Natural Science Report, Ochanomizu University, Vol. 43, No. 1, 1992 11
お茶の水女子大学 自然科学報告 第43巻 第1号

Almost Complex Foliations and its Application to Contact Geometry

Sanae Morimoto

Department of Mathematics, Faculty of Science,
Ochanomizu University, Tokyo*

(Received April 23, 1992)

§1. Introduction

Let $\mathcal{F}$ be a foliation on a manifold M. By the transversal geometry of a foliation, we mean "the differential geometry" of the leaf space $M/\mathcal{F}$. It is the geometry infinitesimally modeled by the normal bundle Q of $\mathcal{F}$. In this paper, we shall study the transversal geometry corresponding to the geometry of almost complex manifolds. That is, we call a foliation $\mathcal{F}$ an *almost complex foliation* if it has a holonomy-invariant almost complex structure J_Q on Q.

The notion of almost Hermitian foliations, almost Kähler foliations, and Kähler foliations are introduced in §3. Our main results for almost complex foliations are the following:

THEOREM A. *Let $\mathcal{F}$ be an almost Hermitian foliation on M. For the differentiable complex vector bundle (Q, J_Q) we have*

$$C_{\lambda_1}(Q, J_Q) \cdots C_{\lambda_i}(Q, J_Q) = 0 \quad for\ 2(\lambda_1 + \cdots + \lambda_i) > q = \dim Q,$$

where $C_\lambda(Q, J_Q)$ denotes the λ-th Chern class of (Q, J_Q).

THEOREM B. *Let M be a compact orientable manifold and $\mathcal{F}$ be a harmonic, Einstein, almost Kähler foliation on M whose transversal scalar curvature is non-negative. Then $\mathcal{F}$ is a Kähler foliation.*

Theorem B is a foliation version of a Theorem of K. Sekigawa ([3]).

We can find examples of almost Kähler foliations and Kähler foliations in contact Riemannian manifolds. Let M be a contact Riemannian manifold with a contact form η and $\mathcal{F}$ be a foliation defined by integral curves of the characteristic vector field. In §4, we will view contact Riemannian manifolds from the point of the transversal geometry of $\mathcal{F}$. Applying

* Present Adress: 2845-256-A-131, Kuroishi Hirabari, Tenpaku-cho, Tenpaku-ku, Nagoya, JAPAN

Theorem B, we obtain

THEOREM C. *Let M be a $(2n+1)$-dimensional compact K-contact manifold. If the Ricci curvature ρ satisfies that for a constant $c \geqq -2$, $\rho = (2n-c)\eta \otimes \eta + cg$, then M is a Sasakian manifold.*

For basic knowledge of the transversal geometry of foliations, see Ph. Tondeur [4].

§ 2. Invariant connections and characteristic classes of the normal bundle

Let $\mathcal{F}$ be a foliation on a manifold M. It is given by an exact sequence of vector bundles

$$(2.1) \qquad 0 \longrightarrow L \longrightarrow TM \xrightarrow{\pi} Q \longrightarrow 0,$$

where L is the tangent bundle and Q the normal bundle of $\mathcal{F}$. A vector field $Y \in \Gamma TM$ is called an infinitesimal automorphism of $\mathcal{F}$ if $[X, Y] \in \Gamma L$ for any $X \in \Gamma L$. The action of the Lie algebra ΓL on ΓQ is defined by $\Theta(X)s = \pi[X, Y_s]$ for $X \in \Gamma L$, $s \in \Gamma Q$, where $Y_s \in \Gamma TM$ with $\pi(Y_s) = s$. A section $s \in \Gamma Q$ is called an *invariant section* if it satisfies $\Theta(X)s = 0$ for any $X \in \Gamma L$. We can consider an invariant section as a vector field on the leaf space $M/\mathcal{F}$ of the foliation. We denote by $V(\mathcal{F})$ and ΓQ^L the set of infinitesimal automorphisms of $\mathcal{F}$ and that of invariant sections of Q respectively. Then we have an exact sequence of Lie algebras:

$$0 \longrightarrow \Gamma L \longrightarrow V(\mathcal{F}) \xrightarrow{\pi} \Gamma Q^L \longrightarrow 0,$$

which is associated with (2.1) ([4] Chapter 9)

A differential form $\omega \in \Omega^r(M)$ is said to be *basic* if $i(X)\omega = 0$, $\Theta(X)\omega = 0$ for $X \in \Gamma L$. A basic form is considered as a differential form on the leaf space. The set $\Omega_B^*(\mathcal{F})$ of all basic forms constitutes a subcomplex of the de Rham complex $(\Omega^*(M), d)$. Its cohomology $H_B^*(\mathcal{F})$ is called the *basic cohomology* of $\mathcal{F}$. It plays the role of the de Rham cohomology of the leaf space ([4] Chapter 9).

In Q, a partial connection $\mathring{\nabla}$ along L is defined by

$$(2.2) \qquad \mathring{\nabla}_X s = \Theta(X)s = \pi[X, Y_s] \quad \text{for } X \in \Gamma L,\ s \in \Gamma Q.$$

It is known as the *Bott connection*. We consider a connection in Q which extends the Bott connection.

DEFINITION 2.1. A connection ∇ in Q is called an invariant connection if it satisfies the following conditions:

(1) it is the extension of the Bott connection, i.e.,

$$\nabla_X s = \mathring{\nabla}_X s \quad \text{for } X \in \Gamma L,\ s \in \Gamma Q,$$

(2) it is holonomy invariant, i.e.,

$$(\Theta(X)\nabla)_Y s = \Theta(X)(\nabla_Y s) - \nabla_{\Theta(X)Y} s - \nabla_Y(\Theta(X)s) = 0$$
$$\text{for } X \in \Gamma L,\ Y \in \Gamma TM,\ s \in \Gamma Q.$$

Let ∇ be an invariant connection in Q. If $Y \in V(\mathcal{F})$ and $s \in \Gamma Q^L$, then we have $\nabla_Y s \in \Gamma Q^L$. Therefore the connection may be seen as a connection of the leaf space. We denote by R_∇ the curvature tensor field of ∇ defined by

$$R_\nabla(X, Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s. \quad \text{for } X, Y \in \Gamma TM,\ s \in \Gamma Q.$$

Then R_∇ has the following property.

LEMMA 2.2. *R_∇ is an End(Q)-valued basic 2-form, i.e.,*

$$i(X)R_\nabla = 0, \quad \Theta(X)R_\nabla = 0 \quad \textit{for } X \in \Gamma L.$$

PROOF. For $X \in \Gamma L$, $Y, Z \in \Gamma TM$ and $s \in \Gamma Q$, we have

$$\begin{aligned} R_\nabla(X, Y)s &= \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s \\ &= \Theta(X)\nabla_Y s - \nabla_Y \Theta(X)s - \nabla_{\Theta(X)Y} s \\ &= (\Theta(X)\nabla)_Y s = 0 \end{aligned}$$

and

$$\begin{aligned} (\Theta(X)R_\nabla)(Y, Z)s &= \Theta(X)R_\nabla(Y, Z)s - R_\nabla(\Theta(X)Y, Z)s \\ &\quad - R_\nabla(Y, \Theta(X)Z)s - R_\nabla(Y, Z)\Theta(X)s \\ &= \Theta(X)\{\nabla_Y \nabla_Z s - \nabla_Z \nabla_Y s - \nabla_{[Y,Z]} s\} \\ &\quad - \{\nabla_{\Theta(X)Y} \nabla_Z s - \nabla_Z \nabla_{\Theta(X)Y} s - \nabla_{[\Theta(X)Y, Z]} s\} \\ &\quad - \{\nabla_Y \nabla_{\Theta(X)Z} s - \nabla_{\Theta(X)Z} \nabla_Y s - \nabla_{[Y, \Theta(X)Z]} s\} \\ &\quad - \{\nabla_Y \nabla_Z \Theta(X)s - \nabla_Z \nabla_Y \Theta(X)s - \nabla_{[Y,Z]} \Theta(X)s\} \\ &= \nabla_Y(\Theta(X)\nabla_Z s) - \nabla_Z(\Theta(X)\nabla_Y s) - \nabla_{\Theta(X)[Y,Z]} s \\ &\quad + \nabla_Z \nabla_{\Theta(X)Y} s + \nabla_{[\Theta(X)Y, Z]} s - \nabla_Y \nabla_{\Theta(X)Z} s + \nabla_{[Y, \Theta(X)Z]} s \\ &\quad - \nabla_Y \nabla_Z \Theta(X)s + \nabla_Z \nabla_Y \Theta(X)s \\ &= -\nabla_{[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]} s = 0 \end{aligned}$$

The above lemma means that R_∇ is the tensor field on the leaf space.

EXAMPLE. (Riemannian foliation).

Let $\mathcal{F}$ be Riemannian with a bundle-like metric g_M on M inducing the holonomy invariant metric g_Q on $Q \cong L^{\perp}$. The holonomy invariance means that $\Theta(X)g_Q=0$ for all $X \in \Gamma L$. We denote by ∇^M the Riemannian connection associated with g_M. Define a connection ∇ in Q by

$$\nabla_X s = \begin{cases} \overset{\circ}{\nabla}_X s. & \text{for } X \in \Gamma L. \\ \pi(\nabla_X^M Y_s). & \text{for } X \in \Gamma L^{\perp}. \end{cases}$$

where $Y_s \in \Gamma L^{\perp}$ with $\pi(Y_s)=s$. Then ∇ is the unique metric and torsion-free connection in Q ([4] Chapter 5). Moreover we can prove that this connection is holonomy invariant, using the identity:

$$2g_Q(\nabla_Y s, t) = Y g_Q(s, t) + Z_s g_Q(\pi(Y), t) - Z_t g_Q(\pi(Y), s)$$
$$+ g_Q(\pi[Y, Z_s], t) + g_Q(\pi[Z_t, Y], s) - g_Q(\pi[Z_s, Z_t], \pi(Y))$$
$$\text{for } Y \in \Gamma TM,\ s, t \in \Gamma Q,\ Z_s, Z_t \in \Gamma TM \text{ with } \pi(Z_s)=s,\ \pi(Z_t)=t$$

([4] Chapter 5 (5.13)). In particular ∇ is an invariant connection in Q.

Let $P_i(\nabla)$ be the i-th Pontrjagin form of Q defined by an invariant connection ∇. Then $P_i(\nabla)$ is basic. Suppose that both ∇^0 and ∇^1 are invariant. Then we have

PROPOSITION 2.3. *There exists a basic form φ such that $P_i(\nabla^1) - P_i(\nabla^0) = d\varphi$.*

PROOF. We put $\delta = \nabla^1 - \nabla^0$. Then δ can be considered as an *End*(Q)*-valued* 1-form on M. Moreover we have $i(X)\delta=0$, $\Theta(X)\delta=0$ for $X \in \Gamma L$. Namely δ is basic. We define connection ∇^t in Q by $\nabla^t = t\nabla^1 + (1-t)\nabla^0$ for $0 \leqq t \leqq 1$ and denote by R^t the curvature tensor of ∇^t. We easily see that ∇^t is an invariant connection and by Lemma 2.2 R^t is an *End*(Q)*-valued* basic 2-form.

Choosing a basis in Q_x at $x \in M$, we identify Q_x with $\boldsymbol{R}^q$. Then we have $R^t(X, Y) \in gl(\boldsymbol{R}^q)$, the set of $q \times q$-real matrices. Suppose P is a symmetric, $Ad_{GL_q(\boldsymbol{R})}$*-invariant,* k-linear form on $gl(\boldsymbol{R}^q)$;

$$P : gl(\boldsymbol{R}^q) \times \cdots \times gl(\boldsymbol{R}^q) \longrightarrow \boldsymbol{R}$$

We define an exterior $2k$-form $P(R^t)$ on M by

$$P(R^t)(X_1, \cdots, X_{2k}) = \sum \operatorname{sgn} \sigma P(R^t(X_{\sigma(1)}, X_{\sigma(2)}), \cdots, R^t(X_{\sigma(2k-1)}, X_{\sigma(2k)})) \tag{2.3}$$

where the summation is taken over all permutations σ of $(1, 2, \cdots, 2k)$ and $\operatorname{sgn} \sigma$ denotes the sign of the permutation σ. We note that by $Ad_{GL_q(\boldsymbol{R})}$*-invariance,* the right hand side of (2.3) is evaluated independently on the choice of bases. Let $A \in gl(\boldsymbol{R}^q)$ and consider the polynomial

$$\det\left(\lambda I-\frac{A}{2\pi}\right)=\sum_{k=0}^{q}\tilde{\sigma}_k(A)\lambda^{q-k}.$$

Let $\tilde{P}_k$ be a symmetric k-linear form such that $\tilde{P}_k(A,\cdots,A)=\tilde{\sigma}_k(A)$, $A\in gl(\boldsymbol{R}^q)$. Then $\tilde{P}_k$ is an $Ad_{GL_q(\boldsymbol{R})}$-*invariant* form. Then i-th Pontrjagin form $P_i(\nabla^t)$ is given by $\tilde{P}_{2i}(R^t)$ ([2] Chapter 12).

Given a symmetric, $Ad_{GL_q(\boldsymbol{R})}$-*invariant*, *k-linear* form P, we construct the exterior $(2k-1)$-form:

$$\varphi(X_1,\cdots,X_{2k-1})$$
$$=k\int_0^1\sum \operatorname{sgn}\sigma P(\delta(X_{\sigma(1)}),R^t(X_{\sigma(2)},X_{\sigma(3)}),\cdots,R^t(X_{\sigma(2k-2)},X_{\sigma(2k-1)}))dt.$$

Then it is known that $d\varphi=P(\nabla^1)-P(\nabla^0)$ ([2], Chapter 12). Since δ and R^t are *End(Q)-valued* basic forms, φ is a basic $(2k-1)$-form.

§3. Almost complex foliations

In this section, we study the transversal geometry of foliations corresponding to the geometry of almost complex manifolds.

We call $\mathcal{F}$ an *almost complex foliation* when it has a holonomy invariant almost complex structure J_Q on the normal bundle Q. The geometric meaning of the holonomy invariance is the following: Let U_α be a distinguished chart and $f_\alpha: U_\alpha\to\boldsymbol{R}^q$ $(q=2n)$ the associated submersion. Then we can define an almost complex structure J_α on $f_\alpha(U_\alpha)$ such that $f_{\alpha*}J_Q=J_\alpha f_{\alpha*}$. If $U_\alpha\cap U_\beta\neq\emptyset$, we have $\gamma_{\beta\alpha*}J_\alpha=J_\beta\gamma_{\beta\alpha*}$, where $\gamma_{\beta\alpha}$ is a local diffeomorphism of $\boldsymbol{R}^{2n}$ such that $f_\beta=\gamma_{\beta\alpha}f_\alpha$ in $U_\alpha\cap U_\beta$. Conversely, consider an atlas $\mathcal{U}=\{U_\alpha\}$ of distinguished charts and $f_\alpha: U_\alpha\to\boldsymbol{R}^{2n}$ submersions related by transition function $\gamma_{\beta\alpha}$ such that there exist almost complex structures J_α on $f_\alpha(U_\alpha)$ and they satisfy $\gamma_{\beta\alpha*}J_\alpha=J_\beta\gamma_{\beta\alpha*}$. Then $\mathcal{F}$ is an almost complex foliation.

$\mathcal{F}$ is called a *holomorphic foliation* if there exists an atlas $\mathcal{U}=\{U_\alpha\}$ of distinguished charts and $f_\alpha: U_\alpha\to\boldsymbol{C}^n$ submersions related by transition functions $\gamma_{\beta\alpha}$ such that $\gamma_{\beta\alpha}$ are holomorphic. Naturally, a holomorphic foliation has a holonomy invariant almost complex structure J_Q on Q and then is necessarily almost complex.

Given a holonomy invariant almost complex structure J_Q, we define a tensor field N_{J_Q} of type $(1,2)$ on the normal bundle Q by

$$N_{J_Q}(s,t)=-\pi[\sigma(s),\sigma(t)]+\pi[\sigma(J_Qs),\sigma(J_Qt)]$$
$$-J_Q\pi[\sigma(J_Qs),\sigma(t)]-J_Q\pi[\sigma(s),\sigma(J_Qt)]$$

for $s,t\in\Gamma Q$, where $\sigma: Q\to TM$ denotes a splitting of the sequence (2.1).

We remark that if J_Q is holonomy invariant, N_{J_Q} is defined independently on the choice of splittings. N_{J_Q} is holonomy invariant and the following is easily seen.

PROPOSITION 3.1. *An almost complex foliation $\mathcal{F}$ is holomorphic if and only if N_{J_Q} vanishes identically.*

An almost complex foliation (resp. a holomorphic foliation) is called an *almost Hermitian foliation* (resp. a *Hermitian foliation*) if it has a holonomy invariant metric g_Q on Q whic is invariant by J_Q, i.e., $g_Q(J_Q s, J_Q t) = g_Q(s, t)$ for $s, t \in \Gamma Q$. The *fundamental 2-form* Φ of an almost Hermitian foliation $\mathcal{F}$ is defined by

$$\Phi(X, Y) = g_Q(\pi X, J_Q \pi Y) \quad \text{for } X, Y \in \Gamma TM.$$

Then Φ is a basic 2-form of $\mathcal{F}$. An almost Hermitian foliation (resp. a Hermitian foliation) is called an *almost Kähler foliation* (resp. a *Kähler foliation*) if the fundamental 2-form Φ is closed. Similarly to the case of Kähler manifolds, the following is shown:

PROPOSITION 3.2. *An almost Hermitian foliation $\mathcal{F}$ with J_Q and g_Q is a Kähler foliation if and only if $\nabla J_Q = 0$ holds, where ∇ denotes the unique metric and torsion-free connection associated with g_Q.*

Applying the results of § 2, we will show a vanishing theorem for Chern classes of an almost Hermitian foliation. Let $\mathcal{F}$ be an almost Hermitian foliation on M with a holonomy invariant almost complex structure J_Q and a holonomy invariant Hermitian metric g_Q. We denote by ∇ the unique metric and torsion-free connection associated with g_Q. As it is shown in § 2, ∇ is an invariant connection. Since $\nabla_X J_Q = \Theta(X) J_Q = 0$ for $X \in \Gamma L$, ∇J_Q may be considered as a tensor field on Q and it is holonomy invariant. Define a new connection ∇' in Q by

$$\nabla'_X s = \nabla_X s - \frac{1}{2} J_Q (\nabla_X J_Q) s \quad \text{for } s \in \Gamma Q,\ X \in \Gamma TM.$$

Then the following facts are proved by straightforward calculation:

(i) J_Q is parallel with respect to ∇', i.e., $\nabla' J_Q = 0$

(ii) ∇' is an invariant connection.

We denote by $C_k(Q, J_Q)$ the k-th Chern class of the differentiable complex vector bundle (Q, J_Q) over M. We calculate Chern classes of (Q, J_Q), using the connection ∇' and by Lemma 2.2, we have

THEOREM A. *If $2(\lambda_1 + \cdots + \lambda_i) > q = \dim Q$, we have*

$$C_{\lambda_1}(Q, J_Q) \cdots C_{\lambda_i}(Q, J_Q) = 0 .$$

K. Sekigawa has shown a condition for an almost Kähler manifold to be a Kähler manifold. That is,

THEOREM ([3]). *Let $M=(M, J, \langle\ ,\ \rangle)$ be a compact Einstein almost Kähler manifold whose scalar curvature is non-negative. Then M is a Kähler manifold.*

Now we will prove an analogous theorem for an almost Kähler foliation.

We prepare some notion of Riemannian foliation. Let $\mathcal{F}$ be a Riemannian foliation on M with a holonomy invariant metric g_Q. We denote by ∇ the unique metric and torsion-free connection in Q. Its curvature tensor R_∇ can be considered as a tensor on Q and satisfies the same identities as Riemannian's one. Therefore we can define the Ricci curvature ρ_∇ and the scalar curvature τ_∇ similarly to the usual ones in Riemannian geometry. We call $\mathcal{F}$ an *Einstein foliation* if the transversal scalar curvature τ_∇ is constant and the Ricci curvature ρ_∇ satisfies $\rho_\nabla=(\tau_\nabla/q)g_Q$, $q=\dim Q$.

Let g_M be a bundle-like metric on M inducing g_Q on Q. If all leaves of $\mathcal{F}$ are minimal submanifolds of (M, g_M), $\mathcal{F}$ is said to be *harmonic* ([4] Chapter 6). Assume that the tangent bundle L of $\mathcal{F}$ is oriented. The *characteristic form* $\chi_\mathcal{F}$ of $\mathcal{F}$ is defined in the following fashion. It is a *p-form* ($p=\dim L$) on M, which for $Y_1, \cdots, Y_p \in \Gamma TM$ is given by

$$\chi_\mathcal{F}(Y_1, \cdots, Y_p) = \det (g_M(Y_i, E_j)_{ij}) ,$$

where $\langle E_1, \cdots, E_p \rangle$ is a local oriented orthonormal frame of L. A foliation $\mathcal{F}$ on a Riemannian manifold (M, g_M) is harmonic if and only if $d\chi_\mathcal{F}(Y, X_1, \cdots, X_p)=0$ for $Y \in \Gamma L^\perp, X_1, \cdots, X_p \in \Gamma L$ ([4] Chapter 6).

THEOREM B. *Let M be a compact orientable manifold and $\mathcal{F}$ be a harmonic, Einstein, almost Kähler foliation on M whose transversal scalar curvature is non-negative. Then $\mathcal{F}$ is a Kähler foliation.*

PROOF. Put $2n=\dim Q$. The tangent bundle L of $\mathcal{F}$ is orientable, since M and the normal bundle Q are both orientable. We fix the orientation of L as follows: for the Riemannian volume form μ of (M, g_M) and the fundamental 2-form Φ of $\mathcal{F}$, $\mu=(1/n!)\Phi^n \wedge \chi_\mathcal{F}$ holds.

We denote by ∇' the connection introduced in the proof of Theorem A. Let $P_1(\nabla)$ and $P_1(\nabla')$ be the first Pontrjagin forms corresponding to ∇ and ∇' respectively. Then by Proposition 2.3, there exists a basic 3-form φ such that $P_1(\nabla)-P_1(\nabla')=d\varphi$. Since $\mathcal{F}$ is harmonic, we have $d\chi_\mathcal{F}(Y, X_1, \cdots, X_p)=0$ for $Y \in \Gamma L^\perp, X_1, \cdots X_p \in \Gamma L$. On the other hand, $\varphi \wedge \Phi^{n-2}$ is a basic form. Therefore $\varphi\wedge\Phi^{n-2}\wedge d\chi_{\mathcal{F}}=0$ holds. Hence we have

$$\text{(B-1)}\qquad \int_M (P_1(\nabla)-P_1(\nabla'))\wedge\Phi^{n-2}\wedge\chi_{\mathcal{F}}=0\,.$$

In fact, the left hand side of (B-1)

$$=\int_M d\varphi\wedge\Phi^{n-2}\wedge\chi_{\mathcal{F}}$$

$$=\int_M \{d(\varphi\wedge\Phi^{n-2}\wedge\chi_{\mathcal{F}})+\varphi\wedge d\Phi^{n-2}\wedge\chi_{\mathcal{F}}+\varphi\wedge\Phi^{n-2}\wedge d\chi_{\mathcal{F}}\}$$

$$=0\,.$$

Now we will represent $(P_1(\nabla)-P_1(\nabla'))\wedge\Phi^{n-2}$ by the curvature tensor R_∇ and ∇J. For detailed calculation, see [3]. We denote the fiber metric by $\langle\,,\,\rangle$ instead of g_Q, if there is no confusion. We introduce a tensor field ρ^*_∇ of type $(0,2)$ defined by

$$\rho^*_\nabla(s,t)=\frac{1}{2}\text{ trace of } u\longrightarrow R_\nabla(s,Jt)Ju$$

$$\text{for } s,t,u\in Q_x \text{ at } x\in M\,.$$

Let $\{e_i\}$ be an orthonormal basis of Q_x at any point $x\in M$. From now on, we shall adopt the following notational convention:

$$R_{hijk}=\langle R_\nabla(e_h,e_i)e_j,e_k\rangle\,,$$

$$R_{\bar{h}ijk}=\langle R_\nabla(Je_h,e_i)e_j,e_k\rangle\,,$$

$$\cdots\cdots\cdots\cdots$$

$$R_{\bar{h}\bar{i}\bar{j}\bar{k}}=\langle R_\nabla(Je_h,Je_i)Je_j,Je_k\rangle\,,$$

$$\rho_{ij}=\rho_\nabla(e_i,e_j),\cdots,\rho_{\bar{i}\bar{j}}=\rho_\nabla(Je_i,Je_j)\,,$$

$$\rho^*_{ij}=\rho^*_\nabla(e_i,e_j),\cdots,\rho^*_{\bar{i}\bar{j}}=\rho_\nabla(Je_i,Je_j)\,,$$

$$J_{ij}=\langle Je_i,e_j\rangle\,,\quad \nabla_iJ_{jk}=\langle(\nabla_{e_i}J)e_j,e_k\rangle\,.$$

For a Riemannian foliation, we have the following identities:

$$R_{hijk}=-R_{ihjk}\,,$$

$$R_{hijk}=-R_{hikj}\,,$$

$$R_{hijk}+R_{ijhk}+R_{jhik}=0\,,$$

$$R_{hijk}=R_{jkhi}\,,$$

$$\nabla_hR_{ijkl}+\nabla_iR_{jhkl}+\nabla_jR_{hikl}=0\,.$$

By direct calculation, we get

$$\text{(B-2)}\qquad P_1(\nabla)\wedge\Phi^{n-2}=\frac{1}{32\pi^2n(n-1)}\{\sum R_{a\bar{a}ij}R_{b\bar{b}\bar{i}j}-2\sum R_{abij}R_{\bar{a}\bar{b}ij}\}\Phi^n$$

$$P_1(\nabla')\wedge\Phi^{n-2}=\frac{1}{32\pi^2 n(n-1)}\{\sum R'_{a\bar{a}ij}R'_{b\bar{b}ij}-2\sum R'_{abij}R'_{\bar{a}\bar{b}ij}\}\Phi^n,$$

where R' denotes the curvature tensor of ∇'. R' is related to R_∇ by

(B-3) $$R'(s,t)u=\frac{1}{2}(R_\nabla(s,t)u-JR_\nabla(s,t)Ju),$$

$$-\frac{1}{4}\{(\nabla_s J)(\nabla_t J)u-(\nabla_t J)(\nabla_s J)u\},\quad \text{for } s,t,u\in\Gamma Q.$$

By (B-2) and (B-3), we have

$$(P_1(\nabla)-P_1(\nabla'))\wedge\Phi^{n-2}=\frac{1}{32\pi^2 n(n-1)}(f_1-f_2+f_3-2f_4)\wedge\Phi^n,$$

where differentiable functions $f_1,\cdots,f_4$ are defined by

$$f_1=\sum R_{abij}(R_{\bar{a}\bar{b}ij}-R_{\bar{a}\bar{b}\bar{i}\bar{j}}),$$
$$f_2=\sum(\rho^*_{ji}-\rho^*_{ij})^2,$$
$$f_3=\sum R_{a\bar{a}ij}(\nabla_{\bar{b}}J_{ik})\nabla_b J_{jk},$$
$$f_4=\sum R_{abij}(\nabla_{\bar{b}}J_{ik})\nabla_{\bar{a}}J_{jk}.$$

The formula (B-1) implies

(B-4) $$\int_M(f_1-f_2+f_3-2f_4)\mu=0.$$

Now, we evaluate $f_3(x)$ at $x\in M$. Let σ be a symmetric bilinear form on Q_x defined by

$$\sigma(s,t)=\sum_{ij}\langle(\nabla_{e_i}J)e_j,s\rangle\langle(\nabla_{e_i}J)e_j,t\rangle.$$

Let λ_i be eigenvalues of σ and define $f(x)$ by $f(x)=\sum_{i,j}(\lambda_i-\lambda_j)^2$. Then we get

(B-5) $$f_3(x)=-\frac{\tau_\nabla}{n}\|\nabla J\|^2-\frac{1}{4n}f(x)-\frac{1}{2n}\|\nabla J\|^4.$$

Next, we will prove

(B-6) $$\int_M(f_1-2f_4)\mu=-\frac{1}{4}\int_M f_5\mu,$$

where $f_5=\sum\langle R(e_i\wedge e_j-Je_i\wedge Je_j),e_a\wedge e_b-Je_a\wedge Je_b\rangle^2$. In the above equation, R means the curvature operator which acts on $\wedge^2 Q_x$ as follows:

$$\langle R(s\wedge t),u\wedge v\rangle=-\langle R_\nabla(s,t)u,v\rangle,\quad \text{for } s,t,u,v\in Q_x.$$

We put $\xi\in\Gamma Q$ by

$$\xi(x)=\sum_a(\sum_{b,i,j,k}R_{abij}(\nabla_b J_{ik})J_{jk})e_a.$$

Then ξ is an invariant section. The transversal divergence $\mathrm{div}_B\xi$ of ξ is defined to be the unique scalar satisfying $\Theta(\xi)\nu=(\mathrm{div}_B\xi)\nu$, $\nu=\Phi^n/n!$ being the transversal volume form. The transversal divergence theorem ([4] Chapter 9 Theorem 9.25) then implies $\int_M (\mathrm{div}_B \xi)\mu = 0$. On the other hand, we have $f_1 - 2f_4 = -2\,\mathrm{div}_B \xi - (1/4) f_5$. Therefore (B-6) is proved.

By (B-4), (B-5), and (B-6), we have finally

$$\int_M \left(\frac{1}{4} f_5 + f_2\right)\mu = -\int_M \left(\frac{\tau_\nabla}{n}\|\nabla J\|^2 + \frac{1}{4n} f + \frac{1}{2n}\|\nabla J\|^4\right)\mu .$$

If τ_∇ is non-negative, then ∇J vanishes identically on M and by proposition 3.2, $\mathcal{F}$ is a Kähler folianion. This completes the proof of Theorem B.

4. The application to the contact geometry.

In this section, we shall find examples of almost Kähler foliations and Kähler foliations in contact Riemannian manifolds and apply Theorem B.

We begin with the review of contact Riemannian geometry. For basic properties of contact Riemannian manifolds, refer to Blair [1]. $(2n+1)$-dimensional manifold M is a *contact manifold* if it carries a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M and η is called a *contact form*. On a contact manifold, there is a unique vector field ξ such that

$$\eta(\xi) = 1 ,$$
$$i(\xi) d\eta = 0 .$$

It is called a *characteristic vector field*. The $2n$-dimensional subbundle D of TM is defined by

$$D = \bigcup_{x \in M} D_x , \quad D_x = \{X \in T_x M \mid \eta(X) = 0\} .$$

Then the tangent bundle TM has the direct sum decomposition:

$$TM = D + \boldsymbol{R}\xi .$$

We note that if $X \in \Gamma D$, then $\Theta(\xi) X \in \Gamma D$. It is well-known that there exist a Riemannian metric g and a (1, 1)-tensor field ϕ, which satisfy

$$g(\xi, X) = \eta(X)$$
$$\phi^2 X = -X + \eta(X)\xi$$
$$d\eta(X, Y) = g(X, \phi Y)$$
$$\text{for } X, Y \in \Gamma TM .$$

The quadruple (ϕ, ξ, η, g) is called a *contact metric structure* and a manifold M with such a structure is called a *contact Riemannian manifold*. The next properties follow from the above condition:

(a) The decomposition $TM = D + \boldsymbol{R} \cdot \xi$ is orthogonal.

(b) The integral curve of ξ is a geodesic of (M, g)

(c) $\phi\xi=0$, $\phi(D)\subset D$ and ϕ is an almost complex structure on D.

(d) The metric g restricted on D is invariant by ϕ,

i.e., $g(\phi X, \phi Y)=g(X, Y)$ for $X, Y\in D$

A contact Riemannian manifold is *K-contact* if the characteristic vector field ξ is a Killing vector field with respect to g, i.e., $\Theta(\xi)g=0$. This condition is equivalent to $\Theta(\xi)\phi=0$.

On a contact Riemannian manifold M, we define a tensor field N of type (1, 2) by

$$N(X, Y)=\phi^2[X, Y]+[\phi X, \phi Y]-\phi[\phi X, Y]-\phi[X, \phi Y]+2d\eta(X, Y)\xi$$

$$\text{for } X, Y\in \Gamma TM.$$

When this tensor field N vanishes on M identically, we say that the contact metric structure (ϕ, ξ, η, g) is a *Sasakian structure* and that the contact Riemannian manifold with such structure is a *Sasakian manifold*. It is known that if N vanishes identically, $\Theta(\xi)\phi=0$ and hence, in particular, a Sasakian manifold is K-contact ([1] Chapter 4). Many examples of K-contact and Sasakian manifolds are shown in [1].

The 1-dimensional foliation $\mathcal{F}$ is given by integral curves of ξ on a contact Riemannian manifold with a contact metric structure (ϕ, ξ, η, g). So we will view contact Riemannian manifolds from the point of the transversal geometry of $\mathcal{F}$. We identify D with the normal bundle Q of $\mathcal{F}$. Then ϕ and g induce an almost complex structure J_Q and a Hermitian metric g_Q on Q. The fundamental form Φ is given by $\Phi=d\eta$.

PROPOSITION 4.1. *Let M be a contact Riemannian manifold. Then the following holds;*

(1) *M is a K-contact manifold if and only if $\mathcal{F}$ is an almost Kähler foliation with respect to (J_Q, g_Q).*

(2) *M is a Sasakian manifold if and only if $\mathcal{F}$ is a Kähler foliation with respect to (J_Q, g_Q).*

PROOF. (1) We see that ξ is a Killing vector field if and only if g_Q is holonomy invariant, i.e., $\Theta(\xi)g_Q=0$. In fact, we have

$$(\Theta(\xi)g)(\xi, \xi)=0\,,$$

$$(\Theta(\xi)g)(\xi, X)=(\Theta(\xi)\eta)(X)=0\,,$$

$$(\Theta(\xi)g)(X, Y)=(\Theta(\xi)g_Q)(X, Y)\,,$$

$$\text{for } X, Y\in \Gamma D\,.$$

Therefore (1) is proved.

(2) If M is a Sasakian manifold, by (1) $\mathcal{F}$ is an almost Kähler foliation with respect to (J_Q, g_Q). Moreover we have $N_{J_Q}(X, Y)=N(X, Y)$ for $X, Y\in\Gamma D$. In fact,

$$\begin{aligned} N_{J_Q}(X, Y) &= -\pi[X, Y]+\pi[\phi X, \phi Y]-\phi\pi[\phi X, Y]-\phi\pi[X, \phi Y], \\ &= -[X, Y]-2d\eta(X, Y)\xi+[\phi X, \phi Y]+2d\eta(\phi X, \phi Y)\xi, \\ &\quad -\phi[\phi X, Y]-\phi[X, \phi Y], \\ &= \phi^2[X, Y]+[\phi X, \phi Y]-\phi[\phi X, Y]-\phi[X, \phi Y]+2d\eta(X, Y)\xi, \\ &= N(X, Y). \end{aligned}$$

Therefore $\mathcal{F}$ is a Kähler foliation.

Conversely, suppose that $\mathcal{F}$ is a Kähler foliation with respect to (J_Q, g_Q). By the above argument, we have $N(X, Y)=N_{J_Q}(X, Y)=0$ for $X, Y\in \Gamma D$. Since M is K-contact by (1), we get $N(\xi, X)=-\phi(\Theta(\xi)\phi)X=0$. Hence M is a Sasakian manifold.

Applying Theorem B to a K-contact manifold, we obtain theorem analogous to Sekigawa's one ([3]). Let M be a $(2n+1)$-dimensional K-contact manifold with a contact metric structure (ϕ, ξ, η, g). Then by Proposition 4.1 (1), $\mathcal{F}$ is an almost Kähler foliation with respect to (J_Q, g_Q). We denote by ∇^M and ∇ the Riemannian connection associated with g and the unique metric and torsion-free connection in the normal bundle Q respectively. Identifying D with Q, we obtain the following relation ([1] Chapter 4 §3).

$$\begin{aligned} &\nabla^M_X Y=\nabla_X Y+g(\phi X, Y)\xi, \\ &\nabla^M_X \xi=-\phi X, \\ &\nabla^M_\xi X=\nabla_\xi X-\phi X, \\ &\nabla^M_\xi \xi=0, \\ &\text{for } X, Y\in\Gamma D. \end{aligned}$$

The next equations follow from the above:

$$\begin{aligned} &R(X, \xi)\xi=X, \\ &g(R(X, Y)Z, W)=g(R_\nabla(X, Y)Z, W)-g(\phi Y, Z)g(\phi X, W), \\ &\qquad\qquad +g(\phi X, Z)g(\phi Y, W)+2g(\phi X, Y)g(\phi Z, W), \\ &\qquad\qquad \text{for } X, Y, Z, W\in\Gamma D, \end{aligned}$$

where R and R_∇ denote the curvature tensor of ∇^M and ∇ respectively. Let ρ and ρ_∇ denote the Ricci curvatures of R and R_∇ respectively. The above implies

$$\begin{aligned} &\rho(\xi, \xi)=2n, \\ &\rho(\xi, X)=0, \end{aligned}$$

$$\rho(X, Y)=\rho_{\nabla}(X, Y)-2g(X, Y),$$
$$\text{for } X, Y\in \Gamma D.$$

Applying Theorem B and Proposition 4.1, we obtain

THEOREM C. *Let M be a $(2n+1)$-dimensional compact K-contact manifold. If the Ricci curvature ρ satisfies that for a constant $c\geqq -2$, $\rho=(2n-c)\eta\otimes\eta+cg$, then M is a Sasakian manifold.*

References

[1] D.E. Blair: *Contact manifolds in Riemannian geometry,* Lect. Notes in Math. **509** (1976), Springer.
[2] S. Kobayashi and K. Nomizu: Foundations of Differential geometry, Interscience Publishers, 1969.
[3] K. Sekigawa: *On some compact Einstein almost Kähler manifolds,* J. Math. Soc. Japan. **39** (1987), 677-684.
[4] Ph. Tondeur: Foliations on Riemannian manifolds, Springer Universitext, 1988.