# Certain Vectors in a G. H. Manifold

**Kimiko Fujii**

Department of Mathematics, Faculty of Science,
Ochanomizu University, Tokyo*
(Received September 6, 1991)

## Introduction

Let $M(g,J)$ be a Hermitian manifold with complex dimension $n$, where $g$ is a Riemann metric, $J$ is a complex structure. $M$ is called a *locally conformal Kähler* (l. c. K.) manifold if its metric is conformally related to a Kähler metric in some neighborhood of every point of $M$. The main characterization of a l. c. K. manifold is that the equation

$$d\Omega=\omega\wedge\Omega$$

holds, where $\Omega$ is a fundamental 2-form and $\omega$ is a closed 1-form in $M$. Because $\omega$ is uniquely determined, we call it a *Lee form.* On the other hand, if $M$ is a l. c. K. manifold, then

$$\nabla_i J_j{}^k=\frac{1}{2}(\tilde{\omega}_j\delta_i{}^k-\omega_j J_i{}^k-g_{ij}\tilde{\omega}^k+J_{ij}\omega^k),$$

where $\nabla_i$ is the covariant differentiation with respect to the Christoffel symbols $\left\{{h \atop ij}\right\}$ of $g_{ij}$ and $\tilde{\omega}_i=J_i{}^j\omega_j$. Particularly we call $M$ a *Generalized Hopf* (G. H.) manifold if the Lee form is parallel. In this paper we prove that in a compact G. H. manifold a vector $v$ is covariant analytic iff $v$ is harmonic and orthogonal to $\omega$. Next we take a look at the following two facts. One is that in a G. H. manifold the foliation defined by $\omega=0$ is a Sasakian space with a contact form $\tilde{\omega}$ [3]. The other is that in a Sasakian space a transversal foliation defined by a contact form $\eta$ has a Kähler foliation structure [7]. Thus we think that certain vectors in a G. H. manifold have similar properties to contravariant analytic and killing vectors in a Kähler manifold. Then we define $V_1$-contravariant analytic and $V_1$-killing vectors. In addition we define a $V_1$-Einstein space. Thus we can get the second main theorem: In a compact $V_1$-Einstein G. H. manifold, a $V_1$-contravariant analytic vector is uniquely decomposed into the sum of a $V_1$-killing vector and a $V_1$-killing vector transformed by

* Present Adress: Industrial Computer Development Department, Information Systems Engeneering Laboratory, Toshiba Coorporation Fuchu Works 1, Fuchu-Shi Tokyo.

$J$. This theorem corresponds to the well-known theorem of Matsushima in a Kähler case.

I would like to express my sincere thanks to Prof. Y. Ogawa for his kind suggestions and many fruitful advices.

## 1. Notations

In this paper, we assume:

(1) $M^{2n}(g,J)$ is a G.H. manifold.

(2) $g(\ ,\ )$ is denoted by $\langle\ ,\ \rangle$.

(3) The Einstein's convention holds.

(4) $\tilde{v}_i = J_i{}^j v_j$.

LEMMA 1.1. *In a compact G.H. manifold if $v$ is harmonic and orthogonal to $\omega$ and $\tilde{\omega}$, then $\tilde{v}$ is harmonic, too.*

PROOF. We calculate

$$\nabla^i \nabla_i \tilde{v}_j - R_{ij}\tilde{v}^i = (1-n)\langle v, \omega\rangle \tilde{\omega}_j - \nabla_j \langle v, \omega\rangle - J_j{}^i \nabla_i \langle v, \tilde{\omega}\rangle = 0 . \tag{1.1}$$

Thus we can get Lemma 1.1. q.e.d.

## 2. A covariant analytic vector in a G.H. manifold

A *covariant analytic vector* $v=(v^i)$ in a Hermitian manifold is defined by

$$J_i{}^j \nabla_j v_k - J_k{}^j \nabla_i v_j - (\nabla_i J_k{}^j - \nabla_k J_i{}^j) v_j = 0 . \tag{2.1}$$

In a G.H. manifold (2.1) can be written as

$$\nabla_i v_j + J_i{}^k J_j{}^l \nabla_k v_l + \frac{1}{2}(\tilde{v}_i \tilde{\omega}_j + \tilde{\omega}_i \tilde{v}_j + v_i \omega_j + \omega_i v_j) - \langle v, \omega\rangle g_{ij} = 0 . \tag{2.2}$$

Contracting (2.2) with $g^{ij}$, we get

$$\nabla_i v^i = (n-1)\langle v, \omega\rangle . \tag{2.3}$$

Thus we get easily the following lemma.

LEMMA 2.1. *In a compact G.H. manifold, a covariant analytic vector $v=(v^i)$ satisfies the following properties.*

$$\int_M \tilde{\omega}^i \tilde{v}^j \nabla_i v_j dM = \frac{1}{2}\int_M \{\langle v, \omega\rangle^2 + \langle v, \tilde{\omega}\rangle^2 - \|v\|^2\} dM , \tag{2.4}$$

$$\int_M \langle v, \tilde{\omega}\rangle J^{ij} \nabla_i v_j dM = (n-1)\int_M \{\langle v, \tilde{\omega}\rangle^2 - \langle v, \omega\rangle^2\} dM . \tag{2.5}$$

THEOREM 2.2. *In a compact G. H. manifold, the necessary and sufficient conditions for a vector $v$ to be covariant analytic are that $v$ is harmonic and orthogonal to $\omega$. In this case $v$ is also orthogonal to $\tilde{\omega}$.*

PROOF. Putting the left hand side of (2.2) as $T_{ij}$, we compute $\int_M \|T\|^2 dM$.

$$\int_M \|T\|^2 dM = \int_M \{2\|\nabla v\|^2 + \|v\|^2 + 2J^{ik}J^{jl}(\nabla_k v_l)(\nabla_i v_j) + 2v^i\omega^j\nabla_i v_j$$
$$+ 2\tilde{v}^i\tilde{\omega}^j\nabla_i v_j + 2\omega^i v^j\nabla_i v_j + 2\tilde{\omega}^i\tilde{v}^j\nabla_i v_j + (2n-3)\langle v, \omega\rangle^2$$
$$-4\langle v, \omega\rangle\nabla_i v^i - \langle v, \tilde{\omega}\rangle^2\} dM. \tag{2.6}$$

Suppose $v$ is covariant analytic. Substituting (2.2)-(2.5) into (2.6), we get

$$\int_M \{\|\nabla v\|^2 - (n-1)\langle v, \omega\rangle^2 + R_{ij}v^i v^j\} dM = 0, \tag{2.7}$$

where $R_{ij}$ is the Ricci curvature of $g_{ij}$. On the other hand for any vector $v$, we have

$$\int_M \{(\nabla_i v_j)(\nabla^j v^i) - (\nabla_i v^i)^2 + R_{ij}v^i v^j\} dM = 0. \tag{2.8}$$

Comparing (2.7) with (2.8) we get

$$\int_M \left\{\frac{1}{2}(\nabla_i v_j - \nabla_j v_i)(\nabla^i v^j - \nabla^j v^i) + (n-1)(n-2)\langle v, \omega\rangle^2\right\} dM = 0, \tag{2.9}$$

from which we get $\nabla_i v_j - \nabla_j v_i = 0$, hence $v$ is closed. Moreover we compute respectively

$$\int_M v^i\omega^j\nabla_i v_j dM = \int_M \omega^i v^j \nabla_i v_j dM, \tag{2.10}$$

$$\int_M \tilde{v}^i\tilde{\omega}^j\nabla_i v_j dM = \int_M \tilde{\omega}^i\tilde{v}^j\nabla_i v_j dM, \tag{2.11}$$

from which we get $v$ is orthogonal to $\omega$ and $\tilde{\omega}$. From (2.3), we can see $\nabla_i v^i = 0$. Thus we conclude that $v$ is harmonic.

Conversely suppose $v$ is harmonic and orthogonal to $\omega$. Because $v$ is closed, we have

$$\int_M \tilde{v}^i\tilde{\omega}^j\nabla_i v_j dM = \int_M \tilde{\omega}^i\tilde{v}^j\nabla_i v_j dM$$
$$= \int_M \left\{\left(n - \frac{1}{2}\right)\langle v, \tilde{\omega}\rangle^2 + \frac{1}{2}\langle v, \omega\rangle^2 - \frac{1}{2}\|v\|^2\right\} dM. \tag{2.12}$$

On the other hand, the condition for a vector $v$ to be harmonic is given by

$$\int_M \{\|\nabla v\|^2 + R_{ij}v^i v^j\} dM = 0\,. \tag{2.13}$$

Substituting (2.12), (2.13), $\nabla_i v^i = 0$ and $\langle v, \omega\rangle = 0$ into the right hand side of (2.6), we have

$$\int_M \|T\|^2 dM = -2(n-1)(n-2)\int_M \langle v, \tilde{\omega}\rangle^2 dM\,, \tag{2.14}$$

which shows $T=0$, i.e. $v$ is a covariant analytic vector. q. e. d.

## 3. A $V_1$-contravariant analytic vector in a G. H. manifold.

A *$V_1$-contravariant analytic vector* $v=(v^i)$ is defined by

$$\mathcal{L}_v(J_i{}^j + \omega_i\tilde{\omega}^j - \tilde{\omega}_i\omega^j) = -(v_i - \langle v, \omega\rangle\omega_i - \langle v, \tilde{\omega}\rangle\tilde{\omega}_i)\tilde{\omega}^j\,. \tag{3.1}$$

This equation can be written as

$$\begin{aligned}\frac{1}{2}\tilde{\omega}_j\tilde{v}_l - \frac{1}{2}v_j\omega_l + \frac{1}{2}\tilde{v}_j\tilde{\omega}_l + \frac{a}{2}\omega_j\omega_l - a\tilde{\omega}_j\tilde{\omega}_l + \frac{b}{2}\omega_j\tilde{\omega}_l + b\tilde{\omega}_j\omega_l - J_j{}^iJ_l{}^k\nabla_i v_k + \nabla_j v_l \\ -\omega_j\tilde{\omega}^iJ_l{}^k\nabla_i v_k + \tilde{\omega}_j\omega^iJ_l{}^k\nabla_i v_k - (\nabla_j a)\omega_l - (\nabla_j b)\tilde{\omega}_l = 0\,,\end{aligned} \tag{3.2}$$

where $a=\langle v, \omega\rangle, b=\langle v, \tilde{\omega}\rangle$.

Now we investigate some properties of a $V_1$-contravariant analytic vector in a G. H. manifold.

PROPOSITION 3.1. *In a G. H. manifold $M$, a $V_1$-contravariant analytic vector $v=(v^i)$ has the following properties: there exist scalars $\alpha, \gamma$ in $M$ satisfying $\nabla_i\langle v, \omega\rangle = \alpha\omega_i$, $\nabla_i\langle v, \tilde{\omega}\rangle = \gamma\omega_i$ and*

$$\mathcal{L}_\omega(v^i - \langle v, \omega\rangle\omega^i - \langle v, \tilde{\omega}\rangle\tilde{\omega}^i) = 0\,,$$

$$(\Longleftrightarrow \omega^j\nabla_j v_l = \alpha\omega_l + \gamma\tilde{\omega}_l)\,, \tag{3.3}$$

$$\mathcal{L}_{\tilde{\omega}}(v^i - \langle v, \omega\rangle\omega^i - \langle v, \tilde{\omega}\rangle\tilde{\omega}^i) = 0\,,$$

$$\left(\Longleftrightarrow \tilde{\omega}^j\nabla_j v_l = -\frac{1}{2}\tilde{v}_l - \frac{1}{2}\langle v, \tilde{\omega}\rangle\omega_l + \frac{1}{2}\langle v, \omega\rangle\tilde{\omega}_l\right). \tag{3.4}$$

PROOF. Transvecting (3.2) with $\tilde{\omega}^l J_i{}^j, \omega^l J_i{}^j$ respectively, we get

$$da = \alpha\omega + \beta\tilde{\omega}, \qquad db = \gamma\omega + \delta\tilde{\omega}\,, \tag{3.5}$$

where $a=\langle v, \omega\rangle, b=\langle v, \tilde{\omega}\rangle, \alpha, \beta, \gamma, \delta$ are scalars in $M$. Differentiating (3.5), we find

$$0 = d\alpha\wedge\omega + d\beta\wedge\tilde{\omega} + \beta\wedge d\tilde{\omega}\,, \tag{3.6}$$

$$0 = d\gamma\wedge\omega + d\delta\wedge\tilde{\omega} + \delta\wedge d\tilde{\omega}\,. \tag{3.7}$$

Calculating the value for $u$, $\tilde{u}$ satisfying $\langle u, \omega\rangle = \langle u, \tilde{\omega}\rangle = 0$ on (3.6) and

(3.7), we get

$$0=\beta\|u\|^2, \qquad 0=\delta\|u\|^2, \tag{3.8}$$

which means $\beta=0$ and $\delta=0$. On the other hand transvecting (3.2) with $\omega^j$, $\tilde{\omega}^j$ respectively, we obtain Proposition 3.1. q. e. d.

Using Propositon 3.1, we can get the following proposition.

PROPOSITION 3.2. *In a G. H. manifold $M$, $v=(v^i)$ is a $V_1$-contravariant analytic vector iff the following properties hold: there are scalars $\alpha, \gamma$ in $M$ satisfying such that*

$$\nabla_i a=\alpha\omega_i, \qquad \nabla_i b=\gamma\omega_i, \tag{3.9}$$

$$\begin{aligned}\nabla_i v_j-J_i{}^kJ_j{}^l\nabla_k v_l+\frac{1}{2}(\tilde{\omega}_i\tilde{v}_j-\omega_i v_j+\tilde{v}_i\tilde{\omega}_j-v_i\omega_j)+(a-\alpha)(\omega_i\omega_j-\tilde{\omega}_i\tilde{\omega}_j)\\+(b-\gamma)(\omega_i\tilde{\omega}_j-\tilde{\omega}_i\omega_j)=0,\end{aligned} \tag{3.10}$$

$$\mathcal{L}_{\tilde{\omega}^j}(v^i-a\omega^i-b\tilde{\omega}^i)=0, \tag{3.11}$$

*where $a=\langle v,\omega\rangle$, $b=\langle v,\tilde{\omega}\rangle$.*

We can get Theorem 3.3 and Corollary 3.4 from Proposition 3.2 easily.

THEOREM 3.3. *In a G. H. manifold, when $v$ is a $V_1$-contravariant analytic vector, $\tilde{v}$ is $V_1$-contravariant analytic, too.*

COROLLARY 3.4. *In a G. H. manifold, for a scalar $a$ (resp. $b$) in $M$, $a\omega^i$ ($b\tilde{\omega}^i$) is $V_1$-contravariant analytic vector iff there is a scalar $\alpha(\gamma)$ in $M$ such that $\nabla_i a=\alpha\omega_i(\nabla_i b=\gamma\omega_i)$.*

THEOREM 3.5. *In a compact G. H. manifold the necessary and sufficient conditions for a vector $v=(v^i)$ with $\langle v,\omega\rangle=\langle v,\tilde{\omega}\rangle=0$ to be $V_1$-contravariant analytic are*

$$\nabla^i\nabla_i v_j+R_{ij}v^i+v_j=0, \tag{3.12}$$

$$\mathcal{L}_{\tilde{\omega}^j}v^i=0,\left(\Longleftrightarrow \tilde{\omega}^j\nabla_j v_l=\frac{1}{2}\tilde{v}_l\right). \tag{3.13}$$

PROOF. First suppose a vector $v$ with $\langle v,\omega\rangle=\langle v,\tilde{\omega}\rangle=0$ to be $V_1$-contravariant analytic. Applying the operator $\nabla^i$ to (3.10), we find (3.12) easily. Because we assume that $v$ satisfies $\langle v,\omega\rangle=\langle v,\tilde{\omega}\rangle=0$, we get (3.13) easily.

Conversely putting the left hand side of (3.10) as $P_{ij}$, we calculate $\int_M \|P\|^2 dM$ and have

$$\int_M \|P\|^2 dM = -2\int_M v^i\left\{\nabla^j\nabla_j v_i + R_{ij}v^j + v_i + nJ_i{}^l\left(\tilde{\omega}^k\nabla_k v_l + \frac{1}{2}\tilde{v}_l\right)\right\} dM, \quad (3.14)$$

which shows that if $v$ satisfies (3.12) and (3.13), then $v$ is $V_1$-contravariant analytic. q. e. d.

## 4. A $V_0$-killing vector and a $V_1$-killing vector in a G. H. manifold.

A *$V_0$-killing vector* $v=(v^i)$ is defined by

$$\mathcal{L}_v(g_{ij}-\omega_i\omega_j-\tilde{\omega}_i\tilde{\omega}_j)=0\,. \quad (4.1)$$

This equation can be written as

$$\nabla_i v_j + \nabla_j v_i - 2a\tilde{\omega}_i\tilde{\omega}_j + b(\tilde{\omega}_i\omega_j+\omega_i\tilde{\omega}_j) + \tilde{v}_i\tilde{\omega}_j + \tilde{v}_j\tilde{\omega}_i$$
$$-(\omega_i\nabla_j a+\omega_j\nabla_i a)-(\tilde{\omega}_i\nabla_j b+\tilde{\omega}_j\nabla_i b)=0\,, \quad (4.2)$$

where $a=\langle v,\omega\rangle, b=\langle v,\tilde{\omega}\rangle$. From (4.2) we get the following Corollary 4.1.

COROLLARY 4.1. *In a G. H. manifold $M$, for scalars $a, b$ in $M$, $a\omega^i$, $b\tilde{\omega}^i$ are $V_0$-killing vectors.*

Transvecting (4.2) with $\omega^j, \tilde{\omega}^j, (1/2)g^{ij}$ respectively, we find

$$\mathcal{L}_\omega(v^i-\langle v,\omega\rangle\omega^i-\langle v,\tilde{\omega}\rangle\tilde{\omega}^i)=0\,, \quad (4.3)$$

$$\mathcal{L}_{\tilde{\omega}}(v^i-\langle v,\omega\rangle\omega^i-\langle v,\tilde{\omega}\rangle\tilde{\omega}^i)=0\,, \quad (4.4)$$

$$\nabla_i v^i=\langle\nabla a,\omega\rangle+\langle\nabla b,\tilde{\omega}\rangle\,. \quad (4.5)$$

Particularly we call a vector $v=(v^i)$ *$V_1$-killing* when it satisfies $\nabla_i a=\langle\nabla a,\omega\rangle\omega_i$ and $\nabla_i b=\langle\nabla b,\omega\rangle\omega_i$, where $a=\langle v,\omega\rangle, b=\langle v,\tilde{\omega}\rangle$.

Now we investigate some relations between a $V_1$-killing vector and a $V_1$-contravariant analytic vector.

THEOREM 4.2. *In a compact G. H. manifold if $v$ is a $V_1$-killing vector, then $v$ is a $V_1$-contravariant analytic vector.*

PROOF. Suppose $v=(v^i)$ to be a $V_1$-killing vector. $v$ can be uniquely decomposed as

$$v=a\omega+b\tilde{\omega}+u\,, \quad (4.6)$$

where $a=\langle v,\omega\rangle, b=\langle v,\tilde{\omega}\rangle$. From Corollary 4.2, $a\omega$ and $b\tilde{\omega}$ are $V_0$-killing. Because these scalars $a, b$ satisfy

$$\nabla_i a=\langle\nabla a,\omega\rangle\omega_i, \qquad \nabla_i b=\langle\nabla b,\omega\rangle\omega_i\,, \quad (4.7)$$

and from Corollary 3.4, we can see that $a\omega$ and $b\tilde{\omega}$ are $V_1$-contravariant analytic. Thus it is enough to show that $u$ is $V_1$-contravariant analytic. We can easily see that $u$ is also $V_1$-killing because of the linearity of a $V_1$-killing vector. Hence $u$ satisfies

$$\nabla_i u_j + \nabla_j u_i + \tilde{u}_i \tilde{\omega}_j + \tilde{\omega}_i \tilde{u}_j = 0\,. \tag{4.8}$$

Applying the operator $\nabla^i$ to (4.8), we find

$$\nabla^i \nabla_i u_j + R_{ij} u^i + u_j + (J^{il} \nabla_i u_l)\tilde{\omega}_j = 0\,. \tag{4.9}$$

Transvecting (4.9) with $u^j$, we get

$$u^j\{\nabla^i \nabla_i u_j + R_{ij} u^i + u_j\} = 0 \tag{4.10}$$

On the other hand,

$$\tilde{\omega}^k \nabla_k u_l + \frac{1}{2}\tilde{u}_l = 0 \tag{4.11}$$

because $u$ is $V_1$-killing. From (4.10), (4.11) and (3.14), we can see that $u$ is $V_1$-contravariant analytic. Consequently $v$ is $V_1$-contravariant analytic.

q. e. d.

THEOREM 4.3. *In a compact G. H. manifold, let $v=(v^i)$ be a $V_1$-contravariant analytic vector. If $v$ satisfies*

$$\nabla_i v^i = \langle \nabla \langle v, \omega \rangle, \omega \rangle\,, \tag{4.12}$$

*then $v$ is $V_1$-killing.*

PROOF. Suppose $v$ to be a $V_1$-contravariant analytic vector. $v$ can be uniquely decomposed as

$$v = a\omega + b\tilde{\omega} + u\,, \tag{4.13}$$

where $a = \langle v, \omega \rangle, b = \langle v, \tilde{\omega} \rangle$. We apply the operator $\nabla_i$ to (4.13) and find

$$\nabla_i v^i = \langle \nabla a, \omega \rangle + \nabla_i u^i\,, \tag{4.14}$$

because

$$\nabla_i a = \langle \nabla a, \omega \rangle \omega_i, \qquad \nabla_i b = \langle \nabla b, \omega \rangle \omega_i\,. \tag{4.15}$$

From the assumption, we get $\nabla_i u^i = 0$. Putting $S_{ij} = \nabla_i u_j + \nabla_j u_i + \tilde{u}_i \tilde{\omega}_j + \tilde{u}_j \tilde{\omega}_i$, we compute $\int_M \|S\|^2 dM$. Using the Green's theorem and Theorem 3.5, we get

$$\int_M \|S\|^2 dM = -2\int_M \{u^i(\nabla^j \nabla_j u_i + R_{ij} u^j + u_i) + (\nabla_i u^i)^2\} dM = 0\,, \tag{4.16}$$

which shows $u^i$ is $V_1$-killing. On the other hand it is clear that both $a\omega^i$ and $b\tilde{\omega}^i$ are $V_1$-killing. Consequently $v$ is $V_1$-killing. q. e. d.

PROPOSITION 4.4. *In a G. H. manifold if $v$ is a $V_1$-contravariant analytic vector with $\langle v, \omega\rangle = \langle v, \tilde{\omega}\rangle = 0$ and $\tilde{v}$ is $V_1$-killing, then $v$ is closed.*

PROOF. From the definition of a $V_1$-killing vector, we get

$$\nabla_i \tilde{v}_j + \nabla_j \tilde{v}_i - (v_i \tilde{\omega}_j + v_j \tilde{\omega}_i) = 0 . \tag{4.17}$$

Using that $v$ is $V_1$-contravariant analytic, (4.17) can be written as

$$J_i{}^h (\nabla_j v_h - \nabla_h v_j) = 0 , \tag{4.18}$$

which shows $v$ is closed. q. e. d.

PROPOSITION 4.5. *In a compact G. H. manifold if $v$ is $V_1$-killing and closed with $\langle v, \omega\rangle = \langle v, \tilde{\omega}\rangle = 0$, then $v=0$.*

PROOF. Since $v$ is closed, (4.2) is written as

$$\nabla_i v_j = -\frac{1}{2}(\tilde{v}_i \tilde{\omega}_j + \tilde{v}_j \tilde{\omega}_i) . \tag{4.19}$$

Contracting (4.19) with $g^{ij}$, we get $\nabla_i v^i = 0$, which shows $v$ is harmonic. From Theorem 1.1 we can see $\tilde{v}$ is also harmonic. Then we get

$$\nabla_i \tilde{v}_j - \nabla_j \tilde{v}_i = 0 . \tag{4.20}$$

Substituting (4.19) into (4.20) we have

$$v_i \tilde{\omega}_j - v_j \tilde{\omega}_i = 0 , \tag{4.21}$$

which means $v=0$. q. e. d.

## 5. The relation between $V_1$-contravariant analytic and $V_1$-killing vectors in a G. H. manifold.

This section shows that a theorem similar to the theorem of Matsushima in a Kähler case holds. Now we define a *$V_1$-Einstein space* as a G. H. manifold satisfying

$$R_{ij} = \lambda g_{ij} + \mu \omega_i \omega_j + \nu \tilde{\omega}_i \tilde{\omega}_j , \tag{5.1}$$

where $\lambda, \mu, \nu$ are scalars in $M$.

THEOREM 5.1. *If a G. H. manifold $M^{2n}(g, J)$ is a $V_1$-Einstein space, then*

$$\lambda = \frac{2R-n+1}{4(n-1)}, \quad \mu = \frac{-2R+n-1}{4(n-1)}, \quad \nu = \frac{-2R+(2n-1)(n-1)}{4(n-1)}, \tag{5.2}$$

*where $R$ is a scalar curvature of $g_{ij}$. In the case of $n>2$, $\lambda, \mu, \nu$, and $R$ are constant.*

PROOF. From the Kähler property of $R^*$, we easily have

$$J_i{}^k J_j{}^l R_{kl} = R_{ij} + \frac{n-1}{2}\omega_i\omega_j - \frac{n-1}{2}\tilde{\omega}_i\tilde{\omega}_j, \quad R_{ij}\omega^i = 0\,. \tag{5.3}$$

Substituting (5.1) into (5.3), we get respectively

$$\mu - \nu - \frac{1}{2} + \frac{n}{2} = 0, \qquad \lambda + \mu = 0\,. \tag{5.4}$$

Contracting (5.1) with $g^{ij}$, we obtain

$$2n\lambda + \mu + \nu = R\,. \tag{5.5}$$

From (5.4) and (5.5), we get (5.2) easily. On the other hand, from the Bianchi identity, we have

$$\nabla_i R = 2\nabla_j R_i{}^j\,. \tag{5.6}$$

Substituting (5.1) and (5.2) into (5.6), we get

$$-(n-2)\nabla_i R = \omega_i\omega^j\nabla_j R + \tilde{\omega}_i\tilde{\omega}^j\nabla_j R\,. \tag{5.7}$$

Transvecting (5.7) with $\omega^i, \tilde{\omega}^i$, we get respectively

$$\omega^i\nabla_i R = 0, \qquad \tilde{\omega}^i\nabla_i R = 0\,. \tag{5.8}$$

Substituting (5.8) into (5.7) again, we get $(n-2)\nabla_i R = 0$, which shows that if $n > 2$, then $\lambda, \mu, \nu$ are constant. q. e. d.

THEOREM 5.2. *Let $g^*_{ij}$ be a locally conformal Kähler metric and $R^*_{ij}$ be the Ricci curvature of $g^*$. Then $(M, g)$ is an Einstein, iff $(M, g^*)$ is a $V_1$-Einstein and $\nu = 0$. In this case naturally $R^*_{ij} = 0$ holds.*

PROOF. Using the properties of locally conformality, we have

$$R^*_{ij} = R_{ij} + \frac{n-1}{2}\omega_i\omega_j - \frac{n-1}{2}g_{ij}\,, \tag{5.9}$$

$$R^* = e^{\sigma}\left(R - \frac{(n-1)(2n-1)}{2}\right), \tag{5.10}$$

where $\sigma$ is defined by $g^* = e^{-\sigma}g$. From (5.9) and (5.10), we obtain Theorem 5.2 easily. q. e. d.

Finally the second main theorem in this paper will be given.

THEOREM 5.3. *In a compact $V_1$-Einstein G. H. manifold, if $R \neq (1-n)/2$, then any $V_1$-contravariant analytic vector $v = (v^i)$ is uniquely decomposed as*

$$v = \langle v, \omega\rangle\omega + \langle v, \tilde{\omega}\rangle\tilde{\omega} + p + \tilde{q}\,, \tag{5.11}$$

*where $p, q$ are $V_1$-killing vectors and $\langle p, \omega\rangle=\langle p, \tilde{\omega}\rangle=\langle q, \omega\rangle=\langle q, \tilde{\omega}\rangle=0$ hold.*

PROOF. $v$ can be decomposed uniquely and orthogonally as

$$v=\langle v, \omega\rangle\omega+\langle v, \tilde{\omega}\rangle\tilde{\omega}+u. \tag{5.12}$$

From Theorem 3.5 we obtain $\nabla_h\nabla_i u^i$ is $V_1$-contravariant analytic. If we put respectively

$$p^i=u^i+\frac{1}{2\lambda+1}\nabla^i\nabla_h u^h, \tag{5.13}$$

$$q^i=-\frac{1}{2\lambda+1}J_l{}^i\nabla^l\nabla_h u^h, \tag{5.14}$$

then naturally we have $u=p+\tilde{q}$ and $p^i, q^i$ are $V_1$-contravariant analytic. On the other hand, we can see easily

$$\nabla_h p^h=0, \qquad \nabla_h q^h=0, \tag{5.15}$$

$$\langle p, \omega\rangle=\langle p, \tilde{\omega}\rangle=\langle q, \omega\rangle=\langle q, \tilde{\omega}\rangle=0, \tag{5.16}$$

which show both $p^h$ and $q^h$ are $V_1$-killing from Theorem 4.3. Thus we find $v$ can be decomposed as (5.11). Now suppose that we have another decomposition:

$$u=p'+\tilde{q}', \tag{5.17}$$

from which we get

$$(p^i-p'^i)-J_j{}^i(q^j-q'^j)=0. \tag{5.18}$$

Applying the operator $\nabla_i$ to (5.18) and using that both $p^i$ and $p'^i$ are $V_1$-killing, we have

$$J^{ji}\nabla_i(q_j-q'_j)=0. \tag{5.19}$$

Putting $q_j-q'_j=X_j$, (5.19) can be written as $J^{ij}\nabla_i X_j=0$, from which we get

$$\nabla_i\tilde{X}^i=-\nabla_i(J_j{}^i X^j)=J^{ij}\nabla_i X_j=0. \tag{5.20}$$

On the other hand, using Proposition 4.4 and that $X^i$ is $V_1$-killing, we get that $\tilde{X}$ is closed. Thus $\tilde{X}$ is harmonic from Theorem 1.1, and we find that $X$ is harmonic, too. Because $X$ is clearly closed and $V_1$-killing, we have $X=0$ from Proposition 4.5. Thus $p=p', q=q'$. Consequently, the uniqueness of the decomposition is proved.

## 6. Remark

We take an almost l.c.K. manifold $M^{2n}(g, J)$. Then we can also consider another almost complex structure: $F_i{}^j=J_i{}^j+2\omega_i\tilde{\omega}^j-2\tilde{\omega}_i\omega^j$. Then we see the following properties:

(1) $F$ is an almost complex structure.

(2) $F$ is integrable iff $J$ is integrable.

(3) $M^{2n}(g,J)$ is a G. H. manifold iff $M^{2n}(g,F)$ is a G. H. manifold.

(4) If $u$ is the Lee form of $F$, then $u=-\omega$.

(5) If we we put $\bar{u}_i=F_i{}^j u_j$, then $\bar{u}=\tilde{\omega}$.

On the other hand from Theorem 2.2 we find that the property of a covariant analytic vector is independent of the choice of $J$ and $F$. Now we have the fact:

$$J_i{}^j+\omega_i\tilde{\omega}^j-\tilde{\omega}_i\omega^j=F_i{}^j+u_i\bar{u}^j-\bar{u}_iu^j\,.$$

From this point of view, it is natural to consider $V_1$-contravariant analytic vectors by (3.1) in this paper.

## References

[1] Vaisman, I.: "On locally conformal almost Kähler manifolds," Israel J. Math. **24** (1976), pp. 338–351.

[2] Vaisman, I.: "A theorem on locally conformal Kähler manifolds," Proc. Amer. Math. Soc. **75** (1979), pp. 279–283.

[3] Vaisman, I.: "Locally conformal Kähler manifold with parallel Lee form," Rend. Mat. Roma **62** (1979), pp. 263–284.

[4] Vaisman, I.: "Some curvature properties of locally conformal Kähler manifolds.," Trans. Amer. Math. Soc. **259** (1980), pp. 439–447.

[5] Vaisman, I.: "A geometric condition for an l.c. K. manifold to be Kähler," Geometric Dedicata, **10** (1981), pp. 129–134.

[6] Vaisman, I.: "Some curvature properties of complex surfaces," Ann. Mat. Pura Appl. (Series 4) **132** (1982), pp. 1–18.

[7] Matsumoto, K., Ianus, S. and Ornea, L.: "On certain vector fields in locally conformal Kähler manifolds.," Bull. Yamagata Univ. **11** (1987), pp. 335–344.

[8] Kashiwada, T.: "Some properties of locally conformal Kähler manifolds," Hokkaido Math. J. **8** (1979), pp. 191–198.

[9] Kashiwada, T.: "On V-Killing forms in a locally conformal Kähler manifolds with parallel Lee form," Ann. Mat. Pura Appl. **121** (1979), pp. 387–396.

[10] Kashiwada, T.: "On $\beta$-Einstein locally conformal Kähler Manifolds," Report of the Saitama Junior College, **1** (1990), pp. 57–64.

[11] Tricerri, F. and Vaisman, I.: "On some 2-dimensional Hermitian manifolds," Math. Z. **192** (1986), pp. 205–216.

[12] Tricerri, F.: "Some examples of locally Kähler manifolds," Rend. Sem. Mat. Politecn. Torino, **40** (1982), pp. 81–92.

[13] Goldberg, S.I. and Vaisman, I: "On compact locally conformal Kähler manifolds with non-negative sectional curvature," Ann. Fac. des Sc. Toulouse, math **II** (1980), pp. 117–123.

[14] Morimoto, S.: "Almost complex foliations and application to contact geometry," (preprint).

[15] Ogawa, Y.: "On C-killing forms in a compact Sasakian space," Tohoku Math. J. **19** (1967), pp. 467–484.

[16] Ogawa, Y.: "On contravariant C-analytic 1-forms in a compact Sasakian space," Tohoku Math. J. **120** (1968), pp. 333–345.

[17] Yano, K.: "Differential geometry on complex and almost complex space," Pergamon Press, 1965.

[18] Kobayoshi and Nomizu: "Foundations of differential geometry vol I, II," New York, 1963.