&=\frac{n_k}{d_{k+1}-d_k},\quad k=1,2,\cdots,m-1,\\
\beta'&=\frac{n_0}{d_1-d_0},\quad \beta=\frac{n_m}{d_{m+1}-d_m}.
\end{aligned}
\tag{7}
$$

It is to be noted that $a_0=a_{m+1}=\infty$, $d_0=z'$ the z coordinate of the source, $d_{m+1}=z$ the z coordinate of the observation plane. We use further a matrix notation

$$
A=\begin{pmatrix}
\alpha_1 & -\beta_1 & & & \\
-\beta_1 & \alpha_2 & -\beta_2 & & \\
 & -\beta_2 & \ddots & & \\
 & & & \alpha_{m-1} & -\beta_{m-1}\\
 & & & -\beta_{m-1} & \alpha_m
\end{pmatrix}=(a_{jk}).
\tag{8}
$$

Assuming the symmetric matrix to be positive definite, we get

$$w=\frac{(2\pi)^m}{q^m \det A}\exp\Big[-qn_0\sqrt{(d_1-d_0)^2+x'^2+y'^2)}-q\sum_{k=1}^{m-1}n_k(d_{k+1}-d_k)$$

$$-qn_{m+1}\sqrt{((d_{m+1}-d_m)^2+x^2+y^2)}$$

$$+\frac{q}{2}\{A_{11}\beta'^2(x'^2+y'^2)+2A_{1m}\beta\beta'(xx'+yy')+A_{mm}\beta^2(x^2+y^2)\}/\det A\Big] \tag{9}$$

A_{jk} denoting the cofactor of a_{jk}. The inverse Laplace transform of $v_m(\boldsymbol{x})$ becomes

$$u_m(\boldsymbol{x})=\text{Const.}\,\delta\Big[t-t'-n_0\sqrt{((d_1-d_0)^2+x'^2+y'^2)}-\sum_{k=1}^{m-1}n_k(d_{k+1}-d_k)$$

$$-n_{m+1}\sqrt{((d_{m+1}-d_m)^2+x^2+y^2)}$$

$$+\frac{1}{2}\{A_{11}\beta'^2(x'^2+y'^2)+2A_{1m}\beta\beta'(xx'+yy')+A_{mm}\beta^2(x^2+y^2)\}/\det A\Big]$$

$$/\det A \tag{10}$$

the factor Const. denoting a constant quantity. Since the matrix A is assumed to be positive definite, so the inverse matrix A^{-1} is also positive definite, so that the expression inside the braces in (10) is not negative for any values of x', y', x, y.

As $z=d_{m+1}$ increases, α_m decreases, and det A may vanish at a certain value of z. The wave front depicted by (10) is to satisfy

$$A_{11}\beta'^2(x'^2+y'^2)+2A_{1m}\beta\beta'(xx'+yy')+A_{mm}\beta^2(x^2+y^2)=0 \tag{11}$$

at the value of z, so that there ensue two conditions

$$A_{11}(\beta'x')^2+2A_{1m}\beta'x'\beta x+A_{mm}(\beta x)^2=0 \tag{12}$$

$$A_{11}(\beta'y')^2+2A_{1m}\beta'y'\beta y+A_{mm}(\beta y)^2=0\,. \tag{13}$$

For given x' there might result two values of x. But, when det A vanishes,

$$\begin{vmatrix} A_{11} & A_{1m} \\ A_{m1} & A_{mm} \end{vmatrix} \begin{matrix} =\det A\cdot(\text{determinant of } A \text{ devoid of } 1, m \text{ rows and } 1, m \text{ columns}) \\ =0 \end{matrix}$$

by virtue of Jacobi's theorem. So the condition (12) may be reduced to

$$A_{1m}\beta'x'+A_{mm}\beta x=0\,.$$

Similarly the condition (13) may be reduced to

$$A_{1m}\beta'y'+A_{mm}\beta y=0\,.$$

In short, the wave front may converge to a point $(-A_{1m}\beta'x'/A_{mm}\beta, -A_{1m}\beta'y'/A_{mm}\beta, z)$ on a plane perpendicular to the z-axis at z, z being determined by det $A=0$. At the point z where det $A=0$, the integral

$$\int \exp\left[-q\left(\frac{1}{2}\sum_1^m a_{jk}(x_jx_k+y_jy_k)-\beta'(x'x_1+y'y_1)-\beta(xx_m+yy_m)\right)\right]\Pi dx_j dy_j$$

divereges. For a monochromatic wave ($q=i\omega$), if $A_{mm}>0$, integration with respect to variables except for x_m, y_m, succeded by integration with respect to x_m, y_m leads to the factor

$$\delta(A_{1m}\beta'x'+A_{mm}\beta x)\delta(A_{1m}\beta'y'+A_{mm}\beta y)$$

as in (14) of §7. The expression (10) represents that the wave starting from (x', y', z') converges at (x, y, z) with the magnification ratio $-A_{1m}\beta'/A_{mm}\beta$.

Let us consider the preceding calculation from the standpoint of geometrical optics. The optical distance L (5), (6) may be expressed as

$$L=\sum_{k=0}^{m} n_k(d_{k+1}-d_k)+\frac{1}{2}\sum_{j,k=0}^{m+1} a_{jk}(x_jx_k+y_jy_k)$$

in $2m+2$ variables, additional variables being $x_0=x'$, $y_0=y'$, $x_{m+1}=x$, $y_{m+1}=y$. The Fermat's principle requires that

$$\frac{\partial L}{\partial x_k}=0\,,\quad \frac{\partial L}{\partial y_k}=0$$

$$k=1, 2, \cdots, m\,.$$

These conditions are two groups of m simultaneous equations in $x_1, \cdots, x_m$ and $y_1, \cdots, y_m$ respectively. The coefficients of variables constitute the matrix A. If $\det A\neq 0$, there is a unique path for any two end points. But, for imaging to be possible, there must be two paths at least for two end points, consequently $\det A$ must vanish. The condition $\det A=0$ gives the relation between z and z'.

§9. Maxwell's equations

In the preceding analysis we treated light as a scalar wave. If we analyze more exactly, we should treat light as an electromagnetic wave governed by Maxwell's equations

$$\mu\frac{\partial \boldsymbol{H}}{\partial t}+\nabla\times\boldsymbol{E}=0 \tag{1}$$

$$\varepsilon\frac{\partial \boldsymbol{E}}{\partial t}-\nabla\times\boldsymbol{H}=0\,. \tag{2}$$

Using the following notations

$$\sqrt{(\varepsilon\mu)}=n\,,\quad x=x_1\,,\quad y=x_2\,,\quad z=x_3\,,\quad \sqrt{\mu}\boldsymbol{H}=(u_1, u_2, u_3)\,,\quad \sqrt{\varepsilon}\boldsymbol{E}=(u_4, u_5, u_6)$$

and a column vector u having components $u_1, u_2, \cdots, u_6$, we write

$$\left(n\frac{\partial}{\partial t}+\gamma_1\frac{\partial}{\partial x_1}+\gamma_2\frac{\partial}{\partial x_2}+\gamma_3\frac{\partial}{\partial x_3}\right)\boldsymbol{u}=0 \tag{3}$$

where three matrices $\gamma_1, \gamma_2, \gamma_3$ are defined for any a_1, a_2, a_3 by

$$\gamma_1 a_1+\gamma_2 a_2+\gamma_3 a_3=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\times\begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$$

in the Kronecker product form. We denote the Laplace transform of $\boldsymbol{u}$ by $\boldsymbol{v}$

$$\boldsymbol{v}=\int_0^\infty e^{-qt}\boldsymbol{u}dt$$

which satisfies

$$L\boldsymbol{v}=\left(nq+\gamma_1\frac{\partial}{\partial x_1}+\gamma_2\frac{\partial}{\partial x_2}+\gamma_3\frac{\partial}{\partial x_3}\right)\boldsymbol{v}=n\boldsymbol{u}_0 \tag{4}$$

$\boldsymbol{u}_0$ denoting the initial value of $\boldsymbol{u}$. The operator M conjugate to L turns out to be defined by

$$M(\boldsymbol{w})=nq\boldsymbol{w}-\frac{\partial \boldsymbol{w}}{\partial x_1}\gamma_1-\frac{\partial \boldsymbol{w}}{\partial x_2}\gamma_2-\frac{\partial \boldsymbol{w}}{\partial x_3}\gamma_3 \tag{5}$$

for any row vector $\boldsymbol{w}$. We introduce a solution $\Gamma(\boldsymbol{y}, \boldsymbol{x})$ to the matrix equation

$$M(\Gamma)=nq\Gamma-\frac{\partial\Gamma}{\partial x_1}\gamma_1-\frac{\partial\Gamma}{\partial x_2}\gamma_2-\frac{\partial\Gamma}{\partial x_3}\gamma_3=1\cdot\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{6}$$

which solution goes to zero as $|\boldsymbol{x}-\boldsymbol{y}|\to\infty$.

Analogous to (6) in § 2, we have

$$\Gamma L\boldsymbol{v}-M(\Gamma)\boldsymbol{v}=\sum_{k=1}^{3}\frac{\partial}{\partial x_k}(\Gamma\gamma_k\boldsymbol{v}) \tag{7}$$

so that we have, integrating in the domain D, defined by $\phi>0$,

$$\int\{\Gamma(\boldsymbol{y},\boldsymbol{x})L\boldsymbol{v}(\boldsymbol{x})-M(\Gamma(\boldsymbol{y},\boldsymbol{x}))\boldsymbol{v}(\boldsymbol{x})\}\theta(\phi(\boldsymbol{x}))d\boldsymbol{x}=\int\sum\frac{\partial}{\partial x_k}(\Gamma(\boldsymbol{y},\boldsymbol{x})\gamma_k\boldsymbol{v}(\boldsymbol{x}))\theta(\phi(\boldsymbol{x}))d\boldsymbol{x},$$

$$\text{the left member}=\int\Gamma(\boldsymbol{y},\boldsymbol{x})n\boldsymbol{u}_0(\boldsymbol{x})\theta(\phi(\boldsymbol{x}))d\boldsymbol{x}-\boldsymbol{v}(\boldsymbol{y})\theta(\phi(\boldsymbol{y})),$$

$$\text{the right member}=-\int\Gamma(\boldsymbol{y},\boldsymbol{x})\sum\gamma_k\frac{\partial\phi}{\partial x_k}\boldsymbol{v}(\boldsymbol{x})\delta(\phi(\boldsymbol{x}))d\boldsymbol{x},$$

consequently we have

$$\boldsymbol{v}(\boldsymbol{y})\theta(\phi(\boldsymbol{y}))=\int\Gamma(\boldsymbol{y},\boldsymbol{x})n\boldsymbol{u}_0(\boldsymbol{x})\theta(\phi(\boldsymbol{x}))d\boldsymbol{x}+\int\Gamma(\boldsymbol{y},\boldsymbol{x})\Phi(\boldsymbol{x})\boldsymbol{v}(\boldsymbol{x})\delta(\phi(\boldsymbol{x}))d\boldsymbol{x} \quad (8)$$

$$\Phi(\boldsymbol{x})=\sum\gamma_k\phi_k=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\times\begin{pmatrix}0 & -\phi_3 & \phi_2\\ \phi_3 & 0 & -\phi_1\\ -\phi_2 & \phi_1 & 0\end{pmatrix},\quad \phi_k=\frac{\partial\phi}{\partial x_k},\quad k=1,2,3. \quad (9)$$

We denote quantities in the domain $\phi<0$ by symbols used in the domain $\phi>0$ added with a prime. The boundary conditions to be imposed upon $\boldsymbol{u}, \boldsymbol{u}'$ at the boundary $\phi=0$ are the continuity of tangential components of field strengths,

$$\nabla\phi\times\boldsymbol{E}=\nabla\phi\times\boldsymbol{E}', \qquad \nabla\phi\times\boldsymbol{H}=\nabla\phi\times\boldsymbol{H}' \quad (10)$$

or, in matrix form

$$\begin{pmatrix}0 & -\phi_3 & \phi_2\\ \phi_3 & 0 & -\phi_1\\ -\phi_2 & \phi_1 & 0\end{pmatrix}\begin{pmatrix}E_x\\ E_y\\ E_z\end{pmatrix}=\begin{pmatrix}0 & -\phi_3 & \phi_2\\ \phi_3 & 0 & -\phi_1\\ -\phi_2 & \phi_1 & 0\end{pmatrix}\begin{pmatrix}E'_x\\ E'_y\\ E_2\end{pmatrix} \quad (11)$$

etc. These conditions may be written in terms of u

$$\Lambda\Phi\boldsymbol{u}=\Lambda'\Phi\boldsymbol{u}', \quad \Lambda=\begin{pmatrix}1/\sqrt{\varepsilon} & 0\\ 0 & 1\sqrt{\mu}\end{pmatrix}\times 1_3, \quad \Lambda'=\begin{pmatrix}1/\sqrt{\varepsilon'} & 0\\ 0 & 1/\sqrt{\mu'}\end{pmatrix}\times 1_3 \quad (12)$$

1_3 denoting the unit matrix of dimension 3. As to vectors $\boldsymbol{v}, \boldsymbol{v}'$, the same condition (12) holds.

In the domain $\phi<0$, we introduce $\Gamma'(\boldsymbol{z},\boldsymbol{x})$ that satisfies

$$M'(\Gamma'(\boldsymbol{z},\boldsymbol{x}))=1\cdot\delta(\boldsymbol{z}-\boldsymbol{x}).$$

Maxwell's equations

$$|L'\boldsymbol{v}'=\left(n'q+\gamma_1\frac{\partial}{\partial x_1}+\gamma_2\frac{\partial}{\partial x_2}\gamma_3+\frac{\partial}{\partial x_3}\right)\boldsymbol{v}'=0$$

have a solution

$$\boldsymbol{v}'(\boldsymbol{z})\theta(-\phi(\boldsymbol{z}))=-\int\Gamma'(\boldsymbol{z},\boldsymbol{x})\Phi(\boldsymbol{x})\boldsymbol{v}'(\boldsymbol{x})\delta(\phi(\boldsymbol{x}))d\boldsymbol{x}$$

where $\Phi(\boldsymbol{x})\boldsymbol{v}(\boldsymbol{x})$ is to be replaced by $\Lambda'^{-1}\Lambda\Phi\boldsymbol{v}(\boldsymbol{x})$ according to (12),

$$=-\int\Gamma'(\boldsymbol{z},\boldsymbol{x})\Lambda'^{-1}\Lambda\Phi(\boldsymbol{x})\boldsymbol{v}(\boldsymbol{x})\delta(\phi(\boldsymbol{x}))d\boldsymbol{x}.$$

As the first approximation to (8), we take

$$\boldsymbol{v}(\boldsymbol{x})=\int\Gamma(\boldsymbol{x},\boldsymbol{y})n\boldsymbol{u}_0(\boldsymbol{y})\theta(\phi(\boldsymbol{y}))d\boldsymbol{y}$$

and get

$$\boldsymbol{v}'(\boldsymbol{z})\theta(-\phi(\boldsymbol{z}))=-\int\Gamma'(\boldsymbol{z},\boldsymbol{x})\Lambda'^{-1}\Lambda\Phi(\boldsymbol{x})\Gamma(\boldsymbol{x},\boldsymbol{y})\delta(\phi(\boldsymbol{x}))n\boldsymbol{u}_0(\boldsymbol{y})\theta(\phi(\boldsymbol{y}))d\boldsymbol{x}d\boldsymbol{y}$$

$$=\int K(\boldsymbol{z},\boldsymbol{y})n\boldsymbol{u}_0(\boldsymbol{y})\theta(\phi(\boldsymbol{y}))d\boldsymbol{y}$$

$$K(\boldsymbol{z},\boldsymbol{y})=-\int\Gamma'(\boldsymbol{z},\boldsymbol{x})\Lambda'^{-1}\Lambda\Phi(\boldsymbol{x})\Gamma(\boldsymbol{x},\boldsymbol{y})\delta(\phi(\boldsymbol{x}))d\boldsymbol{x}\,.$$

The solution $\Gamma(\boldsymbol{y},\boldsymbol{x})$ to the equation (6) is computed to be

$$\Gamma(\boldsymbol{y},\boldsymbol{x})=\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}\times\begin{pmatrix}nq-\frac{1}{nq}\frac{\partial^2}{\partial x\partial x}&-\frac{1}{nq}\frac{\partial^2}{\partial x\partial y}&-\frac{1}{nq}\frac{\partial^2}{\partial x\partial z}\\-\frac{1}{nq}\frac{\partial^2}{\partial y\partial x}&nq-\frac{1}{nq}\frac{\partial^2}{\partial y\partial y}&-\frac{1}{nq}\frac{\partial^2}{\partial y\partial z}\\-\frac{1}{nq}\frac{\partial^2}{\partial z\partial x}&-\frac{1}{nq}\frac{\partial^2}{\partial z\partial y}&nq-\frac{1}{nq}\frac{\partial^2}{\partial z\partial z}\end{pmatrix}\right.$$

$$\left.+\begin{pmatrix}0&1\\-1&0\end{pmatrix}\times\begin{pmatrix}0&-\frac{\partial}{\partial z}&\frac{\partial}{\partial y}\\\frac{\partial}{\partial z}&0&-\frac{\partial}{\partial x}\\-\frac{\partial}{\partial y}&\frac{\partial}{\partial x}&0\end{pmatrix}\right\}G(\boldsymbol{y},\boldsymbol{x})$$

$$G(\boldsymbol{y},\boldsymbol{x})=\frac{e^{-nq|\boldsymbol{y}-\boldsymbol{x}|}}{4\pi|\boldsymbol{y}-\boldsymbol{x}|}$$

and $\Gamma'(\boldsymbol{y},\boldsymbol{x})$ is of the same form as $\Gamma(\boldsymbol{y},\boldsymbol{x})$, except that n is replaced by n'.

In the paraxial approximation we see that

$$\Gamma(\boldsymbol{y},\boldsymbol{x})=\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}\times\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}+\begin{pmatrix}0&1\\-1&0\end{pmatrix}\times\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}\right\}nqG(\boldsymbol{y},\boldsymbol{x})$$

$$=JnqG(\boldsymbol{y},\boldsymbol{x})$$

$$\Phi=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\times\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix},$$

J being defined by the above expression, so that we have

$$K(\boldsymbol{z}, \boldsymbol{y}) = -nqn'qJ\Lambda'^{-1}\Lambda J\int G'(\boldsymbol{z}, \boldsymbol{x})G(\boldsymbol{x}, \boldsymbol{y})\delta(\phi(\boldsymbol{x}))d\boldsymbol{x}$$

so the essential feature of propagation of wave is the same as in the scalar theory (Refer to (10) in § 7, (3) in § 8).

References

[1] Born, M. and Wolf, E.: *Principles of Optics*, p. 419, Pergamon Press, 1980.
[2] Stratton, J. A.: *Electromagnetic Theory*, p. 37, McGraw-Hill, 1941.