National Science Report, Ochanomizu University, Vol. 43, No. 2, 1992
お茶の水女子大学 自然科学報告 第43巻 第2号

Self Similar Sets aud Quotient Sets of Infinite Sequences

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(Received September 10, 1992)

§ 1. Introduction.

The invariant set K with respect to some contraction maps $f_1, \cdots, f_k$ on $\boldsymbol{R}^N$ has interesting feature such as self-similarity, to which fractal is closely related. The set K may be considered as an image of the natural embedding φ from the set $E_k^{(\omega)}$ of infinite sequences on k symbols. The set $E_k^{(\omega)}$ equipped with the product topology is a compact, totally disconnected set. If the map φ is not one-to-one, the invariant set K may be connected, though it is totally disconnected when φ is one-to-one. In [1], Hata studied the structure of self-similar sets by using linear homeomorphisms on the unit interval $[0, 1]$ and he gave the conditions for self-similar sets to be connected, to be a simple arc, to have many end points, etc.

In this paper, we show a different approach to study the structure of self-similar sets which are the invariant sets with respect to two contraction maps f_1, f_2 on $\boldsymbol{R}^N$. When the map φ from $E_2^{(\omega)}$ onto the invariant set K with respect to f_1, f_2 is not one-to-one, there exists a pair (u^1, u^2) in $E_2^{(\omega)}$ such that $\varphi(u^1)=\varphi(u^2)$. We introduce an equivalence relation in the set of infinite sequences, when this pair (u^1, u^2) satisfies some condition, which we call (*)-pair. The cosets induced by the (*)-pair (u^1, u^2) are the elements of a quotient set $E_2^{(\omega)}/_{\tilde{u}}$ of $E_2^{(\omega)}$. We shall investigate the topology of the quotient set $E_2^{(\omega)}/_{\tilde{u}}$, and then study the structure of self-similar sets by applying this result to the invariant set with respect to some contraction maps. At § 3 we show that the quotient set $E_2^{(\omega)}/_{\tilde{u}}$ is connected since the equivalence set consists of more than two elements [Theorem 3.6]. § 4 is devoted to the study of end points of $E_2^{(\omega)}/_{\tilde{u}}$. The number of end points of the quotient set depends on the property of equivalence sets. We show that there exists an end point of $E_2^{(\omega)}/_{\tilde{u}}$ for any (*)-pair (u^1, u^2) and that there exist infinitely many end points for some (*)-pair (u^1, u^2) [Theorem 4.4]. We give a necessary and sufficient condition for $E_2^{(\omega)}/_{\tilde{u}}$ to be homeomorphic to the unit interval $[0, 1]$ in Theorem 4.5. In § 5 we apply these results to the invariant set with respect to contraction maps f_1, f_2 on $\boldsymbol{R}^N$. Let f_j $(j=1,2)$ be an one-to-one, contraction map on $\boldsymbol{R}^N$, let $\mathrm{Fix}(f_j)$ be the uniquely determined fixed point of f_j and let $K=K(f_1,f_2)$ be the compact subset of $\boldsymbol{R}^N$ satisfying $K=f_1(K)\cup f_2(K)$. Then there exists a continuous map φ of $E_2^{(\omega)}$ onto K. If $\mathrm{Fix}(f_1)\neq\mathrm{Fix}(f_2)$ and $f_1(K)\cap f_2(K)\neq\emptyset$, then there exist a (*)-pair (u^1,u^2) in $E_2^{(\omega)}$ such that $\varphi(u^1)=\varphi(u^2)$ [Proposition 5.1]. By using this (*)-pair (u^1,u^2), we define the quotient set $E_2^{(\omega)}/_{\tilde{u}}$ and investigate the relation between the invariant set $K(f_1,f_2)$ and the quotient set $E_2^{(\omega)}/_{\tilde{u}}$. If $f_1(K)\cap f_2(K)$ is a singleton, K is homeomorphic to $E_2^{(\omega)}/_{\tilde{u}}$ [Theorem 5.2] and the structure of K depends on the pair (u^1,u^2). Some pair (u^1,u^2) makes $K(f_1,f_2)$ a simple arc and some other pair (u^1,u^2) induces $K(f_1,f_2)$ to have infinitely many end points [Theorem 5.3]. So the investigation of $E_2^{(\omega)}/_{\tilde{u}}$ is useful in discussing the topology of K and also may be helpful in drawing fractals.

§ 2. Preliminaries.

(1) Let $E_2^{(\omega)}$ be the set of infinite sequences on 2 symbols $\{x=(x_1\cdots x_n\cdots)\mid x_i\in\{1,2\}\}$ and $E_2^{(*)}$ be the set of finite sequences $\{\alpha=(\alpha_1\cdots\alpha_n)\mid n\in\boldsymbol{N}, \alpha_i\in\{1,2\}\}$. Define the addition

$$\oplus : E_2^{(*)}\times(E_2^{(*)}\cup E_2^{(\omega)}) \longrightarrow (E_2^{(*)}\cup E_2^{(\omega)}) \text{ by}$$

$$(\alpha_1\cdots\alpha_n)\oplus(\beta_1\cdots\beta_m)=(\alpha_1\cdots\alpha_n\beta_1\cdots\beta_m)$$

and

$$(\alpha_1\cdots\alpha_n)\oplus(x_1\cdots x_k\cdots)=(\alpha_1\cdots\alpha_n x_1\cdots x_k\cdots)\,.$$

(2) Let $|\alpha|$ be the length n of $\alpha=(\alpha_1\cdots\alpha_n)\in E_2^{(*)}$ and $|x|=\infty$ for $x\in E_2^{(\omega)}$. We shall say that $\alpha\in E_2^{(*)}$ is a cycle of $x=E_2^{(\omega)}$ if α satisfies the relation $x=\alpha\oplus x$ and we say that α_0 is the *minimal cycle of* x if α_0 is a cycle of x and $|\alpha_0|\leqq|\alpha|$ for any cycle α of x. We say that $x\in E_2^{(\omega)}$ is the *minimal cycle of* x if there exists no $\alpha\in E_2^{(*)}$ such that $x=\alpha\oplus x$.

(3) For $x\in E_2^{(\omega)}$, the notation $x=\sum_{j=1}^M\oplus z^j$ means the following:

if $M<\infty$, then $\sum_{j=1}^M\oplus z^j=z^1\oplus z^2\oplus\cdots\oplus z^M$ with $z^j\in E_2^{(*)}(1\leqq j\leqq M-1)$ and $z^M\in E_2^{(\omega)}$, and

if $M=\infty$, then $\sum_{j=1}^M\oplus z^j=z^1\oplus z^2\oplus\cdots\oplus z^n\oplus\cdots$ with $z^j\in E_2^{(*)}(1\leqq j<\infty)$.

(4) For $n\in\boldsymbol{N}\cup\{0\}$, we shall define maps $\sigma^n : E_2^{(\omega)}\to E_2^{(\omega)}$ and $P_n : E_2^{(\omega)}\to E_2^{(*)}\cup\{\emptyset\}$ by

$$\sigma^n(x_1\cdots x_k\cdots)=(x_{n+1}x_{n+2}\cdots)$$

$$P_n(x_1\cdots x_k\cdots)=\begin{cases}(x_1\cdots x_n) & (n\geqq 1)\\ \emptyset & (n=0)\,.\end{cases}$$

(5) For $x, y \in E_2^{(\omega)}$, let α and β be minimal cycles of x and y respectively and $Q(x, y)=\{\sum_{j=1}^{M} \oplus z^j \in E_2^{(\omega)} | z^j \in \{\alpha, \beta, x, y\}\}$. We say that x and $y \in E_2^{(\omega)}$ are *mutually prime* if the existence of $n, m \in \boldsymbol{N}$ such that $P_m \sigma^n v \in \{\alpha, \beta\}$ for $v \in Q(x, y)$ implies $P_n v = z^1 \oplus \cdots \oplus z^l$ with $l \in \boldsymbol{N}$ and $z^j \in \{\alpha, \beta\}$ $(1 \leqq j \leqq l)$.

(6) For $u^1=(u_n^1), u^2=(u_n^2) \in E_2^{(\omega)}$, let α^1 and α^2 be minimal cycles of u^1 and u^2 respectively. We say that (u^1, u^2) is *a (*)-pair* if the following relations (*1), (*2) and (*3) hold.

(*1) $u_1^1=1, u_1^2=2$.

(*2) u^1 and u^2 are mutually prime.

(*3) Either $|\alpha^1| \geqq 2$ or $|\alpha^2| \geqq 2$ is satisfied.

§ 3. The quotient set $E_2^{(\omega)}/_{\tilde{u}}$.

The set $E_2^{(\omega)}$, equipped with the product topology, is a compact set which has a fundamental basis $\{U_n(x) | n \in \boldsymbol{N}\}$ of neighborhoods of $x \in E_2^{(\omega)}$, where $U_n(x)=\{y \in E_2^{(\omega)} | P_n x=P_n y\}$.

Hereafter, let (u^1, u^2) be a (*)-pair and α^1, α^2 be the minimal cycles of u^1, u^2 respectively and let $Qu=Q(u^1, u^2)$, that is, $Qu=\{\sum_{j=1}^{M} \oplus \beta^j \in E_2^{(\omega)} | \beta^j \in \{\alpha^1, \alpha^2, u^1, u^2\}\}$. Then we have

LEMMA 3.1. *Qu is a closed set in $E_2^{(\omega)}$.*

PROOF. Since every point of $E_2^{(\omega)}$ is a closed set, it is obvious that $Qu=\{u^1, u^2\}$ is closed in $E_2^{(\omega)}$ if $|\alpha^1|=|\alpha^2|=\infty$. In case of $|\alpha^1|=n_1<\infty$ and $|\alpha^2|=n_2<\infty$, suppose for $y=(y_n) \in E_2^{(\omega)}, P_n y$ belongs to $P_n Qu$ for all $n \in \boldsymbol{N}$. Then $P_{n_{j_1}} y=\alpha^{j_1}$ with $j_1=y_1$ and $P_{n_{j_1}+n_{j_2}} y=\alpha^{j_1} \oplus \alpha^{j_2}$ with $j_2=y_{n_{j_1}+1}$ $(j_k \in \{1, 2\})$. By repeating this process, we obtain that $y=\sum_{k=1}^{\infty} \oplus \alpha^{j_k}$ belongs to Qu. So $y \notin Qu$ implies the existence of $n \in \boldsymbol{N}$ such that $U_n(y) \cap Qu=\emptyset$. Hence Qu is closed. If either $|\alpha^1|<\infty$, $|\alpha^2|=\infty$ or $|\alpha^1|=\infty$, $|\alpha^2|<\infty$ holds, we can prove that Qu is closed in a similar way. □

DEFINITION. We shall write $x \underset{\tilde{u}}{} y$ for $x, y \in E_2^{(\omega)}$ if either $x=y$ holds or there exists $n \in \boldsymbol{N} \cup \{0\}$ such that $P_n x=P_n y$ and $\sigma^n x, \sigma^n y \in Qu$.

LEMMA 3.2. *The relation $\tilde{u}$ satisfies the equivalence relation.*

PROOF. It is clear that $x \,{}_{\tilde{u}}\, x$ holds and $x \,{}_{\tilde{u}}\, y$ implies $y \,{}_{\tilde{u}}\, x$. So we shall show that $x \,{}_{\tilde{u}}\, y$ and $y \,{}_{\tilde{u}}\, z$ implies $x \,{}_{\tilde{u}}\, z$. If either $x=y$ or $y=z$ holds, it is obvious. So suppose that there exist n_1 and n_2 such that $n_1 \geqq n_2 \geqq 0$, $P_{n_1} x = P_{n_1} y$, $P_{n_2} y=P_{n_2} z$ and $\sigma^{n_1} x$, $\sigma^{n_1} y$, $\sigma^{n_2} y$, $\sigma^{n_2} z \in Qu$. Then $\sigma^{n_2} y \in Qu$ implies $\sigma^{n_2} y=\sum_{j=1}^{M} \oplus z^j$ with $z^j \in \{\alpha^1, \alpha^2\}$ for $1 \leqq j<M$. If $n_1=n_2$, it is obvious that

$x_{\tilde{u}} z$. So suppose $n_1 > n_2$. Then there exists $l \geqq 0$ such that $\sigma^{n_2} y = z^1 \oplus \cdots \oplus z^l \oplus \gamma \oplus \sigma^{n_1} y$ with $|\gamma| \leqq |z^{l+1}|$. Since u^1 and u^2 are mutually prime, $\gamma = z^{l+1}$ holds and $\sigma^{n_2} x = P_{n_1 - n_2}(\sigma^{n_2} y) \oplus \sigma^{n_1} x = \sum_{j=1}^{l+1} \oplus z^j \oplus \sigma^{n_1} x$ belongs to Qu. Since it is obvious that $P_{n_2} x = P_{n_2} z$ we have $x_{\tilde{u}} z$. Therefore the relation $\tilde{u}$ satisfies the equivalence relation. □

DEFINITION. For $x \in E_2^{(\omega)}$, let Qx be the equivalence class $\{y \in E_2^{(\omega)} : x_{\tilde{u}} y\}$ and let $E_2^{(\omega)}/_{\tilde{u}}$ be the set of equivalence classes, which we call the quotient set induced by (u^1, u^2). Let q be the canonical map of $E_2^{(\omega)}$ onto $E_2^{(\omega)}/_{\tilde{u}}$.

For $n \in \boldsymbol{N}$ and $0 \leqq j \leqq n-1$, define

$$H(x;\ j) = \{h \in E_2^{(\omega)} \mid P_j h = P_j x, \sigma^j h \in Qu, q(h) \neq q(x)\}$$

$$J(x;\ n) = \{j \in \{0, 1, \cdots, n-1\} \mid \exists h \in H(x;\ j) \ s.t.\ P_n x = P_n h\}$$

$$V_n(x) = \{y \in E_2^{(\omega)} \mid P_n Qy = \{P_n x\} \ or \ y = x\}$$

$$\tilde{U}_n(q(x)) = \{q(y) \in E_2^{(\omega)}/_{\tilde{u}} \mid P_n Qy \subset P_n Qx\}.$$

Let Λ_x be the set of $\boldsymbol{N}$-valued functions on Qx. For $\eta \in \Lambda_x$, define

$$\tilde{V}_\eta(q(x)) = \cup \{q(V_{\eta(x')}(x')) \mid x' \in Qx\}.$$

REMARK. It holds that $\tilde{U}_n(q(x)) = \cup \{q(V_n(x')) \mid x' \in Qx\}$.

PROPOSITION 3.3. *Concerning the topology in $E_2^{(\omega)}/_{\tilde{u}}$, we have the following.*

(1) *The family $\{\tilde{V}_\eta(q(x)) \mid q(x) \in E_2^{(\omega)}/_{\tilde{u}}, \eta \in \Lambda_x\}$ is a basis for the quotient topology in $E_2^{(\omega)}/_{\tilde{u}}$, that is, for any $\eta \in \Lambda_x, q^{-1}(\tilde{V}_\eta(q(x)))$ is open in $E_2^{(\omega)}$ and for $W \subset E_2^{(\omega)}/_{\tilde{u}}$ satisfying that $q^{-1}(W)$ is open in $E_2^{(\omega)}$, there exists $\eta \in \Lambda_x$ such that $\tilde{V}_\eta(q(x)) \subset W$.*

(2) *The family $\{\tilde{U}_n(q(x)) \mid n \in \boldsymbol{N}, q(x) \in E_2^{(\omega)}/_{\tilde{u}}\}$ is a basis for some topology in $E_2^{(\omega)}/_{\tilde{u}}$.*

(3) *The topology induced by $\{\tilde{V}_\eta(q(x))\}$ is finer than that by $\{\tilde{U}_n(q(x))\}$.*

(4) i) *When Qu consists of two elements $\{u^1, u^2\}$, the topology induced by $\{\tilde{V}_\eta(q(x))\}$ is equivalent to that by $\{\tilde{U}_n(q(x))\}$.*

ii) *For $x \in E_2^{(\omega)}$ satisfying that $Qx = \{x\}$, we have $\tilde{U}_n(q(x)) = \tilde{V}_{\eta_n}(q(x))$ for any $n \in \boldsymbol{N}$, where $\eta_n(x') = n$ for any $x' \in Qx$.*

PROOF. (1) By the definition, we have that $V_n(x) = U_n(x) \setminus \cup \{H(x;\ j) \mid j \in J(x;\ n)\}$. So $V_n(x)$ is an open set, since $H(x;\ j)$ is a closed set by the closedness of Qu and $J(x;\ n)$ is a finite set. For $\eta \in \Lambda_x$, we see that $q^{-1}(\tilde{V}_\eta(q(x))$ is open by the relation $q^{-1}(\tilde{V}_\eta(q(x))) = \cup \{V_{\eta(x')}(x') \mid x' \in Qx\}$.

For $W \subset E_2^{(\omega)}/_{\tilde{u}}$ with $q(x) \in W$, let $q^{-1}(W)$ be open in $E_2^{(\omega)}$. For any

$x'\in Qx$, there exists $n_{x'}\in \boldsymbol{N}$ such that $U_{n_{x'}}(x')\subset q^{-1}(W)$. Define $\eta\in \Lambda_x$ by $\eta(x')=n_{x'}$ for any $x'\in Qx$. Then $V_{\eta(x')}(x')\subset U_{\eta(x')}(x')\subset q^{-1}(W)$ and so $q(\cup\{V_{\eta(x')}(x')|x'\in Qx\})\subset W$. We get $\tilde{V}_\eta(q(x))\subset W$ by the relation

$$q(\cup\{V_{\eta(x')}(x')|x'\in Qx\})=\cup\{q(V_{\eta(x')}(x'))|x'\in Qx\}\,.$$

(2) $q(y)\in\tilde{U}_n(q(x))$ implies $\tilde{U}_n(q(y))\subset\tilde{U}_n(q(x))$. So it is obvious that $\{\tilde{U}_n\}$ is a basis for some topology.

(3) Define $\eta_n\in\Lambda_x$ by $\eta_n(x')=n$ for any $x'\in Qx$. Then $\tilde{U}_n(q(x))=\tilde{V}_{\eta n}(q(x))$, which implies that the topology induced by $\{\tilde{V}_\eta(q(x))\}$ is finer than that by $\{\tilde{U}_n(q(x))\}$.

(4) It is obvious by definition. □

Let $(E_2^{(\omega)}/_{\tilde{u}},\tilde{U})$ be the topological space $E_2^{(\omega)}/_{\tilde{u}}$, where the family $\{\tilde{U}_n(q(x))|q(x)\in E_2^{(\omega)}/_{\tilde{u}},n\in\boldsymbol{N}\}$ is a basis for the topology.

PROPOSITION 3.4. *Suppose* $\phi: E_2^{(\omega)}\to K\subset\boldsymbol{R}^N$ *is continuous and* $q(x)=q(y)$ *implies* $\phi(x)=\phi(y)$. *Then* $\tilde{\phi}:(E_2^{(\omega)}/_{\tilde{u}},\tilde{U})\to K$, *defined by* $\tilde{\phi}(q(x))=\phi(x)$, *is contiuous.*

PROOF. $\tilde{\phi}(q(x))=\phi(x)$ is well-defined since $q(x)=q(y)$ implies $\phi(x)=\phi(y)$. Since ϕ is uniformly continuous on a compact set $E_2^{(\omega)}$, for any neighborhood $V\subset\boldsymbol{R}^N$ of $0\in\boldsymbol{R}^N$ there exist $n\in\boldsymbol{N}$ such that $\phi(U_n(x))\subset V+\phi(x)$ holds for any $x\in E_2^{(\omega)}$. Then it holds that $q^{-1}(\tilde{U}_n(q(x)))\subset\cup\{U_n(x')|x'\in Qx\}$, which implies $\tilde{\phi}(\tilde{U}_n(q(x)))\subset V+\tilde{\phi}(q(x))$ and so $\tilde{\phi}$ is continuous. □

LEMMA 3.5. *Let A be an open proper subset of* $(E_2^{(\omega)}/_{\tilde{u}},\tilde{U})$. *For* $q(x)\in A$ *such that* $Qx=\{x\}$, *suppose* $n\in\boldsymbol{N}$ *satisfies* $\tilde{U}_n(q(x))\subset A$ *and* $\tilde{U}_{n-1}(q(x))\not\subset A$. *Then either of the following holds.*

(1) *$q(h)$ does not belong to A for any* $h\in H(x;\,n-1)$.

(2) *There exist $m(>n)$ and* $y\in q^{-1}(A)$ *satisfying* $P_{n-1}y=P_{n-1}x$,

$$Qy=\{y\},\ \tilde{U}_m(q(y))\subset A \quad \textit{and} \quad \tilde{U}_{m-1}(q(y))\not\subset A\,.$$

PROOF. $\tilde{U}_n(q(x))\subset A$ and $\tilde{U}_{n-1}(q(x))\not\subset A$ implies $H(x;\,n-1)\neq\emptyset$. Suppose $q(h)\in A$ for some $h\in H(x;\,n-1)$. Then we shall show that (2) holds. $Qx=\{x\}$ implies $\tilde{U}_{n-1}(q(x))\subset\tilde{U}_n(q(h))$ for $h\in H(x;\,n-1)$. By the relation $q(h)\in A$, there exists $m(>n)$ satisfying $\tilde{U}_m(q(h))\subset A$ and $\tilde{U}_{m-1}(q(h))\not\subset A$. So there exists $z\in E_2^{(\omega)}$ satisfying $P_{m-1}Qz\subset P_{m-1}Qh$, $P_{m-1}Qz=\{P_{m-1}z\}$ and $q(z)\notin A$. Let $h'\in Qh$ satisfy $P_{m-1}z=P_{m-1}h'$. We can find $y\in E_2^{(\omega)}$ such that $P_my=P_mh'$ and $Qy=\{y\}$. Then $y\in q^{-1}(A)$ and m are required ones. □

THEOREM 3.6. *Let (u^1,u^2) be a $(*)$-pair.*

Then the quotient set $(E_2^{(\omega)}/_{\tilde{u}},\tilde{U})$ *induced by* (u^1,u^2) *is a connected, compact, Hausdorff space, where the family* $\{\tilde{U}_n(q(x))|q(x)\in E_2^{(\omega)}/_{\tilde{u}},n\in\boldsymbol{N}\}$

is a basis for the topology.

PROOF. To prove the connectedness of the set $(E_2^{(\omega)}/_{\tilde{u}}, \tilde{U})$, we shall show that nonempty subset B of $E_2^{(\omega)}/_{\tilde{u}}$ is not open if $A=B^c$ (=the complement of B) is a nonempty open set. Since A is nonempty, there exist $x^1 \in q^{-1}(A)$ and $n_1 \in \boldsymbol{N}$ such that $Qx^1=\{x^1\}$, $\tilde{U}_{n_1}(q(x^1)) \subset A$ and $\tilde{U}_{n_1-1}(q(x^1)) \not\subset A$. If $q(h)$ does not belong to A for $h \in H(x^1; n_1-1)$, $q(h)$ is not an interior point of B, which implies that B is not open. If $q(h)$ belongs to A for $h \in H(x^1; n_1-1)$, there exist $n_2(>n_1)$ and $x^2 \in q^{-1}(A)$ such that $P_{n_1-1}x^1 = P_{n_1-1}x^2$, $Qx^2=\{x^2\}$, $\tilde{U}_{n_2}(q(x^2)) \subset A$ and $\tilde{U}_{n_2-1}(q(x^2)) \not\subset A$, by Lemma 3.5. If $q(h')$ does not belong to A for $h' \in H(x^2; n_2-1)$, it follows that B is not open in the same way as above. So we consider the case that there exist sequences $\{n_j\} \subset \boldsymbol{N}$ and $\{x^j\} \subset q^{-1}(A)$ satisfying $n_j > n_{j-1}$, $P_{n_{j-1}-1}x^j = P_{n_{j-1}-1}x^{j-1}$, $\tilde{U}_{n_j}(q(x^j)) \subset A$ and $\tilde{U}_{n_j-1}(q(x^j)) \not\subset A$. Consider $z=(z_k) \in E_2^{(\omega)}$ with $z_k = x_k^j$ for $n_{j-1} \leqq k \leqq n_j - 1$. Then z belongs to $q^{-1}(A^c)$ and $q(z)$ is not an interior point of B, which implies that B is not open. So $E_2^{(\omega)}/_{\tilde{u}}$ is connected.

The compactness of $E_2^{(\omega)}/_{\tilde{u}}$ follows from the compactness of $E_2^{(\omega)}$, the relation $q(U_n(x)) \subset \bigcup_{j \in J(x;\, n)} q(H(x;\, j)) \cup \tilde{U}_n(q(x))$ and the fact that $J(x; n)$ consists of finite elements. It is easily obtained that $E_2^{(\omega)}/_{\tilde{u}}$ is a Hausdorff space. □

In a similar way to Theorem 3.6, the following is obtained.

COROLLARY 1. *For any $x \in E_2^{(\omega)}$ and any $n \in \boldsymbol{N}$, the set $\tilde{U}_n(q(x))$ is a connected open set in the set $(E_2^{(\omega)}/_{\tilde{u}}, \tilde{U})$.*

Since $\{\tilde{U}_n(q(x)) | n \in \boldsymbol{N}\}$ is a basis for $(E_2^{(\omega)}/_{\tilde{u}}, \tilde{U})$, the following is obtained.

COROLLARY 2. *The set $(E_2^{(\omega)}/_{\tilde{u}}, \tilde{U})$ is locally connected, that is, for any open set $\tilde{W}$ containing $q(x) \in (E_2^{(\omega)}/_{\tilde{u}}, \tilde{U})$, there exists an open set $\tilde{V}$ containing $q(x)$, which is contained in a connected component of $\tilde{W}$.*

§ 4. End Points of $E_2^{(\omega)}/_{\tilde{u}}$.

DEFINITION. $q(x) \in E_2^{(\omega)}/_{\tilde{u}}$ is called *an end point* of $E_2^{(\omega)}/_{\tilde{u}}$ if there exists $N \in \boldsymbol{N}$ satisfying that the boundary $\partial \tilde{U}_n(q(x))$ of $\tilde{U}_n(q(x))$ is a singleton for any $n \geqq N$.

The following lemma characterizes the boundary $\partial \tilde{U}_n(q(x))$.

LEMMA 4.1. $\partial \tilde{U}_n(q(x)) = \bigcup \{q(H(x';\, j)) | x' \in Qx, j \in J(x';\, n)\}$.

PROOF. $x' \in Qx$, $j \in J(x'; n)$ and $h \in H(x'; j)$ we have $g(h) \notin (\tilde{U}_n(q(x)))$. For any $m(>n)$, choose $y \in E_2^{(\omega)}$ such that $Qy=\{y\}$ and $P_m y = P_m h$. Then

$q(y)\in\tilde{U}_n(q(x))\cap\tilde{U}_m(q(h))$, which implies

$$\cup\{q(H(x';\ j))|x'\in Qx, j\in J(x';\ n)\}\subset\partial\tilde{U}_n(q(x)).$$

Conversely, if $q(y)\notin\cup\{q(H(x';\ j))|x'\in Qx, j\in J(x';\ n)\}\cup\tilde{U}_n(q(x))$, then $q(y)$ is an interior point of $(\tilde{U}_n(q(x)))^c$, which implies

$$\partial\tilde{U}_n(q(x))\subset\cup\{q(H(x';\ j))|x'\in Qx, j\in J(x';\ n)\}. \quad \square$$

PROPOSITION 4.2. *If* $|\alpha^1|\geqq 2$ *and* $|\alpha^2|\geqq 2$, *the following are equivalent.*

(1) $q(x)\in E_2^{(\omega)}/_{\tilde{u}}$ *is an end point of* $E_2^{(\omega)}/_{\tilde{u}}$.

(2) $Qx=\{x\}$ *and there exists* $N\in\boldsymbol{N}$ *such that* $J(x;\ n)=\{n-1\}$ *for all* $n\geqq N$.

PROOF. By Lemma 4.1, it is clear that (2) implies (1).

(1)⇒(2): If $Qx\neq\{x\}$, there exists $k\in\boldsymbol{N}$ such that $Qx=H(x;\ k)$. For any $N>k$ there exists $n>N$ such that $\partial\tilde{U}_n(q(x))$ is not a singleton, since $|\alpha^2|\geqq 2$ is satisfied. So (1) implies $Qx=\{x\}$.

Suppose $q(H(x;\ j))=q(H(x;\ n))$ with $j<n$. Then $P_{n-j}\sigma^j x=\beta^1\oplus\cdots\oplus\beta^l$ with $l\geqq 1, \beta^i\in\{\alpha^1,\alpha^2\}$ $(1\leqq i\leqq l)$. We have $j\leqq n-2$ and $q(H(x;\ j))\neq q(H(x;\ n-1))$, since $|\alpha^1|\geqq 2$ and $|\alpha^2|\geqq 2$. So $\partial\tilde{U}_n(q(x))$ is not a singleton. Hence (1) implies that there exists $N\in\boldsymbol{N}$ such that $J(x;\ n)=\{n-1\}$ for all $n\geqq N$. $\square$

LEMMA 4.3. *If* $|\alpha^1|=1$ *and* $2\leqq|\alpha^2|=n_2<\infty$ *then* $\alpha^2_{n_2}=2$.

PROOF. Consider $v=\alpha^2\oplus u^1$. If $\alpha^2_{n_2}=1$, we have $\sigma^{n_2-1}v=u^1$. Since u^1 and u^2 are mutually prime, $P_{n_2-1}v$ must be $z^1\oplus\cdots\oplus z^l$ with $l\geqq 1, z^j\in\{\alpha^1,\alpha^2\}$ $(1\leqq j\leqq l)$, which is a contradiction. $\square$

COROLLARY TO PROPOSITION 4.2. *In case that* $|\alpha^1|=1$ *and* $u^2_2=1$, (1) *and* (2) *in Proposition* 4.2 *are equivalent.*

PROOF. To prove the above corollary, it is enough to show that (1) implies (2) for $x\in E_2^{(\omega)}$ with $Qx=\{x\}$. For any $N\in\boldsymbol{N}$, there exists $n>N$ such that $x_n=2$, since $Qx=\{x\}$ and $|\alpha^1|=1$. If $x_n=2$ and $x_{n+1}=1$, $\partial\tilde{U}_{n+1}(q(x))$ is not a singleton by $u^2_2=1$ and Lemma 4.3. So there exists $n_0\in\boldsymbol{N}$ such that $x_n=2$ for $n\geqq n_0$, which implies that $J(x;\ n)=\{n-1\}$ for all $n>n_0$. $\square$

REMARK. If $|\alpha^1|=1$ and $u^2_2=2$, there exists $x\in E_2^{(\omega)}$ such that $q(x)$ is an end point of $E_2^{(\omega)}/_{\tilde{u}}$ but there exists no $N\in\boldsymbol{N}$ such that $J(x;\ n)=\{n-1\}$ for all $n\geqq N$. For example, $x=\sum_{j=1}^{\infty}\oplus(211)$ satisfies this condition.

As for the existence of end points, we have

THEOREM 4.4. (1) *For any* (*)-*pair* (u^1,u^2), *there exists an end point* *of* $E_2^{(\omega)}/_{\tilde{u}}$.

(2) *There exist infinitely many end points unless* $J(u^1;\ n)=J(u^2;\ n)$ $=\{n-1\}$ *holds for all* $n\geqq 2$.

PROOF. Let $x^1=\sum_{j=1}^{\infty}\oplus(1)$, $x^2=\sum_{j=1}^{\infty}\oplus(2)$, $x^3=\sum_{j=1}^{\infty}\oplus(12)$, $x^4=\sum_{j=1}^{\infty}\oplus(112)$ and $x^5=\sum_{j=1}^{\infty}\oplus(221)$.

(1) We shall show that for any (*)-pair (u^1, u^2), one of $q(x^1)\sim q(x^5)$ is an end point of $E_2^{(\omega)}/_{\tilde{u}}$.

By Proposition 4.2, it is obtained that

$q(x^1)$ is an end point of $E_2^{(\omega)}/_{\tilde{u}}$ if $u_2^1=2$,

$q(x^2)$ is an end point of $E_2^{(\omega)}/_{\tilde{u}}$ if $u_2^2=1$,

$q(x^3)$ is an end point of $E_2^{(\omega)}/_{\tilde{u}}$ if $|\alpha^1|\geqq 2$, $|\alpha^2|\geqq 2$, $u_2^1=1$ and $u_2^2=2$,

$q(x^4)$ is an end point of $E_2^{(\omega)}/_{\tilde{u}}$ if $|\alpha^1|=1$ and $u_2^2=2$,

and

$q(x^5)$ is an end point of $E_2^{(\omega)}/_{\tilde{u}}$ if $|\alpha^2|=1$ and $u_2^1=2$.

(2) Unless $J(u^1;\ n)=J(u^2;\ n)=\{n-1\}$ holds for all $n\geqq 2$, there exist $n_0\in \boldsymbol{N}$ and j_0 $(1\leqq j_0\leqq n_0-2)$ such that either $j_0\in J(u^1;\ n_0)$ or $j_0\in J(u^2;\ n_0)$ holds, since $0\notin J(u^j;\ n_0)$ by the definition of $J(u^j;\ n_0)$ $(j=1, 2)$. It is enough to consider the case of $j_0\in J(u^2;\ n_0)$.

(a) In case $u_2^1=2$:

If there exists $m\in\boldsymbol{N}$ such that

$$\sigma^m u^1=x^1 \tag{4.1}$$

then $m\geqq 2$. If there exists $m\in\boldsymbol{N}$ such that

$$\sigma^m u^2=x^1 \tag{4.2}$$

then $m\geqq 3$.

Let m_1 and m_2 be the smallest number satisfying (4.1) and (4.2) respectively, if they exist. Choose $y\in E_2^{(\omega)}$ such that

(i) $P_2\sigma y\neq P_2\sigma^{m_1-2}u^1$ (ii) $P_3 y\neq P_3\sigma^{m_2-3}u^2$ and (iii) $\sigma^2 y=(2)\oplus x^1$, where the condition (i) [resp. (ii)] is unnecessary if there exists no m satisfying (4.1) [resp. (4.2)]. For any $k\in\boldsymbol{N}$, let $z^k\in E_2^{(\omega)}$ satisfy $\sigma^k z^k=y$. Then $q(z^k)$ is an end point of $E_2^{(\omega)}/_{\tilde{u}}$. Hence there exist infinitely many end points of $E_2^{(\omega)}/_{\tilde{u}}$.

(b) In case $u_2^1=1$ and $u_2^2=1$:

We can show that there exist infinitely many end points of $E_2^{(\omega)}/_{\tilde{u}}$ in the same way as (a) by replacing x_1 with x_2.

(c) In case $|\alpha^1|\geqq 2$, $|\alpha^2|\geqq 2$, $u_2^1=1$ and $u_2^2=2$:

If there exists $m_3\in\boldsymbol{N}$ [resp. $m_4\in\boldsymbol{N}$] such that

$$\sigma^{m_3}u^1=x^3 \qquad [\text{resp. } \sigma^{m_4}u^2=x^3] \tag{4.3}$$

the $m_3\geqq 1$ [resp. $m_4\geqq 3$]. Let m_3 and m_4 be the smallest number satisfying

(4.3) if they exist and let $m_3=\infty$ or $m_4=\infty$ if there exists no $m\in\boldsymbol{N}$ satisfying (4.3). If $m_3=1$ [resp. $2\leqq m_3\leqq\infty$], choose $y\in E_2^{(\omega)}$ such that $\sigma y=(22)\oplus x^3$ and $P_3y\neq P_3\sigma^{m_3-3}u^2$ [resp. $\sigma^2y=x^3$, $P_2y\neq P_2\sigma^{m_1-2}u^1$, $P_2y\neq P_2\sigma^{m_2-2}u^2$ and $P_2y\neq(12)$], where the corresponding condition is unnecessary if either $m_3=\infty$ or $m_4=\infty$ holds.

(d) In case $|\alpha^1|=1$ and $u_2^2=2$:

If there exists $m_5\in\boldsymbol{N}$ such that $\sigma^{m_5}u^2=x^4$, then $m_5\geqq 3$. Choose $y\in E_2^{(\omega)}$ such that $\sigma^2y=x^4$, $P_2y\neq P_2\sigma^{m_5-2}u^2$ and $P_2y\neq(12)$.

(e) In case $|\alpha^2|=1$ and $u_2^1=1$:

If there exists $m_6\in\boldsymbol{N}$ such that $\sigma^{m_6}u^1=x^5$, then $m_6\geqq 2$. Choose $y\in E_2^{(\omega)}$ such that $\sigma^2y=x^5$, $P_2y\neq P_2\sigma^{m_6-2}u^1$ and $P_2y\neq(21)$.

So also in case (c)$\sim$(e) as (a) and (b), let $z^k\in E_2^{(\omega)}$ satisfy $\sigma^kz^k=y$ for any $k\in\boldsymbol{N}$. Then $q(z^k)$ is an end point of $E_2^{(\omega)}/_{\tilde{u}}$. Hence there exist infinitely many end points of $E_2^{(\omega)}/_{\tilde{u}}$. □

THEOREM 4.5. *The following are equivalent.*

(1) *The set* $(E_2^{(\omega)}/_{\tilde{u}}, \tilde{U})$ *is homeomorphic to the unit interval* $[0,1]$.

(2) *For any* $n\geqq 2$, $J(u^1;n)=J(u^2;n)=\{n-1\}$ *holds.*

PROOF. (1)⇒(2): By (1), there exists a homeomorphism $\tau:[0,1]\to E_2^{(\omega)}/_{\tilde{u}}$. For $t\in(0,1)$, there exists $\varepsilon>0$ such that $(t-\varepsilon,t+\varepsilon)\subset(0,1)$. Since $\tau((t-\varepsilon,t+\varepsilon))$ is an open set in $E_2^{(\omega)}/_{\tilde{u}}$, there exists $n_0\in\boldsymbol{N}$ such that $\tilde{U}_n(\tau(t))\subset\tau(t-\varepsilon,t+\varepsilon)$ for $n\geqq n_0$. Since $\tilde{U}_n(\tau(t))$ is a connected open set by Corollary 1 to Theorem 3.6, $\tau^{-1}(\tilde{U}_n(\tau(t)))$ is also connected open set in $(0,1)$ and so $\partial(\tau^{-1}(\tilde{U}_n(\tau(t))))$ consists of two points. Hence $\partial(\tilde{U}_n(\tau(t)))$ consists of two points, which implies that $\tau(t)$ is not an end point. Therefore the set of end points of $E_2^{(\omega)}/_{\tilde{u}}$ consists of $\tau(0)$ and $\tau(1)$, which implies (2) by Theorem 4.4.

(2)⇒(1): For $x=(x_n)\in E_2^{(\omega)}$, define

$$e_n(x)=\begin{cases}\sum_{j=1}^{n-1}(x_j-u_2^{x_j}+1) & (n\geqq 2)\\ 0 & (n=1)\end{cases}$$

$$a_n(x)=\frac{1}{2}+(-1)^{e_n(x)}\left(x_n-\frac{3}{2}\right)$$

$$r(x)=\sum_{n=1}^{\infty}\frac{a_n(x)}{2^n}$$

Then $a_n(x)\in\{0,1\}$ for any $n\in\boldsymbol{N}$ and $r:E_2^{(\omega)}\to[0,1]$ is a continuous, onto mapping.

We recall that $r(x)=r(y)$ $(x\neq y)$ if and only if

$$(4.4)\quad \begin{cases} \text{there exists } n_0 \in \boldsymbol{N} \text{ such that } a_j(x)=a_j(y) \ (j \leqq n_0-1) \\ a_{n_0}(x)=0,\ a_{n_0}(y)=1,\ a_j(x)=1,\ a_j(y)=0 \ (j \geqq n_0+1) \\ [\text{resp. } a_{n_0}(y)=0,\ a_{n_0}(x)=1,\ a_j(y)=1,\ a_j(x)=0 \ (j \geqq n_0+1)]. \end{cases}$$

We see that (2) implies the following

$$(4.5)\qquad u_j^1 \neq u_{2j-1}^{u_2^1}, \quad u_j^2 \neq u_{2j-1}^{u_2^2} \qquad \text{for} \quad j \geqq 3.$$

We shall show that $r(x)=r(y)$ is equivalent to $q(x)=q(y)$ for $x, y \in E_2^{(\omega)}$.

Suppose $q(x)=q(y)$ for $x, y \in E_2^{(\omega)}$. Then since $Qu=\{u^1, u^2\}$ holds by (2), we may take $x=P_{n_0-1}x \oplus u^1$ and $y=P_{n_0-1}x \oplus u^2$ with some $n_0 \in \boldsymbol{N}$. Then $a_j(x)=a_j(y)$ for $j \leqq n_0-1$ and $e_{n_0}(x)=e_{n_0}(y)$. If $e_{n_0}(x)=e_{n_0}(y)=0$ (mod 2), we have $x_{n_0}=u_1^1=1$ and $a_{n_0}(x)=1/2+(u_1^1-(3/2))=0$. By (4.5), we have $e_{n_0+1}(x)=2-u_2^1$ (mod 2) and $e_{n_0+j}(x)=1-u_{2j}^{u_1^1}$ (mod 2) for $j \geqq 2$. Hence $a_{n_0+j}(x)=1$ for $j \geqq 1$. In the same way, we get $a_{n_0}(y)=1$ and $a_{n_0+j}(y)=0$ for $j \geqq 1$. Hence (4.4) is satisfied. If $e_{n_0}(x)=e_{n_0}(y)=1$ (mod 2), (4.4) is also satisfied in the same way, which implies $r(x)=r(y)$.

On the other hand, suppose $r(x)=r(y)$ for $x, y \in E_2^{(\omega)}$. If $x=y$, then $q(x)=q(y)$. So suppose there exists $n_0 \in \boldsymbol{N}$ such that $x_j=y_j$ for $j \leqq n_0-1$ and $x_{n_0} \neq y_{n_0}$. Then $a_j(x)=a_j(y)$ for $j \leqq n_0-1$, $e_{n_0}(x)=e_{n_0}(y)$ and $a_{n_0}(x) \neq a_{n_0}(y)$. It is enough to consider the case $a_{n_0}(x)=0$ and $a_{n_0}(y)=1$. Then $r(x)=r(y)$ implies $a_j(x)=1$ and $a_j(y)=0$ for $j \geqq n_0+1$ by (4.4). $a_{n_0}(x)=1/2+(-1)^{e_{n_0}(x)}(x_{n_0}-3/2)=0$ implies $e_{n_0}(x) \neq x_{n_0}$ (mod 2). Hence $e_{n_0+1}(x)=u_2^{x_{n_0}}$ (mod 2) and so $x_{n_0+1}=u_2^{x_{n_0}}$ by $a_{n_0+1}(x)=1$. In the same way, we get $e_{n_0+j}(x)=1-u_2^{x_{n_0+j-1}}$ (mod 2) and $x_{n_0+j} \neq u_2^{x_{n_0+j-1}}$ for $j \geqq 2$. Hence by using (4.5), we get $x_{n_0+j}=u_{j+1}^{x_{n_0}}$ by induction and we have $x=P_{n_0-1}x \oplus u^{x_{n_0}}$. In the same way, we get $y=P_{n_0-1}x \oplus u^{y_{n_0}}$, which implies $q(x)=q(y)$.

So we have proved that $q(x)=q(y)$ is equivalent to $r(x)=r(y)$. Since $(E_2^{(\omega)}/_{\tilde{u}}, \tilde{U})$ is compact and the unit interval $[0,1]$ is a Hausdorff space and $r: E_2^{(\omega)} \to [0,1]$ is a continuous, onto mapping, $(E_2^{(\omega)}/_{\tilde{u}}, \tilde{U})$ is homeomorphic to $[0,1]$ by Proposition 3.4. □

REMARK. It is easily seen that the condition (2) in Theorem 4.5 is equivalent to the following (3).

(3) (u^1, u^2) is one of the followings (i)~(iv)

(i) $u^1=(1)\oplus \sum_{i=1}^{\infty} \oplus(2) \qquad u^2=(2)\oplus \sum_{i=1}^{\infty} \oplus(1)$

(ii) $u^1=(12)\oplus \sum_{i=1}^{\infty} \oplus(1) \qquad u^2=(22)\oplus \sum_{i=1}^{\infty} \oplus(1)$

(iii) $u^1=(11)\oplus \sum_{i=1}^{\infty} \oplus(2) \qquad u^2=(21)\oplus \sum_{i=1}^{\infty} \oplus(2)$

(iv) $u^1=(1)\oplus\sum_{i=1}^{\infty}\oplus(12)$ $\qquad u^2=(2)\oplus\sum_{i=1}^{\infty}\oplus(21)$

§ 5. The invariant set with respect to contraction maps on R^N.

Let f_j $(j=1,2)$ be an one-to-one, contraction map on $\boldsymbol{R}^N$, with Lipschitz constant $r_j\in(0,1)$ and Fix (f_j) be the uniquely determined fixed point of f_j. When a nonempty compact subset K of $\boldsymbol{R}^N$ satisfies $K=f_1(K)\cup f_2(K)$, we shall write $K=K(f_1,f_2)$, which is uniquely determined by the fixed point theorem. For $x_j\in\{1,2\}$ $(j=1,2,\cdots,n)$, we shall write

$$f_{x_1\cdots x_n}=f_{x_1}f_{x_2}\cdots f_{x_n}$$

and

$$K_{x_1\cdots x_n}=f_{x_1\cdots x_n}(K).$$

Then $\operatorname{diam}(K_{x_1\cdots x_n})\leqq r_{x_1}r_{x_2}\cdots r_{x_n}\operatorname{diam}(K)\to 0$ as $n\to\infty$.

For $x=(x_n)\in E_2^{(\omega)}$, we have

$$K\supset K_{x_1}\supset K_{x_1x_2}\supset\cdots\supset K_{x_1\cdots x_n}\supset\cdots$$

and the set $\bigcap_{n=1}^{\infty}K_{x_1\cdots x_n}$ consists of one point, say k_x. By [2, § 3.1], we have $K=\bigcup_{x\in E_2^{(\omega)}}\{k_x\}$.

Define $\varphi: E_2^{(\omega)}\to K$ by $\varphi(x)=k_x$. Then φ is continous and $f_j\circ\varphi(x)=\varphi((j)\oplus x)$.

PROPOSITION 5.1. *If* Fix $(f_1)\neq$ Fix (f_2) *and* $f_1(K)\cap f_2(K)\neq\emptyset$, *then there exists a* (*)-*pair* (u^1,u^2) *such that* $\varphi(u^1)=\varphi(u^2)$.

PROOF. Since $\varphi(E_2^{(\omega)})=K$ and $f_1(K)\cap f_2(K)\neq\emptyset$, there exist $x=(x_n)$ and $y=(y_n)\in E_2^{(\omega)}$ such that $\varphi(x)=\varphi(y)$ with $x_1=1$ and $y_1=2$. When there exists $j\geqq 1$ such that

$$(5.1)\qquad f_{x_1\cdots x_j}\varphi(x)=\varphi(x)\qquad [\text{resp. } f_{y_1\cdots y_j}\varphi(y)=\varphi(y)],$$

let j_1 [resp. j_2] be the smallest one and put $\alpha^1=(x_1\cdots x_{j_1})$ [resp. $\alpha^2=(y_1\cdots y_{j_2})$] and $u^1=\sum_{i=1}^{\infty}\oplus\alpha^1$ [resp. $u^2=\sum_{i=1}^{\infty}\oplus\alpha^2$]. If there does not exist $j\in\boldsymbol{N}$ satifying (5.1), let $u^1=x$ [resp. $u^2=y$]. Then $\varphi(x)=\varphi(u^1)$ and $\varphi(y)=\varphi(u^2)$ implies $\varphi(u^1)=\varphi(u^2)$ and u^1 and u^2 are mutualy prime. The relations $\varphi(\sum_{i=1}^{\infty}\oplus(j))=\operatorname{Fix}(f_j)$ $(j=1,2)$ and Fix $(f_1)\neq$ Fix (f_2) imply that either $j_1\geqq 2$ or $j_2\geqq 2$ is satisfied. So (u^1,u^2) is a (*)-pair. □

THEOREM 5.2. *Let f_1 and f_2 be one-to-one, contraction maps on* $\boldsymbol{R}^N$ *and let* Fix $(f_1)\neq$ Fix (f_2). *Then*

(1) *$K=K(f_1,f_2)$ is either connected or totally disconnected.*

(2) *If $f_1(K)\cap f_2(K)$ consists of one point $\{k_0\}$, then there exists a* (*)-

pair (u^1, u^2) *such that* $(E_2^{(\omega)}/_{\tilde{u}}, \tilde{U})$ *is homeomorphic to* K.

PROOF. (1) If $\varphi : E_2^{(\omega)} \to K$ is one-to-one, then K is totally disconnected since $E_2^{(\omega)}$ is totally disconnected and φ is a homeomorphism.

If φ is not one-to-one, there exists a (*)-pair (u^1, u^2) satisfying $\varphi(u^1)=\varphi(u^2)$ by Proposition 5.1. Since $(E_2^{(\omega)}/_{\tilde{u}}, \tilde{U})$ is connected by Theorem 3.6 and the mapping $\tilde{\varphi} : (E_2^{(\omega)}/_{\tilde{u}}, \tilde{U}) \to K$, defined by $\tilde{\varphi}(q(x))=\varphi(x)$, is continuous by Proposition 3.4, K is connected.

(2) If $f_1(K) \cap f_2(K)=\{k_0\}$, there exist a (*)-pair (u^1, u^2) and the mapping $\tilde{\varphi} : (E_2^{(\omega)}/_{\tilde{u}}, \tilde{U}) \to K$ in the same way as (1). It is enough to show that $\tilde{\varphi}$ is one-to-one.

a) For $z \in E_2^{(\omega)}$ we shall show that $q(z)=q(u^1)$ if $\varphi(z)=\varphi(u^1)$. If $z=u^1$, it is obvious. So suppose that there exists $n_1 \in \boldsymbol{N}$ such that $P_{n_1}z=P_{n_1}u^1$ and $z_{n_1+1} \neq u^1_{n_1+1}$. Since f_1 and f_2 are one-to-one and $f_{P_{n_1}z}\varphi(\sigma^{n_1}z)=f_{P_{n_1}z}\varphi(\sigma^{n_1}u^1)$, we have $\varphi(\sigma^{n_1}z)=\varphi(\sigma^{n_1}u^1)=k_0=\varphi(u^1)$. By the construction of u^1 in Proposition 5.1, we have $P_{n_1}u^1=\alpha^1 \oplus \cdots \oplus \alpha^1$. Hence $P_{n_1}z=\alpha^1 \oplus \cdots \oplus \alpha^1$, $(\sigma^{n_1}u^1)_1=1$ and $(\sigma^{n_1}z)_1=2$. If $\sigma^{n_1}z=u^2$, we have $z=\alpha^1 \oplus \cdots \oplus \alpha^1 \oplus u^2$, which implies $z \in Qu$ and $q(z)=q(u^1)$. If there exists $n_2 \in \boldsymbol{N}$ such that $P_{n_2}(\sigma^{n_1}z)=P_{n_2}(u^2)$ and $(\sigma_{n^1}z)_{n_2+1} \neq u^2_{n_1+1}$, we have $P_{n_2}(\sigma^{n_1}z)=P_{n_2}(u^2)=\alpha^2 \oplus \cdots \oplus \alpha^2$ by using $\varphi(\sigma^{n_1}z)=\varphi(u^2)$. By repeating this process, we have $z \in Qu$ and $q(z)=q(u^1)$.

b) For $z^1, z^2 \in E_2^{(\omega)}$ we shall show that $q(z^1)=q(z^2)$ if $\varphi(z^1)=\varphi(z^2)$. We may suppose that there exists $n_0 \geqq 0$ such that $P_{n_0}z^1=P_{n_0}z^2$ and $z^1_{n_0+1} \neq z^2_{n_0+1}$. Since f_1 and f_2 are one-to-one and $f_{P_{n_0}z^1}\varphi(\sigma^{n_0}z^1)=f_{P_{n_0}z^2}\varphi(\sigma^{n_0}z^2)$, we have $\varphi(\sigma^{n_0}z^1)=\varphi(\sigma^{n_0}z^2)$. Since $(\sigma^{n_0}z^1)_1 \neq (\sigma^{n_0}z^2)_1$, we have $\varphi(\sigma^{n_0}z^1)=\varphi(\sigma^{n_0}z^2)=k_0=\varphi(u^1)$. By using a), we get $q(\sigma^{n_0}z^1)=q(\sigma^{n_0}z^2)=q(u^1)$ and $\sigma^{n_0}z^1$, $\sigma^{n_0}z^2 \in Qu$, which implies $q(z^1)=q(z^2)$ and $\tilde{\varphi}$ is one-to-one. Hence $(E_2^{(\omega)}/_{\tilde{u}}, \tilde{U})$ is homeomorphic to K. □

By using Theorems 4.4, 4.5 and 5.2, we have the following.

THEOREM 5.3. *Let* f_1 *and* f_2 *be one-to-one, contraction maps on* $\boldsymbol{R}^N$. *If* $\mathrm{Fix}(f_1) \neq \mathrm{Fix}(f_2)$ *and* $f_1(K) \cap f_2(K)$ *is a singleton, then we have the following.*

(1) *The following* (a), (b) *and* (c) *are equivalent.*

(a) $K(f_1, f_2)$ *is a simple arc.*

(b) *There exists a* (*)*-pair* $(u^1, u^2) \in E_2^{(\omega)}$ *such that* $\varphi(u^1)=\varphi(u^2)$ *and* $J(u^1; n)=J(u^2; n)=\{n-1\}$ *holds for any* $n \geqq 2$.

(c) f_1 *and* f_2 *satisfy one of the following* (i)~(iv)

(i) $f_1(\mathrm{Fix}(f_2))=f_2(\mathrm{Fix}(f_1))$

(ii) $f_1f_2(\mathrm{Fix}(f_1))=f_2f_2(\mathrm{Fix}(f_1))$

(iii) $f_1f_1(\mathrm{Fix}(f_2))=f_2f_1(\mathrm{Fix}(f_2))$

(iv) $f_1(\mathrm{Fix}(f_1f_2))=f_2(\mathrm{Fix}(f_2f_1))$

(2) *$K(f_1, f_2)$ has always end points.*

Moreover if it is not a simple arc, then there are infinitely many end points in $K(f_1, f_2)$.

The assumption that $f_1(K)\cap f_2(K)$ is a singleton plays an important role in Theorem 5.3. Next we shall consider the condition for the set $f_1(K)\cap f_2(K)$ to be a singleton in case of maps on a complex plane $\boldsymbol{C}$ which is isomorphic to $\boldsymbol{R}^2$.

PROPOSITION 5.4. *Let f_1 and f_2 be one-to-one, contraction maps on a complex plane and let f_1 be expressed by $f_1(z)=\gamma\bar{z}+\omega$ with $\gamma,\omega\in\boldsymbol{C}$. Suppose $f_1(K)\cap f_2(K)=\{k_0\}$, $\mathrm{Fix}(f_1)\neq\mathrm{Fix}(f_2)$ and (u^1,u^2) is a $(*)$-pair satisfying $\varphi(u^1)=\varphi(u^2)$, where u^1 and u^2 are expressed with some x^0, $y^0\in E_2^{(\omega)}$ as follows:*

$$u^1=\underbrace{(11\cdots1)}_{m\ \mathrm{times}}\oplus(2)\oplus x^0,\qquad u^2=(2)\oplus y^0.$$

Then $m\leqq 2$.

PROOF. By assumption, it holds that $\varphi(u^1)=\varphi(u^2)=\{k_0\}$. Since $\tilde{\varphi}:(E_2^{(\omega)}/_{\tilde{u}},\tilde{U})\to K$ is a homeomorphism by Theorem 5.2, the set K is connected and locally connected by Theorem 3.6 and its Corollary. Since K is a complete metric set, K is arcwise connected [3, p. 36].

Let $x^1=\sum_{i=1}^{\infty}\oplus(1)$ and suppose $m\geqq 3$.

a) In case that either m is an even number or γ is a real number:

By the equation $f_1f_1(z)=|\gamma|^2z+\gamma\bar{\omega}+\omega$ and $f_1(\varphi(x_1))=\varphi(x_1)$, we have $\varphi((11)\oplus x)-\varphi(x^1)=|\gamma|^2(\varphi(x)-\varphi(x^1))$ for any $x\in E_2^{(\omega)}$. So $\varphi(x^1),\varphi((11)\oplus u^1)$, $\varphi(u^1),\varphi((112)\oplus x^0)$ and $\varphi((2)\oplus x^0)$ are on one line in this order since $m\geqq 3$. $\varphi((2)\oplus x^0)$ can be joined to $\varphi(u^1)=\varphi((2)\oplus y^0)$ by a curve C_2 in $f_2(K)$, since K is arcwise connected and $\varphi((2)\oplus x^0)$ and $\varphi(u^1)=\varphi((2)\oplus y^0)$ belong to $f_2(K)$. The curve $f_1f_1C_2$ in $f_1(K)$ connects $\varphi((112)\oplus x^0)$ and $\varphi((11)\oplus u^1)$. Since f_1f_1 is a contraction map with ratio $|\gamma|^2$ and it holds that $f_1f_1\varphi(x^1)=\varphi(x^1)$, we have $f_1f_1C_2\cap C_2\backslash\varphi(u^1)\neq\emptyset$, which is a contradiction, since $f_1(K)\cap f_2(K)=\varphi(u^1)$ holds by assumption.

b) In case that γ is not a real number and m is an odd number:

$\varphi(x^1),\varphi((1)\oplus u^1),\varphi((112)\oplus x^0)$ and $\varphi((2)\oplus x^0)$ are on one line in this order since $m\geqq 3$. $\varphi(x^1),\varphi((11)\oplus u^1),\varphi(u^1)$ and $\varphi((12)\oplus x^0)$ are on another line in this order. In the same way as a), we can join $\varphi((2)\oplus x^0)$ to $\varphi(u^1)=\varphi((2)\oplus y^0)$ by a curve C_2 in $f_2(K)$. The curve f_1C_2 [resp. $f_1f_1C_2$] joins $\varphi((12)\oplus x^0)$ to $\varphi((1)\oplus u^1)$ [resp. $\varphi((112)\oplus x^0)$ to $\varphi((11)\oplus u^1)$] in $f_1(K)$. Since f_1 [resp. f_1f_1] is a contraction map with ratio γ, [resp. $|\gamma|^2$] and $f_1\varphi(x^1)=\varphi(x^1)$ [resp. $f_1f_2\varphi(x^1)=\varphi(x^1)$], either $C_2\cap f_1C_2\backslash\varphi(u^1)$ or $C_2\cap f_1f_1C_2\backslash\varphi(u^1)$ is nonempty, which is a contradiction, since $f_1(K) \cap f_2(K) = \varphi(u^1)$ holds by assumption. Therefore $m \leqq 2$ holds. □

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