On the Isomonodromic Deformation Related to
the Sixth Painlevé Equation

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§ 1. Introduction

It was R. Fuchs who derived the sixth Painlevé equation

$$\frac{d^2\lambda}{dt^2} = \frac{1}{2}\left(\frac{1}{\lambda}+\frac{1}{\lambda-1}+\frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{\lambda-t}\right)\frac{d\lambda}{dt}$$
$$+ \frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left(c_1+c_2\frac{t}{\lambda^2}+c_3\frac{t-1}{(\lambda-1)^2}+c_4\frac{t(t-1)}{(\lambda-t)^2}\right)$$

by the isomonodromic deformation for the linear differential equation

$$\frac{d^2y}{dx^2} = \left(\frac{\alpha}{x}+\frac{\beta}{x-1}+\frac{\gamma}{x-t}+\frac{\delta}{x-\lambda}+\frac{a}{x^2}+\frac{b}{(x-1)^2}+\frac{c}{(x-t)^2}+\frac{3}{4(x-\lambda)^2}\right)y$$

which is of Fuchsian type.   Here a, b and c are complex constants and α, β, γ, λ and δ are functions of t satisfying

$$\alpha+\beta+\gamma+\delta=0 ,$$

$$\beta+t\gamma+\lambda\delta \equiv \text{constant} ,$$

and $x=\lambda$ is a regular singularity with exponents $-1/2$ and $3/2$ around which there exists a fundamental system of solutions without logarithmic term.

The other five Painlevé equations were obtained by R. Garnier as deformation equations for linear differential equations of the second order with irregular points. We restrict ourselves to the sixth Painlevé equation and its related equations.

K. Okamoto gave a new standpoint. First he converted the sixth Painlevé equation into a Hamiltonian system, which we call the sixth Painlevé system and denote by P_{VI} ([O1]), and then, normalizing the linear differential equation above, he derived the Painlevé system P_{VI} as deformation system for the normalized equation. ([O2], [O3])

The normalized linear differential equation is written as

$$\frac{d^2y}{dx^2}+\left(\frac{1-\kappa_0}{x}+\frac{1-\kappa_1}{x-1}+\frac{1-\theta}{x-t}-\frac{1}{x-\lambda}\right)\frac{dy}{dx}$$

$$+\left(\frac{\chi(\chi+\kappa_\infty)}{x(x-1)}-\frac{t(t-1)H}{x(x-1)(x-t)}+\frac{\lambda(\lambda-1)\mu}{x(x-1)(x-\lambda)}\right)y=0$$

and its Rieman scheme is given by

$$\left\{\begin{array}{ccccc} x=0 & x=1 & x=t & x=\lambda & x=\infty \\[2mm] 0 & 0 & 0 & 0 & \chi \\[2mm] \kappa_0 & \kappa_1 & \theta & 2 & \chi+\kappa_\infty \end{array}\right\}$$

where $\kappa_0+\kappa_1+\theta+\kappa_\infty+2\chi=1$ (Fuchs relation). The singularity $x=\lambda$ is supposed to be apparent. From this assumption, we have

$$H=\frac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)}\left(\mu^2-\left(\frac{\kappa_0}{\lambda}+\frac{\kappa_1}{\lambda-1}+\frac{\theta-1}{\lambda-t}\right)\mu+\frac{\chi(\chi+\kappa_\infty)}{\lambda(\lambda-1)}\right).$$

Okamoto proved that the Painlevé system P_{VI} is given by

$$P_{\mathrm{VI}}\qquad\left\{\begin{array}{l}\dfrac{d\lambda}{dt}=\dfrac{\partial H}{\partial\mu}\\[4mm]\dfrac{d\mu}{dt}=-\dfrac{\partial H}{\partial\lambda}\end{array}\right.$$

and P_{VI} is a deformation system for the normalized equation.

Considering the equation

$$\frac{d^2y}{dx^2}+\left(\frac{1-\kappa_0}{x}+\frac{1-\kappa_1}{x-1}+\frac{1-\theta}{x-t}-\frac{n}{x-\lambda}\right)\frac{dy}{dx}$$

$$+\left(\frac{\chi(\chi+\kappa_\infty)}{x(x-1)}-\frac{t(t-1)H}{x(x-1)(x-t)}+\frac{\lambda(\lambda-1)\mu}{x(x-1)(x-\lambda)}\right)y=0,$$

where $n=1,2,3,\cdots$, T. Kimura tried to obtain the deformation systems. He succeeded in the case $n=2$ and gave a conjecture saying that the equation above admits the deformation system for $n=1,2,3,\cdots$. ([K1], [K2])

The purpose of this paper is to prove that Kimura's conjecture is true for $n=3$ and to give the deformation system in an explicit form.

§2. Condition for $x=\lambda$ to be apparent

We consider the equation

$$(2.1)\qquad \frac{d^2y}{dx^2}+\left(\frac{1-\kappa_0}{x}+\frac{1-\kappa_1}{x-1}+\frac{1-\theta}{x-t}-\frac{3}{x-\lambda}\right)\frac{dy}{dx}$$

$$+\left(\frac{\chi(\chi+\kappa_\infty)}{x(x-1)}-\frac{t(t-1)H}{x(x-1)(x-t)}+\frac{\lambda(\lambda-1)\mu}{x(x-1)(x-\lambda)}\right)y=0,$$

where the Fuchs relation reads $\kappa_0+\kappa_1+\theta+\kappa_\infty+2\chi=-1$, and suppose that $x=\lambda$ is an apparent singularity. We shall search for a condition for $x=\lambda$ to be apparent, or we shall determine H as function of t, λ and μ. To apply the Frobenius method, we write (2.1) in the form

$$(x-\lambda)^2 y'' + (x-\lambda)p(x)y' + q(x)y = 0,$$

where

$$p(x) = -3 + \left(\frac{1-\kappa_0}{x} + \frac{1-\kappa_1}{x-1} + \frac{1-\theta}{x-t}\right)(x-\lambda),$$

$$q(x) = \mu(x-\lambda) + \left(-\left(\frac{t-1}{x} - \frac{t}{x-1} + \frac{1}{x-t}\right)H\right.$$

$$\left. + \left(\frac{\lambda-1}{x} - \frac{\lambda}{x-1}\right)\mu + \frac{\chi(\chi+\kappa_\infty)}{x(x-1)}\right)(x-\lambda)^2$$

Putting

$$P(x) = \frac{1-\kappa_0}{x} + \frac{1-\kappa_1}{x-1} + \frac{1-\theta}{x-t}.$$

$$Q(x) = \frac{t-1}{x} - \frac{t}{x-1} + \frac{1}{x-t} = \frac{t(t-1)}{x(x-1)(x-t)},$$

$$R(x) = \frac{\lambda-1}{x} - \frac{\lambda}{x-1},$$

$$S(x) = \frac{\chi(\chi+\kappa_\infty)}{x(x-1)},$$

we have

$$p(x) = -3 + P(\lambda)(x-\lambda) + P'(\lambda)(x-\lambda)^2 + \frac{P''(\lambda)}{2}(x-\lambda)^3 + \frac{P'''(\lambda)}{6}(x-\lambda)^4 + \cdots,$$

$$q(x) = \mu(x-\lambda) + (-Q(\lambda)H + R(\lambda)\mu + S(\lambda))(x-\lambda)^2$$

$$+ (-Q'(\lambda)H + R'(\lambda)\mu + S'(\lambda))(x-\lambda)^3$$

$$+ \frac{1}{2}(-Q''(\lambda)H + R''(\lambda)\mu + S''(\lambda))(x-\lambda)^4 + \cdots.$$

We define

$$f_0(\rho) = \rho(\rho-1) - 3\rho,$$

$$f_1(\rho) = P(\lambda)\rho + \mu,$$

$$f_2(\rho) = P'(\lambda)\rho + (-Q(\lambda)H + R(\lambda)\mu + S(\lambda)),$$

$$f_3(\rho) = \frac{1}{2}P''(\lambda)\rho + (-Q'(\lambda)H + R'(\lambda)\mu + S'(\lambda)),$$

$$f_4(\rho) = \frac{1}{6}P'''(\lambda)\rho + \frac{1}{2}(-Q''(\lambda)H + R''(\lambda)\mu + S''(\lambda)).$$

Then a condition $x = \lambda$ to be apparent is given by

$$\Delta = \begin{vmatrix} f_1(0) & f_0(1) & 0 & 0 \\ f_2(0) & f_1(1) & f_0(2) & 0 \\ f_3(0) & f_2(1) & f_1(2) & f_0(3) \\ f_4(0) & f_3(1) & f_2(2) & f_1(3) \end{vmatrix} = 0.$$

We see easily that Δ is a polynomial in H and μ with rational coefficients in λ and t and that the degrees with respect to H and μ are 2 and 4 respectively. Hence H is an algebraic function in t, λ and μ which we denote by $H(t, \lambda, \mu)$.

§ 3. Transformed linear differential equation

We transform the equation (2.1) into an equation of the form

(3.1) $$\frac{d^2 z}{dx^2} = A(x) z$$

by a change of variable $y = a(x) z$. We can write $A(x)$ as

$$A(x) = \frac{\alpha}{x} + \frac{\beta}{x-1} + \frac{\gamma}{x-t} - \frac{\nu}{x-\lambda} + \frac{a}{x^2} + \frac{b}{(x-1)^2} + \frac{c}{(x-t)^2} + \frac{15}{4(x-\lambda)^2},$$

where a, b and c are constants given by

$$a = \frac{\kappa_0^2 - 1}{4}, \qquad b = \frac{\kappa_1^2 - 1}{4}, \qquad c = \frac{\theta^2 - 1}{4}$$

and α, β, γ and ν are given by

$$\alpha = (t-1)H - (\lambda-1)\mu - \frac{(1-\kappa_0)(1-\kappa_1)}{2} - \frac{(1-\kappa_0)(1-\theta)}{2t} + \frac{3(1-\kappa_0)}{2\lambda} + \chi(\chi+\kappa_\infty),$$

$$\beta = -tH + \lambda\mu + \frac{(1-\kappa_0)(1-\kappa_1)}{2} - \frac{(1-\kappa_1)(1-\theta)}{2(t-1)} + \frac{3(1-\kappa_1)}{2(\lambda-1)} - \chi(\chi+\kappa_\infty),$$

(3.2) $$\gamma = H + \frac{(1-\kappa_0)(1-\theta)}{2t} + \frac{(1-\kappa_1)(1-\theta)}{2(t-1)} + \frac{3(1-\theta)}{2(\lambda-t)},$$

(3.3) $$\nu = \mu + \frac{3(1-\kappa_0)}{2\lambda} + \frac{3(1-\kappa_1)}{2(\lambda-1)} + \frac{3(1-\theta)}{2(\lambda-t)}.$$

We have

$$\alpha + \beta + \gamma - \nu = 0,$$

$$\beta + t\gamma - \lambda\nu = (\kappa_\infty^2 - \kappa_0^2 - \kappa_1^2 - \theta^2 - 13)/4.$$

Putting

$$d = (\kappa_\infty^2 - \kappa_0^2 - \kappa_1^2 - \theta^2 - 13)/4,$$

we obtain

(3.4) $$\alpha = (t-1)\gamma - (\lambda-1)\nu - d ,$$

(3.5) $$\beta = -t\gamma + \lambda\nu + d .$$

The following proposition will be used in § 5.

PROPOSITION 3.1. *The change of variables*

$$(\lambda, \mu, t, H) \longmapsto (\lambda, \nu, t, \gamma)$$

given by (3.2) *and* (3.3) *is a time dependent canonical transformation, namely*

$$d\nu \wedge d\lambda + dt \wedge d\gamma = d\mu \wedge d\lambda + dt \wedge dH$$

holds.

The proof is strightforward.

From the assumption that $x = \lambda$ is an apparent singularity of (2.1), the singularity $x = \lambda$ of (3.1) is such that there exists a fundamental system of solutions of (3.1) which are expressed as

$$(x-\lambda)^{-3/2} \sum_{n=0}^{\infty} a_n (x-\lambda)^n , \qquad (x-\lambda)^{5/2} \sum_{n=0}^{\infty} b_n (x-\lambda)^n .$$

Although a condition for that is obtained from $\Delta = 0$, we seek the condition immediately by applying the Frobenius method to (3.1). We write (3.1) as

$$(x-\lambda)^2 z'' = (x-\lambda)^2 A(x) z .$$

We put

$$F(x) = \frac{\alpha}{x} + \frac{\beta}{x-1} + \frac{\gamma}{x-t} + \frac{a}{x^2} + \frac{b}{(x-1)^2} + \frac{c}{(x-t)^2} .$$

Then we have

$$(x-\lambda)^2 A(x) = \frac{15}{4} - \nu(x-\lambda) + F(\lambda)(x-\lambda)^2 + F'(\lambda)(x-\lambda)^3 + \frac{F''(\lambda)}{2}(x-\lambda)^4 + \cdots .$$

It is easy to see that

$$\begin{vmatrix} \nu & -3 & 0 & 0 \\ -F(\lambda) & \nu & -4 & 0 \\ -F'(\lambda) & -F(\lambda) & \nu & -3 \\ \dfrac{-F''(\lambda)}{2} & -F'(\lambda) & -F(\lambda) & \nu \end{vmatrix} = 0$$

is a desired condition. We have from (3.4) and (3.5)

$$F^{(j)}(\lambda) = Q^{(j)}(\lambda)\gamma - R^{(j)}(\lambda)\nu - dT^{(j)}(\lambda) + U^{(j)}(\lambda) \qquad (j = 0, 1, 2, \cdots) ,$$

where

$$Q(x) = \frac{t(t-1)}{x(x-1)(x-t)}, \qquad R(x) = \frac{\lambda-1}{x} - \frac{\lambda}{x-1},$$

$$T(x) = \frac{1}{x} - \frac{1}{x-1} \qquad U(x) = \frac{a}{x^2} + \frac{b}{(x-1)^2} + \frac{c}{(x-t)^2}.$$

Therefore γ is determined as an algebraic function of t, λ and ν by the equation

(3.6) $\Xi(t, \lambda, \nu, \gamma) = 0$,

where

$$\Xi = 9F(\lambda)^2 - 10F(\lambda)\nu^2 - 24F'(\lambda)\nu - 18F''(\lambda) + \nu^4.$$

We denote this function by $\Gamma(t, \lambda, \nu)$. We want to obtain the derivatives $\partial\Gamma/\partial\lambda$, $\partial\Gamma/\partial\nu$ and $\partial\Gamma/\partial t$. For this purpose, it is sufficient to calculate the derivetives $\partial\Xi/\partial\gamma$, $\partial\Xi/\partial\lambda$, $\partial\Xi/\partial\nu$ and $\partial\Xi/\partial t$. Some calculations lead us to

$$\frac{\partial\Xi}{\partial\gamma} = 18F(\lambda)Q(\lambda) - 10Q(\lambda)\nu^2 - 24Q'(\lambda)\nu - 18Q''(\lambda),$$

$$\frac{\partial\Xi}{\partial\lambda} = 18F(\lambda)(F'(\lambda) - T(\lambda)\nu) - 10(F'(\lambda) - T(\lambda)\nu) - 24(F''(\lambda) - T'(\lambda)\nu)$$

$$- 18(F'''(\lambda) - T''(\lambda)\nu),$$

$$\frac{\partial\Xi}{\partial\nu} = -18F(\lambda)R(\lambda) + 10R(\lambda)\nu^2 - 20F(\lambda)\nu + 24R'(\lambda)\nu - 24F'(\lambda) + 18R''(\lambda) + 4\nu^3,$$

$$\frac{\partial\Xi}{\partial t} = 18F(\lambda)\left(\left(T(\lambda) + \frac{1}{(\lambda-t)^2}\right)\gamma + \frac{2c}{(\lambda-t)^3}\right) - 10\left(\left(T(\lambda) + \frac{1}{(\lambda-t)^2}\right)\gamma + \frac{2c}{(\lambda-t)^3}\right)\nu^2$$

$$+ 24\left(\left(T'(\lambda) - \frac{2}{(\lambda-t)^3}\right)\gamma - \frac{6c}{(\lambda-t)^4}\right)\nu - 18\left(\left(T''(\lambda) + \frac{6}{(\lambda-t)^4}\right)\gamma + \frac{24c}{(\lambda-t)^5}\right).$$

§4. Isomonodromic deformation for (3.1)

In this section we make the isomonodromic deformation for (3.1) as taking t as a deformation parameter. For this purpose it is sufficient to show that there exists a function $D(x, t, \lambda, \nu)$, rational in x and complex analytic in t, λ and ν, which satisfies the equation

$$\frac{1}{2}\frac{\partial^3 D}{\partial x^3} - 2A\frac{\partial D}{\partial x} - \frac{\partial A}{\partial x}D + \frac{\partial A}{\partial t} = 0.$$

Following Kimura's suggestion, we put

$$D = K(x-\lambda) + L + \frac{M_1}{x-\lambda} + \frac{M_2}{(x-\lambda)^2} + \frac{M_3}{(x-\lambda)^3},$$

where K, L, M_1, M_2 and M_3 are complex analytic in t, λ, ν. Inserting D and $A = -\nu(x-\lambda)^{-1} + 15(x-\lambda)^{-2}/4 + F(x)$ into this equation, we have

$$(4.1) \quad -\frac{3M_1}{(x-\lambda)^4} - \frac{12M_2}{(x-\lambda)^5} - \frac{30M_3}{(x-\lambda)^6}$$

$$-\left(-\frac{2\nu}{x-\lambda} + \frac{15}{2(x-\lambda)^2} + 2F(x)\right)\left(K - \frac{M_1}{(x-\lambda)^2} - \frac{2M_2}{(x-\lambda)^3} - \frac{3M_3}{(x-\lambda)^4}\right)$$

$$-\left(\frac{\nu}{(x-\lambda)^2} - \frac{15}{2(x-\lambda)^3} + F'(x)\right)\left(K(x-\lambda) + L + \frac{M_1}{x-\lambda} + \frac{M_2}{(x-\lambda)^2} + \frac{M_3}{(x-\lambda)^3}\right)$$

$$-\frac{\nu'}{x-\lambda} - \frac{\nu\lambda'}{(x-\lambda)^2} + \frac{15\lambda'}{2(x-\lambda)^3} + \frac{\alpha'}{x} + \frac{\beta'}{x-1} + \frac{\gamma'}{x-t} + \frac{\gamma}{(x-t)^2} + \frac{2c}{(x-t)^3} = 0 \ .$$

We expand the left-hand side into partial fraction and equate the coefficients to zero.

From the coefficients of $1/x^3$, $1/x^2$, $1/(x-1)^3$, $1/(x-1)^2$, $1/(x-t)^3$ and $1/(x-t)^2$, we have

$$(4.2) \quad -\lambda K + L - \frac{M_1}{\lambda} + \frac{M_2}{\lambda^2} - \frac{M_3}{\lambda^3} = 0 \ ,$$

$$(4.3) \quad -(\lambda-1)K + L - \frac{M_1}{\lambda-1} + \frac{M_2}{(\lambda-1)^2} - \frac{M_3}{(\lambda-1)^3} = 0 \ ,$$

$$(4.4) \quad -(\lambda-t)K + L - \frac{M_1}{\lambda-t} + \frac{M_2}{(\lambda-t)^2} - \frac{M_3}{(\lambda-t)^3} + 1 = 0 \ .$$

The coefficient of $1/(x-\lambda)^6$ is equal to zero. From the coefficients of $1/(x-\lambda)^5$ and $1/(x-\lambda)^4$ we have

$$(4.5) \quad 3M_2 - 2\nu M_3 = 0 \ ,$$

$$(4.6) \quad 12M_1 - 5\nu M_2 + 6F(\lambda)M_3 = 0 \ .$$

From the coefficients of $1/x$, $1/(x-1)$ and $1/(x-t)$ we have

$$(4.7) \quad \frac{d\alpha}{dt} = \alpha K - \left(\frac{\alpha}{\lambda^2} + \frac{2a}{\lambda^3}\right)M_1 + \left(\frac{2\alpha}{\lambda^3} + \frac{6a}{\lambda^4}\right)M_2 - \left(\frac{3\alpha}{\lambda^4} + \frac{12a}{\lambda^5}\right)M_3 \ ,$$

$$\frac{d\beta}{dt} = \beta K - \left(\frac{\beta}{(\lambda-1)^2} + \frac{2b}{(\lambda-1)^3}\right)M_1 + \left(\frac{2\beta}{(\lambda-1)^3} + \frac{6b}{(\lambda-1)^2}\right)M_2$$

$$-\left(\frac{3\beta}{(\lambda-1)^4} + \frac{12b}{(\lambda-1)^5}\right)M_3 \ ,$$

$$(4.9) \quad \frac{d\gamma}{dt} = \gamma K - \left(\frac{\gamma}{(\lambda-t)^2} + \frac{2c}{(\lambda-t)^3}\right)M_1 + \left(\frac{2\gamma}{(\lambda-t)^3} + \frac{6c}{(\lambda-t)^4}\right)M_2$$

$$-\left(\frac{3\gamma}{(\lambda-t)^4} + \frac{12c}{(\lambda-t)^5}\right)M_3 \ .$$

Finally, we have from the coefficients of $1/(x-\lambda)^3$, $1/(x-\lambda)^2$ and $1/(x-\lambda)$

(4.10) $$\frac{d\lambda}{dt} = -L + 2\nu M_1/5 - 8F(\lambda)M_2/15 - 2F'(\lambda)M_3/3 \,,$$

(4.11) $$\nu\frac{d\lambda}{dt} = -\nu L + 2F(\lambda)M_1 + 3F'(\lambda)M_2 + 2F''(\lambda)M_3 \,,$$

(4.12) $$\frac{d\nu}{dt} = \nu K + F'(\lambda)M_1 + F''(\lambda)M_2 + F'''(\lambda)M_3/2 \,.$$

The equalities (4.5) and (4.6) give us

(4.13) $$M_2 = 2\nu M_3/3 \,,$$

(4.14) $$M_1 = (5\nu^2 - 9F(\lambda))M_3/18 \,.$$

From (4.2) and (4.3) we have

$$K = -T(\lambda)M_1 - T'(\lambda)M_2 - T''(\lambda)M_3/2 \,,$$

$$L = -R(\lambda)M_1 - R'(\lambda)M_2 - R''(\lambda)M_3/2 \,,$$

from which we have, using (4.4),

$$Q(\lambda)M_1 + Q'(\lambda)M_2 + Q''(\lambda)M_3/2 = 1 \,.$$

The equalities (4.13) and (4.14) yield

(4.15) $$\frac{\partial \Xi}{\partial \gamma} M_3 = -36 \,.$$

We eliminate $d\lambda/dt$ from (4.10), (4.11). We have then

$$(2\nu^2/5 - 2F(\lambda))M_1 - (8F(\lambda)\nu/15 + 3F'(\lambda))M_2 - (2F''(\lambda) + 2F'(\lambda)\nu/3)M_3 = 0 \,,$$

which coincides with

$$\Xi(t, \lambda, \nu, \gamma)M_3/9 = 0 \,.$$

This means that (4.11) is derived from (4.10) and (3.6).

Consider the equation (4.10). We see that the right-hand side of (4.10) becomes

$$(R(\lambda) + 2\nu/5)M_1 + (R'(\lambda) - 8F(\lambda)/15)M_2 + (R''(\lambda)/2 - 2F'(\lambda)/3)M_3$$

which is equal to

$$\frac{\partial \Xi}{\partial \nu} M_3/36 \,.$$

Using (4.15) and $\partial\Gamma/\partial\nu = -(\partial\Xi/\partial\nu)/(\partial\Xi/\partial\gamma)$, we obtain

(4.16) $$\frac{d\lambda}{dt} = \frac{\partial \Gamma}{\partial \nu} \,.$$

In the same way we have

(4.17) $$\frac{d\nu}{dt} = -\frac{\partial \Gamma}{\partial \lambda} \cdot$$

We shall prove that (4.9) holds. By (4.16) and (4.17) we have

$$\frac{d\gamma}{dt} = \frac{\partial \Gamma}{\partial \lambda}\frac{d\lambda}{dt} + \frac{\partial \Gamma}{\partial \nu}\frac{d\nu}{dt} + \frac{\partial \Gamma}{\partial t} = \frac{\partial \Gamma}{\partial t} \cdot$$

It is sufficient, therefore, to show that

$$\frac{\partial \Gamma}{\partial t} = \gamma K - \left(\frac{\gamma}{(\lambda-t)^2} + \frac{2c}{(\lambda-t)^3}\right)M_1 + \left(\frac{2\gamma}{(\lambda-t)^3} + \frac{6c}{(\lambda-t)^4}\right)M_2$$

$$- \left(\frac{3\gamma}{(\lambda-t)^4} + \frac{12c}{(\lambda-t)^5}\right)M_3 \cdot$$

calculations similar to the above lead us to the conclusion.

Finally we check the equalities (4.7) and (4.8). We expand the left-hand side into a Taylor series at $x=\infty$, and equating the coefficients of of $1/x$ and $1/x^2$, we have

(4.18) $$\alpha' + \beta' + \gamma' - \nu' - (\alpha+\beta+\gamma-\nu)K = 0$$

(4.19) $$\beta' + t\gamma' + \gamma - \lambda'\nu - \lambda\nu' - (\alpha+\beta+\gamma-\nu)(\lambda K - L) = 0 \ .$$

This means that (4.7) and (4.8) can be replaced by (4.18) and (4.19). It is clear that (4.18) and (4.19) are immediate consequences of (3.4) and (3.5).

We arrive thus at the following theorem.

THEOREM 4.1. *An isomonodromic deformation system for* (3.1) *is given*

$$\begin{cases} \dfrac{d\lambda}{dt} = \dfrac{\partial \Gamma}{\partial \nu} \\[2mm] \dfrac{d\nu}{dt} = -\dfrac{\partial \Gamma}{\partial \lambda} \cdot \end{cases}$$

§ 5. Isomonodromic deformation for (2.1)

The proposition 3.1 and the theorem 4.1 yield the following theorem.

THEOREM 5.1. *An isomonodromic deformation for* (2.1) *is governed by*

$$\begin{cases} \dfrac{d\lambda}{dt} = \dfrac{\partial H}{\partial \mu} \\[2mm] \dfrac{d\mu}{dt} = -\dfrac{\partial H}{\partial \lambda} \cdot \end{cases}$$

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