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On Two Theorems of the Nina Bary Type

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Introduction. Unless explicitly stated otherwise, every set considered in this paper will be linear and every function will mean a mapping of the real line $\boldsymbol{R}$ into itself.

A function $\varphi(x)$ will be called AC superposable on a set E, if $\varphi(x)$ is expressible on E in the composite form $\varphi(x)=\theta\circ\psi(x)$, where the inner function $\psi(x)$ and the outer function θ are absolutely continuous on the set E and on the image $\psi[E]$, respectively.

There are two well-known interesting theorems, due to Nina Bary, on superposition of functions (see p. 288 of Saks [6]):

THEOREM (a). *In order that a function $\varphi(x)$ which is continuous on a closed interval I be expressible on this interval as a superposition of two functions of which the inner function $\psi(x)$ is, on I, both continuous and* BV, *while the outer function is* AC *on the set $\psi[I]$, it is necessary and sufficient that the function $\varphi(x)$ fulfil the condition* (T_1) *on I.*

THEOREM (b). *In order that a function which is continuous on a closed interval I be* AC *superposable on this interval, it is necessary and sufficient that the function fulfil on I both the conditions* (T_1) *and* (N), *or what amounts to the same, the condition* (S).

We are interested in investigating what will become of the above theorems if the underlying closed interval I is replaced by a general set. The answer to each of the two problems will be embodied respectively in Theorem 14 and Theorem 27, which will together constitute the main results of this paper. It will turn out that the conditions (T_1) and (S) of Banach, although certainly extensible to the case of a general underlying set, are not strong enough to serve the purpose. In order to tide over this situation, we shall find it appropriate to introduce two new conditions called (U) and (W).

The final section, which is independent of the main theorems, will supplement § 1 with a few results of their own interest.

§ 1. Conditions (F), (G), and (T_1).

DEFINITION. Let $\boldsymbol{N}$ denote the set of the positive integers. A function $\varphi(x)$ will be said to fulfil the *condition* (F) on a set E, if for each non-overlapping infinite sequence of closed intervals, $\langle I_1, I_2, \cdots\rangle$, the upper limit of the sequence $\langle\varphi[E\cap I_n];\ n\in\boldsymbol{N}\rangle$ is a null set.

DEFINITION. A function $\varphi(x)$ will be said to fulfil the *condition* (G) on a set E, if for each compact set C and each number $\varepsilon>0$ there exists an open set $D\supset C$ such that $|\varphi[E\cap(D\setminus C)]|<\varepsilon$.

In this definition we may clearly replace the existence of the set D by that of an elementary figure Z whose interior contains C and which fulfils the inequality $|\varphi[E\cap(Z\setminus C)]|<\varepsilon$.

In the present paper, use will be made here and there of a few fundamental properties of analytic sets. All of them are contained, for instance, in § 35 of Kuratowski [5]. As a rule, we shall employ them without quoting them from the literature.

THEOREM 1. *Let a function $\varphi(x)$ be continuous on an analytic set A, subject to the condition* (F) *on this set, and such that $|\varphi[A]|<+\infty$. Then the function necessarily fulfils the condition* (G) *on A.*

PROOF. With each number ξ we associate two infinite sequences of closed intervals, $\langle I_1, I_2, \cdots\rangle$ and $\langle K_1, K_2, \cdots\rangle$, which are defined by

$$I_{2n-1}=\left[\xi+\frac{1}{n+1}, \xi+\frac{1}{n}\right],\quad I_{2n}=\left[\xi-\frac{1}{n}, \xi-\frac{1}{n+1}\right],\quad K_n=\left[\xi-\frac{1}{n}, \xi+\frac{1}{n}\right].$$

The intervals I_n are non-overlapping, while $\varphi(x)$ fulfils the condition (F) on A. Hence the upper limit of the sequence $\langle\varphi[A\cap I_n];\ n\in\boldsymbol{N}\rangle$ is a null set. But this upper limit contains all, except perhaps one, of the points of the limit of the descending sequence $\langle\varphi[A\cap K_n];\ n\in\boldsymbol{N}\rangle$. On the other hand, the set $\varphi[A\cap K_n]$, which is a continuous image of an analytic set, is measurable for every n and we have $|\varphi[A\cap K_n]|\leqq|\varphi[A]|<+\infty$. Consequently $\lim|\varphi[A\cap K_n]|=0$. We shall write $K_n=K_n(\xi)$ in the sequel.

This being so, let C be any compact set. We shall prove that there exists for each $\varepsilon>0$ a figure Z containing C in its interior and satisfying the relation $|\varphi[A\cap(Z\setminus C)]|<\varepsilon$. As easily seen, C is expressible as the limit of a descending infinite sequence, say $Z_1\supset Z_2\supset\cdots$, of figures each of which contains C in its interior (see p. 8 of [2] for the proof). Now consider any open interval $U=(p, q)$ disjoint with C. From what we have just proved, it follows that

$$\lim_n |\varphi[A\cap K_n(p)]|=0 \quad \text{and} \quad \lim_n |\varphi[A\cap K_n(q)]|=0\,.$$

But if we keep an integer $m>0$ fixed, then for large n the set $U\cap Z_n$ is contained in the union $K_m(p)\cup K_m(q)$. Accordingly $\lim|\varphi[A\cap U\cap Z_n]|=0$. This implies the more general result that if Δ is the union of any finite number of open intervals none of which intersects the set C, then necessarily $\lim|\varphi[A\cap\Delta\cap Z_n]|=0$.

Choosing an open interval $I\supset Z_1$, let us write $D=I\setminus C$, so that D is an open set. If D has at most a finite number of component intervals, then the result obtained just now, in combination with the evident relation $D\cap Z_n=(I\setminus C)\cap Z_n=Z_n\setminus C$, shows that the figure $Z=Z_n$, where n is sufficiently large, conforms to the requirement. We may therefore assume, in the sequel, that the open set D has an infinity of component intervals.

Let us arrange all the components of D in a distinct infinite sequence $\langle J_1, J_2, \cdots\rangle$. If we replace here the open interval J_n by its closure for each n, we obtain a non-overlapping infinite sequence of closed intervals. But the function $\varphi(x)$ fulfils the condition (F) on A. Thus the sequence $\langle\varphi[A\cap J_n]\rangle$ has a null set for its upper limit. This fact implies the relation $\lim|\varphi[A\cap R_n]|=0$, where we write $R_n=J_{n+1}\cup J_{n+2}\cup\cdots$ for brevity. Hence there corresponds to each number $\delta>0$ an integer $k>0$ such that $|\varphi[A\cap R_k]|<\delta$. Keeping δ and k fixed and noting that

$$Z_n\setminus C=(I\setminus C)\cap Z_n=D\cap Z_n=(U_k\cap Z_n)\cup(R_k\cap Z_n)$$

for $n\in\boldsymbol{N}$, where $U_k=J_1\cup\cdots\cup J_k$, we find that, for every n,

$$A\cap(Z_n\setminus C)\subset(A\cap U_k\cap Z_n)\cup(A\cap R_k)\,,$$

$$|\varphi[A\cap(Z_n\setminus C)]|<|\varphi[A\cap U_k\cap Z_n]|+\delta\,.$$

But U_k is a bounded open set disjoint with C and having exactly k component intervals. Consequently, by that has already been proved, there is an index m such that $|\varphi[A\cap U_k\cap Z_m]|<\delta$. We thus obtain the appraisal $|\varphi[A\cap(Z_m\setminus C)]|<2\delta$, which completes the proof since δ is arbitrary.

REMARKS. (i) A scrutiny into the above proof shows that every function $\varphi(x)$ which is monotone (namely nondecreasing or non-increasing) on a set E and subject to the condition $|\varphi[E]|<+\infty$, fulfils the condition (G) on E. Such a function, although not always continuous on E, necessarily fulfils the condition (F) on E, as we find easily.

(ii) On the other hand, the hypothesis $|\varphi[A]|<+\infty$ is not superfluous for the validity of the theorem. To see this, consider the function $\varphi(x)$ which vanishes for $x=0$ and equals x^{-1} for $x\neq 0$. If A is the set of the real numbers $x\neq 0$, then $\varphi(x)$ is continuous on A. Further, $\varphi(x)$ fulfils the

condition (F) on A, as this function is biunique on A. In the case where $C=\{0\}$, however, $|\varphi[A\cap(D\setminus C)]|$ is infinite for any open set $D\supset C$, so that $\varphi(x)$ cannot fulfil the condition (G) on A.

THEOREM 2. *If $\varphi(x)$ is a function which is continuous on an analytic set A, then we have $|\varphi[A]|=\sup|\varphi[Q]|$, where Q is a generic compact set contained in A.*

PROOF. Let $\mathfrak{N}$ denote the set of all the irrational numbers of the unit interval $I=[0,1]$. We shall begin with the following particular case of the assertion: If $\psi(t)$ is a function which is, on the set $\mathfrak{N}$, both continuous and bounded, then for each number $\xi<|\psi[\mathfrak{N}]|$ the set $\mathfrak{N}$ contains a compact set C such that $|\psi[C]|\geqq\xi$.

To establish this proposition, it is convenient to preface the following. Consider any set $M\subset\mathfrak{N}$ and any rational number r of the interval I. Writing $M_n=M\setminus(r-n^{-1}, r+n^{-1})$ for each $n\in\boldsymbol{N}$, we find at once that $M_1\subset M_2\subset\cdots$ and that M is the limit of this ascending sequence. It follows that $\psi[M_1]\subset\psi[M_2]\subset\cdots$ and that this latter sequence converges to $\psi[M]$. Hence we have $|\psi[M]|=\lim|\psi[M_n]|$. This result will be used as a lemma in the sequel.

Let us arrange all the rational numbers of the interval I in a distinct infinite sequence $\langle r_1, r_2, \cdots\rangle$. Given any number $\xi<|\psi[\mathfrak{N}]|$, we can easily construct by induction an infinite sequence of open intervals, $\langle H_1, H_2, \cdots\rangle$, such that for every n we have both

$$r_n\in H_n \quad\text{and}\quad |\psi[\mathfrak{N}\setminus S_n]|>\xi\,, \qquad \text{where}\quad S_n=H_1\cup\cdots\cup H_n\,.$$

In fact, if the first k intervals $H_1, \cdots, H_k$ have already been determined so as to fulfil this condition for $n\leqq k$, then writing $r=r_{k+1}$ and $M=\mathfrak{N}\setminus S_k$ in the above lemma, we find the existence of an open interval H_{k+1} containing r_{k+1} and such that

$$|\psi[\mathfrak{N}\setminus S_{k+1}]|=|\psi[M\setminus H_{k+1}]|>\xi\,.$$

On the other hand, the choice of H_1 is similar and simpler.

Now $\mathfrak{N}\setminus S_n$ is a Borel set and hence $\psi[\mathfrak{N}\setminus S_n]$ must be a measurable set, for every n. But $\psi[\mathfrak{N}\setminus S_n]$ clearly descends for increasing n. We have moreover $|\psi[\mathfrak{N}\setminus S_n]|\leqq|\psi[\mathfrak{N}]|$ for every n, where $|\psi[\mathfrak{N}]|$ is finite since $\psi[\mathfrak{N}]$ is a bounded set. Thus

$$\lim_n|\psi[\mathfrak{N}\setminus S_n]|=|L|\,, \qquad \text{where}\quad L=\lim_n\psi[\mathfrak{N}\setminus S_n]\,.$$

It is obvious that $|L|\geqq\xi$.

Let us write $S=\lim S_n=H_1\cup H_2\cup\cdots$ and $C=\mathfrak{N}\setminus S$, so that the set S

is open and contains all the numbers r_n. Then $C=I\setminus S$, and consequently C is a compact set. We shall go on to show further that $\psi[C]$ coincides with the set L introduced just now. We need only verify the inclusion $L\subset\psi[C]$, the converse inclusion being evident. Let λ be any point of L. Then λ belongs to $\psi[\mathfrak{N}\setminus S_n]$ for every n, and so there is an infinite sequence of points, say $\langle t_1, t_2, \cdots\rangle$, such that

$$t_n\in\mathfrak{N}\setminus S_n \quad \text{and} \quad \lambda=\psi(t_n)$$

for every n. This sequence, which is bounded, must contain a convergent infinite subsequence, say $\langle u_1, u_2, \cdots\cdot\rangle$. Then there is for each n an index $m\geqq n$ such that $u_n=t_m$. It follows that

$$u_n\in\mathfrak{N}\setminus S_n\subset\mathfrak{N}\setminus S_n\subset I\setminus S_n \qquad \text{for each } n\,.$$

On the other hand, the sequence $\langle I\setminus S_n; n\in\boldsymbol{N}\rangle$ is descending and consists of compact sets. The set $\{u_n, u_{n+1}, \cdots\cdot\}$ is therefore contained in $I\setminus S_n$. Hence, if we write $t_0=\lim u_n$, we have $t_0\in I\setminus S_n$ for every n, so that

$$t_0\in\lim_n (I\setminus S_n)=I\setminus S=C=\mathfrak{N}\setminus S\,.$$

But the function $\psi(x)$ is continuous on $\mathfrak{N}$, and thus

$$\psi(t_0)=\lim_n \psi(u_n)=\lim_n \lambda=\lambda\,.$$

This implies the inclusion $L\subset\psi[C]$, whence $L=\psi[C]$.

We have thus shown that there exists for each $\xi<|\psi[\mathfrak{N}]|$ a compact set $C\subset\mathfrak{N}$ fulfilling $|\psi[C]|\geqq\xi$. However, this result still holds good without assuming the boundedness of the function $\psi(t)$ on the set $\mathfrak{N}$, as we shall now go on to establish.

Supposing merely that $\psi(t)$ is continuous on $\mathfrak{N}$, let $\psi_n(t)$ denote for each $n\in\boldsymbol{N}$ the function $[\psi(t)]_n$, where the symbol $[x]_n$ means $x, n, -n$ according as $|x|\leqq n$, $x>n$, $x<-n$ respectively. Since $[x]_n$ is a continuous function of x, the function $\psi_n(t)$ is continuous on $\mathfrak{N}$. On the other hand, writing E_n for the set of the points $t\in\mathfrak{N}$ at which $|\psi(t)|\leqq n$, we find at once that $E_1\subset E_2\subset\cdots\cdot$ and that $\mathfrak{N}$ is the limit of this ascending sequence. Then $|\psi[\mathfrak{N}]|=\lim|\psi[E_n]|$ and hence, if $\xi<|\psi[\mathfrak{N}]|$, there is a k such that $|\psi[E_k]|>\xi$. This, together with $|\psi_k[\mathfrak{N}]|\geqq|\psi_k[E_k]|=|\psi[E_k]|$, shows that $|\psi_k[\mathfrak{N}]|>\xi$. The function $\psi_k(t)$ being continuous on $\mathfrak{N}$ and satisfying $|\psi_k(t)|\leqq k$ for any t, we can apply to $\psi_k(t)$ what has already been proved. The set $\mathfrak{N}$ thus contains a compact set C such that $|\psi_k[C]|\geqq\xi$. But

$$|\psi_k[C]|\leqq|\psi_k[C\cap E_k]|+|\psi_k[C\setminus E_k]|\,,$$

where $|\psi_k[C\setminus E_k]|=0$ since $|\psi_k(t)|=k$ for $t\in\mathfrak{N}\setminus E_k$. Hence

$$|\psi_k[C]| \leqq |\psi_k[C \cap E_k]| = |\psi[C \cap E_k]| \leqq |\psi[C]| .$$

Combining the above results we conclude that $|\psi[C]| \geqq \xi$.

We are now in a position to deduce the theorem. Suppose given a function $\varphi(x)$ which is continuous on an analytic set A. Assuming A nonvoid as we may, we choose a function $f(t)$ which is continuous on $\mathfrak{N}$ and which maps $\mathfrak{N}$ onto A. Then the composite function $\psi(t) = \varphi \circ f(t)$ is continuous on $\mathfrak{N}$. Hence there exists for each $\xi < |\psi[\mathfrak{N}]|$ a compact set $C \subset \mathfrak{N}$ such that $|\psi[C]| \geqq \xi$. On the other hand, writing $K = f[C]$, we evidently have $\psi[C] = \varphi \circ f[C] = \varphi[K]$, so that $|\varphi[K]| \geqq \xi$. But the set K is compact, being a continuous image of a compact set. This completes the proof of the theorem, since $|\varphi[A]| = |\varphi \circ f[\mathfrak{N}]| = |\psi[\mathfrak{N}]|$.

REMARK. The theorem will cease to hold, if the set A is merely assumed to be measurable. This will incidentally be verified later on by a function constructed in the Example just after the proof of Theorem 35.

THEOREM 3. *If a function is continuous over a compact set Q, then this set necessarily contains a Borel set on which the function assumes exactly once each value that the function assumes on Q.*

This proposition is slightly preciser than Lemma (7.1) on p. 282 of Saks [6]; but the proof is word for word the same as in that lemma.

PREFATORY REMARKS. In § 5 of [4], the fluctuation $\Xi(\varphi; M)$ of a function $\varphi(x)$ on a set M was defined in the case in which M is a Borel set and $\varphi(x)$ is continuous on M. The definition can, however, be extended to the case where M is an analytic set, $\varphi(x)$ being still continuous on M. This will now be set forth briefly.

As in [4], let us denote by $\mathrm{N}(y; \varphi; E)$ the *multiplicity* (perhaps $+\infty$) of a real number y with respect to a function $\varphi(x)$ and a set E. Regarded as a function of y, this multiplicity will be called *multiplicity function* associated with $\varphi(x)$ and E.

A *partition* of a set is defined in the same way as in [4]. A partition will be called *analytic*, if all its constituents are analytic sets. We shall deal exclusively with countable partitions, without explicitly mentioning so. If $\varphi(x)$ is a function and $\mathfrak{S}$ a partition of a set E, the quantity $\Theta(y; \varphi; \mathfrak{S})$ is defined, as in [4], for each number y by the formula

$$\Theta(y; \varphi; \mathfrak{S}) = \sum_{X \in \mathfrak{S}} \mathrm{c}(y; \varphi[X]) .$$

Theorem 15 of [4] is now extensible to the following form, the proof remaining quite the same: *Given any function $\varphi(x)$ which is continuous*

on an analytic set A, let $\mathfrak{S}$ be a generic analytic partition of A. Then both $\Theta(y;\varphi;\mathfrak{S})$ and $\mathrm{N}(y;\varphi;A)$ are nonnegative measurable functions of y and we have

$$\sum_{X\in\mathfrak{S}}|\varphi[X]|\longrightarrow\int_{-\infty}^{+\infty}\mathrm{N}(y;\varphi;A)\,dy \qquad as\ \ \mathrm{d}(\mathfrak{S})\longrightarrow 0\,.$$

This integral will be written $\Xi(\varphi;A)$ and called *fluctuation* of the function $\varphi(x)$ over A.

There holds further the following extension of Theorem 16 of [4]. *If a function $\varphi(x)$ is continuous on an analytic set A and if $\mathfrak{S}$ is any analytic partition of A, we have the additivity relation*

$$\Xi(\varphi;A)=\sum_{X\in\mathfrak{S}}\Xi(\varphi;X)\,.$$

In the sequel we shall make free use of the above results, without being at the trouble of quoting them from these remarks.

The following simple fact will also be useful sometimes. *Given a function $\varphi(x)$ which is continuous on an analytic set A, let D be any open set and Q any closed set. Then both $A\cap\varphi^{-1}[D]$ and $A\cap\varphi^{-1}[Q]$ are analytic sets.* Indeed, the function $\varphi(x)$ being continuous on A, each point x of the set $L=A\cap\varphi^{-1}[D]$ is contained in an open interval $I(x)$ such that $A\cap I(x)\subset L$. Accordingly, denoting by U the union of all the $I(x)$, where x ranges over L, we obviously have $L=A\cap U$. Then L must be analytic as the intersection of two analytic sets. Again, if Q is a closed set and if we write $D=\boldsymbol{R}\setminus Q$, then D is open and so the set $L=A\cap\varphi^{-1}[D]$ is expressible in the form $L=A\cap U$, where U is an open set. It follows that

$$A\cap\varphi^{-1}[Q]=A\setminus L=A\setminus(A\cap U)=A\cap(\boldsymbol{R}\setminus U)\,,$$

which shows that $A\cap\varphi^{-1}[Q]$ is analytic. This terminates the prefatory remarks.

DEFINITION. Generalizing the condition (T_1) of Banach which refers to an interval (see p. 277 of Saks [6]), we shall say that a function fulfils the *condition* (T_1) on a set E, if almost every one of its values on E is assumed by the function at most a finite number of times on this set.

A function which is monotone on a set E, is a typical example of such a function. It is obvious that every function which filfils the condition (T_1) on a set, fulfils the condition (F) on this set. But we do not know if the converse of this assertion is true. The converse does hold, however, in the case where the underlying set is analytic and where the function is continuous on this set, as we shall now establish.

THEOREM 4. *Every function $\varphi(x)$ which is continuous on an analytic set A and subject to the condition* (F) *on this set, fulfils the condition* (T_1) *on A.*

PROOF. Supposing, if possible, that the function $\varphi(x)$ of the theorem does not fulfil the condition (T_1) on the set A, we shall derive a contradiction. Let Y be the set of all the values of $\varphi(x)$ each of which is assumed by $\varphi(x)$ infinitely often on A, namely the set of the numbers η such that $N(\eta;\varphi;A)=+\infty$. The set Y must be measurable, since the multiplicity function $N(y;\varphi;A)$ is measurable. But we have $|Y|>0$ by hypothesis. Consequently Y contains a compact set S of positive measure. The function $\varphi(x)$ being continuous on A, the points x of A such that $\varphi(x)\in S$, together form an analytic set, say M, and we evidently have $\varphi[M]=S$. Moreover, $\varphi(x)$ fulfils the condition (F) on M.

Let ε be any positive number $<3^{-1}|S|$, kept fixed in the sequel. We shall associate with each $n\in \boldsymbol{N}$ a compact set $C_n\subset M$ and a figure Z_n, so as to fulfil the following three conditions:

(1) The function $\varphi(x)$ is biunique on the set C_n and the image $\varphi[C_n]$ has measure $|\varphi[C_n]|>|S|-3\varepsilon>0$.

(2) The interior of the figure Z_n contains C_n and we have the inequality $|\varphi[M\cap(Z_n\setminus C_n)]|<2^{-n}\varepsilon$.

(3) The sequence $\langle Z_1, Z_2, \cdots\rangle$ is disjoint.

We shall proceed by induction. The function $\varphi(x)$ is continuous on M, which is an analytic set. Hence, by Theorem 2, there exists in M a compact set K such that $|\varphi[K]|>|S|-\varepsilon$. By Theorem 3, the set K contains a Borel set B_1 on which $\varphi(x)$ assumes each value $y\in\varphi[K]$ exactly once. Using Theorem 2 once more, we can further choose in B_1 a compact set C_1 such that $|\varphi[C_1]|>|\varphi[B_1]|-\varepsilon=|\varphi[C]|-\varepsilon>|S|-2\varepsilon$. The function $\varphi(x)$ is plainly biunique on C_1. Moreover, noting that the set $\varphi[M]=S$ is bounded, we find by Theorem 1 that $\varphi(x)$ fulfils the condition (G) on M. Consequently, there is a figure Z_1 whose interior contains C_1 and which fulfils the inequality $|\varphi[M\cap(Z_1\setminus C_1)]|<2^{-1}\varepsilon$. The choice of the set C_1 and the figure Z_1 is thus complete.

This being so, let $k\in\boldsymbol{N}$ and suppose that two sequences $\langle C_1, \cdots, C_k\rangle$ and $\langle Z_1, \cdots, Z_k\rangle$ have been constructed in such a manner that the three conditions (1), (2), and $C_n\subset M$ are satisfied for every $n=1, \cdots, k$ and that the latter sequence is disjoint. Writing now

$$C=C_1\cup\cdots\cup C_k,\quad Z=Z_1\cup\cdots\cup Z_k,\quad \text{and}\quad X=Z\setminus C$$

for brevity, we find immediately that

$$|\varphi[M\cap X]|\leqq\left|\bigcup_{n\leqq k}\varphi[M\cap(Z_n\setminus C_n)]\right|\leqq\sum_{n\leqq k}|\varphi[M\cap(Z_n\setminus C_n)]|<\varepsilon\,.$$

Further, each value y belonging to the set $S\setminus\varphi[M\cap X]$ is assumed by $\varphi(x)$ infinitely often on the set $M\setminus X$. On the other hand, such a value y is assumed by $\varphi(x)$ at most k times on C, and hence infinitely often on the set $(M\setminus X)\setminus C=M\setminus Z$. Consequently it follows that $\varphi[M\setminus Z]$ contains the set $S\setminus\varphi[M\cap X]$ and therefore that

$$|\varphi[M\setminus Z]|\geqq|S\setminus\varphi[M\cap X]|\geqq|S|-|\varphi[M\cap X]|>|S|-\varepsilon\,.$$

We now argue as in the above choice of C_1 and Z_1. The set $M\setminus Z$, which is analytic, contains a compact set Q such that

$$|\varphi[Q]|>|\varphi[M\setminus Z]|-\varepsilon>|S|-2\varepsilon\,.$$

The set Q then contains a Borel set B_{k+1} on which the function $\varphi(x)$ assumes each value $y\in\varphi[Q]$ exactly once. In this set B_{k+1} we can further choose a compact set C_{k+1} such that

$$|\varphi[C_{k+1}]|>|\varphi[B_{k+1}]|-\varepsilon=|\varphi[Q]|-\varepsilon>|S|-3\varepsilon\,.$$

The function $\varphi(x)$ is plainly biunique on the set C_{k+1}. Moreover, there is a figure Z_{k+1} whose interior contains C_{k+1} and which fulfils the inequality $|\varphi[M\cap(Z_{k+1}\setminus C_{k+1})]|<2^{-k-1}\varepsilon$. Since $C_{k+1}\subset Q\subset M\setminus Z$, we may require that Z_{k+1} is disjoint with the figure Z. This completes the inductive construction of the two sequences $\langle C_n;\, n\in\boldsymbol{N}\rangle$ and $\langle Z_n;\, n\in\boldsymbol{N}\rangle$.

Now the figure Z_n is nonvoid for every n, since

$$|\varphi[Z_n]|\geqq|\varphi[C_n]|>|S|-3\varepsilon>0\,.$$

Thus the set $Z_1\cup Z_2\cup\cdots$ is partitionable into a disjoint infinite sequence of closed intervals, say $\langle I_1, I_2, \cdots\rangle$. If we write $R_n=I_n\cup I_{n+1}\cup\cdots$ for short, then $\varphi[M\cap R_n]=\varphi[M\cap I_n]\cup\varphi[M\cap I_{n+1}]\cup\cdots$ for every n. Hence the limit of the descending sequence $\langle\varphi[M\cap R_n];\, n\in\boldsymbol{N}\rangle$ is the upper limit, say U, of the sequence $\langle\varphi[M\cap I_n];\, n\in\boldsymbol{N}\rangle$. But U must be a null set, since the function $\varphi(x)$ fulfils the condition (F) on M. Moreover, the sets $\varphi[M\cap R_n]$ are measurable and the set $\varphi[M]=S$ is bounded. We thus have

$$\lim_n|\varphi[M\cap R_n]|=|\lim_n\varphi[M\cap R_n]|=|U|=0\,.$$

On the other hand, there certainly corresponds to each n an index $p>0$ such that $Z_p\subset R_n$. It follows that

$$|\varphi[M\cap R_n]|\geqq|\varphi[M\cap Z_p]|\geqq|\varphi[M\cap C_p]|=|\varphi[C_p]|>|S|-3\varepsilon$$

for every n, whence we get $\lim|\varphi[M\cap R_n]|\geqq|S|-3\varepsilon>0$. This contradicts

what we stated above, and the theorem is thus established.

THEOREM 5. *Every function $\theta(t)$ which is absolutely continuous on a set W of finite measure, fulfils the condition* (T_1) *on this set and maps W onto a set of finite measure.*

PROOF. Since every function which is AC on a set is uniformly continuous on this set, we may assume without loss of generality that the function $\theta(t)$ is continuous on the closure S of the set W. It follows that $\theta(t)$ is AC on the whole set S. Plainly we need only consider the case in which S contains at least two points.

Let $L(t)$ be the linear modification of the function $\theta(t)$ with respect to the set S. This means that $L(t)=\theta(t)$ unless t belongs to an open interval contiguous to S, and further that $L(t)$ is linear on each closed interval contiguous to S. Consider any closed interval I pertaining to S (that is, with its end points belonging to S). Then $L(t)$ is AC on I by Theorem 15 of [1], and hence BV on I. Thus the Banach Theorem (6.4) on p. 280 of Saks [6] shows that the function $L(t)$ fulfils the condition (T_1) on the interval I and *a fortiori* that the function $\theta(t)$ fulfils the same condition on the set $S\cap I$.

On the other hand, since $\theta(t)$ is AC on S, there corresponds to each $\varepsilon>0$ a number $\delta>0$ such that for every non-overlapping infinite sequence $\langle K_1, K_2, \cdots\rangle$ of closed intervals, the inequality

$$|K_1|+|K_2|+\cdots<\delta \quad \text{implies} \quad |\theta[S\cap K_1]|+|\theta[S\cap K_2]|+\cdots<\varepsilon .$$

Indeed, to each n for which we have $|\theta[S\cap K_n]|>0$, there obviously corresponds a closed interval L_n pertaining to the set $S\cap K_n$ and fulfilling the inequality $|\theta[S\cap K_n]|\leqq|\theta(L_n)|$. The numbers ε and δ will be kept fixed in the sequel.

Since W is measurable and $|W|$ is finite, there exists a closed interval I_0 pertaining to S and such that $|W\setminus I_0|<\delta$. We can enclose the set $W\setminus I_0$ in a nonvoid open set D disjoint with I_0 and such that $|D|<\delta$. It is obvious that $W\setminus I_0=W\cap D$. Expressing the set D as the union of a non-overlapping infinite sequence $\langle K_1, K_2, \cdots\rangle$ of closed intervals, we thus have the relation

$$|\theta[W\setminus I_0]|=|\theta[W\cap D]|=\left|\bigcup_n \theta[W\cap K_n]\right|\leqq\sum_n|\theta[W\cap K_n]| .$$

This, together with the above choice of δ, implies that

$$|\theta[W\setminus I_0]|\leqq\sum_n|\theta[S\cap K_n]|<\varepsilon .$$

But $\theta(t)$ fulfils the condition (T_1) on $S\cap I_0$ by what was stated above, and hence this function does so too on $W\cap I_0$. Therefore, if E denotes the set of the values of $\theta(t)$ each of which is assumed by $\theta(t)$ infinitely often on W, we must have $|E|<\varepsilon$. This implies $|E|=0$, since ε is arbitrary. We conclude that the function $\theta(t)$ fulfils the condition (T_1) on W.

The set W is measurable and therefore expressible in the form $W=Q\cup T$, where Q is a sigma-compact set and T a null set. Then the set $\theta[Q]$, being a continuous image of Q, is also sigma-compact. On the other hand, the image $\theta[T]$ is null, since every function which is AC on a set fulfils the condition (N) on this set (see the top of p. 225 of Saks [6]). It follows that $\theta[W]$ is a measurable set.

It remains to verify that $|\theta[W]|$ is finite. Let I_0 be the same closed interval as above. By what has already been proved we have

$$|\theta[W]|\leqq|\theta[W\cap I_0]|+|\theta[W\setminus I_0]|\leqq|\theta[W\cap I_0]|+\varepsilon\,.$$

But the function $\theta(t)$, which is AC on W, is bounded on the bounded set $W\cap I_0\subset W$, so that $|\theta[W\cap I_0]|<+\infty$. This completes the proof.

NOTATION. Let $\varphi(x)$ be a function and E a set. By means of the multiplicity function $\mathrm{N}(y;\varphi;E)$ we define a function $\mathrm{P}(y;\varphi;E)$ for $y\in\boldsymbol{R}$, as follows. $\mathrm{P}(y;\varphi;E)$ is equal to $1/\mathrm{N}(y;\varphi;E)$ for $y\in\varphi[E]$, and to 1 for all other numbers y. Thus $0\leqq\mathrm{P}(y;\varphi;E)\leqq1$.

Suppose that a function $\varphi(x)$ is continuous on an analytic set A, and write for short $f(y)=\mathrm{P}(y;\varphi;A)$. The function $\mathrm{N}(y;\varphi;A)$ being measurable as already remarked, we see that $f(y)$ is a measurable function. We have further $0\leqq f(y)\leqq1$. Thus $f(y)$ has an indefinite integral, say $F(y)$. It is obvious that $F(y)$ is continuous and nondecreasing. Moreover, as $f(y)$ is bounded, $F(y)$ is AC on the real line and hence maps every measurable set onto a measurable set.

If we assume further that the function $\varphi(x)$ fulfils the condition (T_1) on the set A, then $\mathrm{N}(y;\varphi;A)<+\infty$, or equivalently $f(y)>0$, for almost every y. In this case, therefore, $F(y)$ is an increasing function, and hence we can consider the inverse mapping of $F(y)$. The range of $F(y)$, namely the image $F[\boldsymbol{R}]$, is clearly an interval which is an open set, and the domain of the inverse mapping coincides with this interval.

The above properties of the function $F(y)$ will be used freely without any quotation hereafter.

THEOREM 6. *Given a function $\varphi(x)$ which is continuous on an analytic set A and subject to the condition (T_1) on this set, let $F(y)$ be an indefinite Lebesgue integral of the function $f(y)=\mathrm{P}(y;\varphi;A)$ and let us write*

$\psi(x)=F\circ\varphi(x)$ for $x\in\boldsymbol{R}$, so that the function $\psi(x)$ is continuous over the set A.

Then we have $\Xi(\psi;M)\leqq|\varphi[M]|$ for every analytic set $M\subset A$, where $\Xi(\psi;M)$ denotes as before the fluctuation of the function $\psi(x)$ over the set M.

PROOF. Let E_n denote for $n\in\boldsymbol{N}$ the set of the points y at which $\mathrm{N}(y;\varphi;M)=n$, so that E_n is a measurable set. If we write

$$E=E_1\cup E_2\cup\cdots,\quad S=S_1\cup S_2\cup\cdots,\quad E_\infty=\varphi[M]\setminus E,\quad S_\infty=\psi[M]\setminus S,$$

where S_n denotes for $n\in\boldsymbol{N}$ the measurable set $F[E_n]$, then E_∞ is plainly a null set and so is also the set S_∞. In fact, we have

$$S_\infty=(F\circ\varphi[M])\setminus F[E]=F[E_\infty]$$

on account of the biuniqueness of $F(y)$, and this implies the nullity of S_∞ since $F(y)$ is AC on the real line and since $|E_\infty|=0$.

Now the function $f(y)$ is estimated for $y\in E_n$ as follows:

$$f(y)=\mathrm{P}(y;\varphi;A)=\frac{1}{\mathrm{N}(y;\varphi;A)}\leqq\frac{1}{\mathrm{N}(y;\varphi;M)}=\frac{1}{n}.$$

On the other hand, we have $F'(y)=f(y)$ at almost every y. Therefore, if D_n denotes for $n\in\boldsymbol{N}$ the set of the points $y\in E_n$ at which $F'(y)=f(y)$, then $E_n\setminus D_n$ is a null set and D_n is measurable. Using again the absolute continuity of the function $F(y)$ on the real line, we see that the difference $F[E_n]\setminus F[D_n]$, which is contained in $F[E_n\setminus D_n]$, must be null. It follows, in virtue of Theorem (6.5) on p. 227 of Saks [6], that

$$|S_n|=|F[E_n]|=|F[D_n]|\leqq n^{-1}|D_n|=n^{-1}|E_n|.$$

To each point t of S_n there corresponds a point y of E_n such that $t=F(y)$. We then have $\mathrm{N}(t;\psi;M)=\mathrm{N}(y;\varphi;M)=n$. As to the fluctuation $\Xi(\psi;M)$ we thus conclude that

$$\Xi(\psi;M)=\int_S\mathrm{N}(t;\psi;M)\,dt=\sum_n n|S_n|\leqq\sum_n|E_n|=|\varphi[M]|,$$

which completes the proof.

THEOREM 7. *In order that a function $\varphi(x)$ which is continuous on an analytic set A, be expressible on A in the form $\varphi(x)=\theta\circ\psi(x)$, where the function $\psi(x)$ is continuous on A and has finite fluctuation $\Xi(\psi;A)$, and where the function $\theta(t)$ is absolutely continuous on the set $\psi[A]$, it is necessary and sufficient that the function $\varphi(x)$ fulfil on A the condition* (T_1) *and the condition $|\varphi[A]|<+\infty$.*

PROOF. (i) Necessity. Let $E(\psi)$ be the set of the numbers t such that $\mathrm{N}(t;\psi;A)=+\infty$. Then $E(\psi)$ is null, i.e. the function $\psi(x)$ fulfils the condition (T_1) on the set A, since we have

$$\int_{-\infty}^{+\infty}\mathrm{N}(t;\psi;A)\,dt=\Xi(\psi;A)<+\infty\,.$$

On the other hand, if we write W for the analytic set $\psi[A]$, the function $\theta(t)$ is AC on W. But plainly $E(\psi)\subset W$, and it follows that $|\theta[E(\psi)]|=0$.

Since $|W|=|\psi[A]|\leqq\Xi(\psi;A)<+\infty$, the function $\theta(t)$ fulfils the condition (T_1) on W by Theorem 5. Therefore, denoting by $L(\theta)$ the set of the numbers y such that $\mathrm{N}(y;\theta;W)=+\infty$, we have $|L(\theta)|=0$.

Now each value that is assumed infinitely often on A by the function $\varphi(x)=\theta\circ\psi(x)$, plainly belongs to the union $L(\theta)\cup\theta[E(\psi)]$, which is a null set by what we have already proved. The function $\varphi(x)$ thus fulfils the condition (T_1) on the set A.

Finally Theorem 5 shows that $|\varphi[A]|=|\theta\circ\psi[A]|=|\theta[W]|<+\infty$, which completes the necessity proof.

(ii) Sufficiency. Given a function $\varphi(x)$ which is continuous on an analytic set A and subject on this set to both the conditions (T_1) and $|\varphi[A]|<+\infty$, let $F(y)$ be a Lebesgue indefinite integral of the function $f(y)=\mathrm{P}(y;\varphi;A)$, and let us write $\psi(x)=F\circ\varphi(x)$ for $x\in\boldsymbol{R}$. The function $\psi(x)$ is evidently continuous on the set A, and Theorem 6 shows that $\Xi(\psi;A)\leqq|\varphi[A]|$. It follows that $\Xi(\psi;A)$ is finite.

We know that the function $F(y)$ is increasing and AC on the real line. The range of $F(y)$ is also $\boldsymbol{R}$, since $f(y)=1$ unless y belongs to the measurable set $\varphi[A]$ of finite measure. Thus the inverse function $\theta(t)$ of the function $t=F(y)$ is increasing and continuous on $\boldsymbol{R}$, and we have the relation $\varphi(x)=\theta\circ\psi(x)$ for $x\in\boldsymbol{R}$ since $\psi(x)=F\circ\varphi(x)$.

Let us show that the function $\theta(t)$ is AC on $\boldsymbol{R}$. For this purpose, let H_n denote for $n\in\boldsymbol{N}$ the set of the points $y\in\varphi[A]$ at which $f(y)<n^{-1}$. Then H_n is measurable and we have $|H_n|\leqq\varphi[A]|<+\infty$. Further the limit, say H, of the descending sequence $H_1\supset H_2\supset\cdots$ is the set of the points $y\in\varphi[A]$ at which $f(y)=0$, that is to say, $\mathrm{N}(y;\varphi;A)=+\infty$. But the function $\varphi(x)$ fulfils the condition (T_1) on A. We thus have $|H|=0$, and it follows that $\lim|H_n|=|H|=0$. Hence, given a $\rho>0$, there exists an index k such that $|H_k|<\rho$. Now consider an arbitrary figure Z and write for short $Y=\theta[Z]$, so that Y is also a figure and we have $Z=F[Y]$. Then

$$|Z|=|F[Y]|=\int_Y f(y)\,dy\geqq k^{-1}|Y\setminus H_k|>k^{-1}(|Y|-\rho)\,,$$

and thus the inequality $|Z|<k^{-1}\rho$ implies $|Y|<2\rho$. The function $\theta(t)$ is

therefore AC on the real line, and this completes the sufficiency proof.

REMARK. The present theorem, although a generalization of Theorem (a) of the Introduction on account of the Banach Theorem (6.4) on p. 280 of Saks [6], is unsatisfactory in that the underlying set is not general, but restricted to an analytic set. It appears to us, however, that such a restriction is inherent in the theorem itself and hence inevitable. In fact, we find it difficult to think out a usable definition of the fluctuation of a function on a set E, if we go outside the case where E is an analytic set and where the function is continuous on E.

§ 2. Extension of Theorem (a) by means of condition (U).

For each set M we shall denote by $\square M$ the intersection of all the closed connected sets that contain M. In other words, $\square M$ means the smallest closed connected set containing M. Thus $\square M$ is either void, or singletonic, or else an interval which is a closed set (but which need not be a closed interval).

DEFINITION. A function $\varphi(x)$ will be said to fulfil the *condition* (U) on a set E, if for each non-overlapping sequence $\langle I_n; n\in \boldsymbol{N}\rangle$ of closed intervals, the upper limit of the sequence $\langle \square\varphi[E\cap I_n]; n\in \boldsymbol{N}\rangle$ is null.

It is obvious that for any function which is continuous on an interval I, the condition (U) on this interval is equivalent to the condition (F) on I and hence to the condition (T_1) on I.

DEFINITION. Given a function $\varphi(x)$, a set E, and a family $\mathfrak{M}$ (perhaps void) of sets, we shall denote by $\overline{\lim}(\square\varphi; E; \mathfrak{M})$ the set of all the numbers y for each of which there exists in $\mathfrak{M}$ an infinity of sets X such that $y\in\square\varphi[E\cap X]$. This set will be called *upper limit* of $\square\varphi[E\cap X]$, where X ranges over $\mathfrak{M}$. In the special case in which $\mathfrak{M}$ is a finite family, this upper limit reduces to the void set.

As we find at once, a function $\varphi(x)$ fulfils the condition (U) on a set E if, and only if, the upper limit $\overline{\lim}(\square\varphi; E; \mathfrak{M})$ is a null set for every non-overlapping family $\mathfrak{M}$ of closed intervals.

THEOREM 8. *Every function $\varphi(x)$ which is subject to the condition* (U) *on a set E and continuous on the closure S of E, fulfils the condition* (U) *on the whole set S.*

PROOF. We shall show that $|\overline{\lim}(\square\varphi; S; \mathfrak{M})|=0$ whenever $\mathfrak{M}$ is a non-

overlapping family of closed intervals. We may plainly suppose that each interval of the family $\mathfrak{M}$ intersects S. Let $\mathfrak{M}_0$ be the family of all the intervals $K\in\mathfrak{M}$ such that $S\cap K$ is a singletonic set. Noting that every non-overlapping family of intervals is countable, we find that the union of $\mathfrak{M}_0$ has a countable set as its intersection with the set S. Then this intersection is mapped by the function $\varphi(x)$ onto a countable set. Hence we may assume that the family $\mathfrak{M}_0$ is void, or in other words, that every interval of $\mathfrak{M}$ contains at least two points of S. Then we may assume further that every $K\in\mathfrak{M}$ pertains to S, namely has its end points belonging to S.

Let us now introduce a temporary concept in reference to the set E. By an *admissible interval* we shall understand any closed interval J which pertains to the closure of the set $E\cap J$. Such an interval clearly pertains to the set S.

As we find without difficulty, any closed interval pertaining to S is expressible as the union of at most three non-overlapping closed intervals each of which is either admissible or contiguous to the closed set S (the two alternatives may occur simultaneously). We now replace each interval $K\in\mathfrak{M}$ by at most three intervals of this description. We thus obtain from $\mathfrak{M}$ a new non-overlapping family of closed intervals, which will be denoted by $\mathfrak{N}$.

If an interval K of the family $\mathfrak{M}$ is expressed as the union of a finite non-overlapping sequence, say $\langle K_1, \cdots, K_n\rangle$, of closed intervals each of which is either admissible or contiguous to the set S, then

$$\square\varphi[S\cap K]=\square\varphi[S\cap K_1]\cup\cdots\cup\square\varphi[S\cap K_n],$$

as we find easily on noting that all the intervals $K, K_1, \cdots, K_n$ pertain to S. From this relation it follows at once that

$$\overline{\lim}\,(\square\varphi;S;\mathfrak{M})=\overline{\lim}\,(\square\varphi;S;\mathfrak{N}).$$

Let $\mathfrak{N}_1$ be the family of all the admissible intervals belonging to $\mathfrak{N}$, and let us write $\mathfrak{N}_2=\mathfrak{N}\setminus\mathfrak{N}_1$. Then obviously

$$\overline{\lim}\,(\square\varphi;S;\mathfrak{N})=\overline{\lim}\,(\square\varphi;S;\mathfrak{N}_1)\cup\overline{\lim}\,(\square\varphi;S;\mathfrak{N}_2).$$

The nullity of $\overline{\lim}\,(\square\varphi;S;\mathfrak{M})$ is thus reduced to the same property of the two sets $\overline{\lim}\,(\square\varphi;S;\mathfrak{N}_i)$, where $i=1,2$.

Given a function $\psi(x)$, a set T, and a figure Z, let I be a generic component interval of Z. We write by definition

$$\square(\psi;T;Z)=\bigcup_I\square\psi[T\cap I] \quad\text{and}\quad \mathrm{d}(\psi;T;Z)=\sum_I\mathrm{d}(\psi[T\cap I]),$$

where $\mathrm{d}(X)$ denotes the diameter of X for any set X, so that $\mathrm{d}(X)=+\infty$ if the set X is not bounded. It is obvious that $|\square(\psi;T;Z)|\leqq \mathrm{d}(\psi;T;Z)$. In the particular case in which the figure Z is void, the set $\square(\psi;T;Z)$ is void and the number $\mathrm{d}(\psi;T;Z)$ vanishes.

In order to ascertain $|\overline{\lim}\,(\square\varphi;S;\mathfrak{N}_1)|=0$, we may assume $\mathfrak{N}_1$ to be an infinite family. Then the intervals of $\mathfrak{N}_1$ can be arranged in an infinite non-overlapping sequence, say $\langle Q_1, Q_2, \cdots\rangle$. Now the function $\varphi(x)$ is continuous on S. Hence, given any $\varepsilon>0$, there exists an infinite sequence of closed intervals pertaining to the set E, say $\langle H_1, H_2, \cdots\rangle$, such that

$$H_n\subset Q_n \quad \text{and} \quad \mathrm{d}(\varphi;S;Q_n\ominus H_n)<2^{-n}\varepsilon$$

for every n, where the reader is referred to p. 59 of Saks [6] for the symbol $\ominus$. From the obvious equality

$$\square\varphi[S\cap Q_n]=\square\varphi[S\cap H_n]\cup\square(\varphi;S;Q_n\ominus H_n)$$

it follows immediately that

$$\begin{aligned}\overline{\lim}\,(\square\varphi;S;\mathfrak{N}_1)&=\overline{\lim_n}\,\square\varphi[S\cap Q_n]\\&=\overline{\lim_n}\,\square\varphi[S\cap H_n]\cup\overline{\lim_n}\,\square(\varphi;S;Q_n\ominus H_n)\,.\end{aligned}$$

But the interval H_n pertains to E, and so the set $S\cap H_n$ is the closure of $E\cap H_n$. Hence $\varphi[S\cap H_n]$ is the closure of $\varphi[E\cap H_n]$, the function $\varphi(x)$ being continuous on S. We thus have $\square\varphi[S\cap H_n]=\square\varphi[E\cap H_n]$. Furthermore, $\varphi(x)$ fulfils the condition (U) on E. Consequently

$$\left|\overline{\lim_n}\,\square\varphi[S\cap H_n]\right|=\left|\overline{\lim_n}\,\square\varphi[E\cap H_n]\right|=0\,.$$

On the other hand, we find at once that

$$\begin{aligned}\left|\overline{\lim_n}\,\square(\varphi;S;Q_n\ominus H_n)\right|&\leqq\left|\bigcup_n\square(\varphi;S;Q_n\ominus H_n)\right|\\&\leqq\sum_n|\square(\varphi;S;Q_n\ominus H_n)|\leqq\sum_n\mathrm{d}(\varphi;S;Q_n\ominus H_n)\,.\end{aligned}$$

Combining the above results, we deduce that

$$\begin{aligned}|\overline{\lim}\,(\square\varphi;S;\mathfrak{N}_1)|&\leqq\left|\overline{\lim_n}\,\square\varphi[S\cap H_n]\right|+\left|\overline{\lim_n}\,\square(\varphi;S;Q_n\ominus H_n)\right|\\&\leqq\sum_n\mathrm{d}(\varphi;S;Q_n\ominus H_n)<\sum_n 2^{-n}\varepsilon=\varepsilon\,.\end{aligned}$$

Since ε is arbitrary, we obtain $|\overline{\lim}\,(\square\varphi;S;\mathfrak{N}_1)|=0$, as desired.

It remains to show the nullity of $\overline{\lim}\,(\square\varphi;S;\mathfrak{N}_2)$. For this purpose, we may assume the family $\mathfrak{N}_2$ infinite. The intervals of $\mathfrak{N}_2$ then can be arranged in a non-overlapping infinite sequence, say $\langle I_1, I_2, \cdots\rangle$. By defini-

tion of $\mathfrak{N}_2$, every I_n is a closed interval contiguous to the set S.

Given any $\varepsilon>0$, we shall temporarily understand by an *ε-suitable pair* any couple of non-overlapping finite sequences of closed intervals, say $\langle C_1, \cdots, C_k\rangle$ and $\langle L_1, \cdots, L_k\rangle$, of equal length $k>0$, such that

(1) $C_n \subset L_n$, $C_n \in \mathfrak{N}_2$, and L_n pertains to E for $n=1, \cdots, k$;

(2) we have $\sum_{n=1}^{k} \mathrm{d}(\varphi; S; L_n \ominus C_n) < \varepsilon$.

Let us show that if $\langle C_1, \cdots, C_k\rangle$ and $\langle L_1, \cdots, L_k\rangle$ form an ε-suitable pair and if every interval of the family $\mathfrak{N}_2$ is contained in the union $L_1 \cup \cdots \cup L_k$ (and hence in one of the intervals $L_1, \cdots, L_k$), then we necessarily have the inclusion

$$\overline{\lim}\,(\square\varphi; S; \mathfrak{N}_2) \subset \bigcup_{n=1}^{k} \square(\varphi; S; L_n \ominus C_n)\,,$$

so that $|\overline{\lim}\,(\square\varphi; S; \mathfrak{N}_2)| < \varepsilon$ on account of the relation

$$\sum_{n=1}^{k} |\square(\varphi; S; L_n \ominus C_n)| \leqq \sum_{n=1}^{k} \mathrm{d}(\varphi; S; L_n \ominus C_n) < \varepsilon\,.$$

The above inclusion may be verified as follows. Let η be any point of the set $\overline{\lim}\,(\square\varphi; S; \mathfrak{N}_2)$. Then there is an infinity of values of n such that $\eta \in \square\varphi[S \cap I_n]$. It is now convenient to introduce a nonce wording in reference to the pair $\langle C_1, \cdots, C_k\rangle$, $\langle L_1, \cdots, L_k\rangle$. Any pair $\langle n, i\rangle$ of positive integers, where $i \leqq k$, will be termed *η-admissible,* if we have both

$$I_n \subset L_i \quad \text{and} \quad \eta \in \square\varphi[S \cap I_n]\,.$$

Plainly there exists for each η an infinity of η-admissible pairs. We can therefore choose an η-admissible pair $\langle r, s\rangle$ such that $I_r \neq C_s$. Then I_r does not overlap C_s and hence we have $I_r \subset L_s \ominus C_s$. Accordingly I_r is contained in a component, say I, of the figure $L_s \ominus C_s$. We thus have

$$\eta \in \square\varphi[S \cap I_r] \subset \square\varphi[S \cap I] \subset \square(\varphi; S; L_s \ominus C_s)\,,$$

whence the announced inclusion.

It follows from the above that if there exists for every $\varepsilon>0$ an ε-suitable pair $\langle C_1, \cdots, C_k\rangle$, $\langle L_1, \cdots, L_k\rangle$ with $L_1 \cup \cdots \cup L_k$ containing the union of $\mathfrak{N}_2$, then necessarily $|\overline{\lim}\,(\square\varphi; S; \mathfrak{N}_2)| = 0$. We may hence assume in the sequel that this hypothesis does not take place. We then can choose an $\varepsilon_0>0$ such that whenever $0<\varepsilon<\varepsilon_0$, there is no ε-suitable pair of the mentioned description.

Now suppose given a positive number $\varepsilon<\varepsilon_0$. We shall construct two infinite sequences of closed intervals, $\langle C_1, C_2, \cdots\rangle$ and $\langle L_1, L_2, \cdots\rangle$, such that for every $k \in \boldsymbol{N}$ the partial sequences $\langle C_1, \cdots, C_k\rangle$ and $\langle L_1, \cdots, L_k\rangle$

together constitute an ε-suitable pair.

We arranged the intervals of the family $\mathfrak{N}_2$ in an infinite sequence $\langle I_1, I_2, \cdots\rangle$. Writing now $I_n=[a_n, b_n]$ for $n\in \boldsymbol{N}$, we put $C_1=I_1=[a_1, b_1]$, so that C_1 is contiguous to the set S. We then set $L_1=[p_1, q_1]$, where p_1 and q_1 are determined as follows. If $a_1\in E$, we simply choose $p_1=a_1$. In the opposite case, a_1 must be an accumulation point from the left for E, and this fact, together with the continuity of the function $\varphi(x)$ on S, shows that E contains a point $p_1<a_1$ fulfilling $\mathrm{d}(\varphi; S; [p_1, a_1])<2^{-1}\varepsilon$. By symmetry, we choose the point q_1 in a similar way. We find at once that $\langle C_1\rangle$ and $\langle L_1\rangle$ constitute an ε-suitable pair.

We now proceed by induction. Suppose that for a $k\in \boldsymbol{N}$ we have an ε-suitable pair $\langle C_1, \cdots, C_k\rangle$, $\langle L_1, \cdots, L_k\rangle$ of length k. As $\varepsilon<\varepsilon_0$, there is an interval $I_m\in\mathfrak{N}_2$ which is contained in none of the intervals $L_1, \cdots, L_k$. We may assume that m is the smallest integer with this property. We then choose as C_{k+1} this interval $I_m=[a_m, b_m]$, so that C_{k+1} is contiguous to S and does not overlap the union $A=L_1\cup\cdots\cup L_k$. This being so, we define the interval $L_{k+1}=[p_{k+1}, q_{k+1}]$ as follows. If $a_m\in E$, we simply take $p_{k+1}=a_m$. In the opposite case, a_m is an exterior point of the figure A and at the same time an accumulation point from the left for E. Hence E contains a point $p_{k+1}<a_m$ such that the interval $[p_{k+1}, a_m]$ does not intersect A and further that

$$\mathrm{d}(\varphi; S; [p_{k+1}, a_m])<\delta, \quad \text{where } 2\delta=\varepsilon-\sum_{n=1}^{k}\mathrm{d}(\varphi; S; L_n\ominus C_n)>0\,.$$

By symmetry, we choose the point q_{k+1} in a similar way. It follows at once that the pair $\langle C_1, \cdots, C_{k+1}\rangle$, $\langle L_1, \cdots, L_{k+1}\rangle$ is ε-suitable.

We have thus constructed two infinite sequences $\langle C_1, C_2, \cdots\rangle$ and $\langle L_1, L_2, \cdots\rangle$ such that the sequences $\langle C_1, \cdots, C_k\rangle$ and $\langle L_1, \cdots, L_k\rangle$ together form an ε-suitable pair for every $k\in \boldsymbol{N}$.

We shall show that each interval of $\mathfrak{N}_2$ is contained in one of the intervals $L_1, L_2, \cdots$. For this purpose, suppose if possible that there is an interval I_g for which this does not take place. We then have $g>1$, since we chose $C_1=I_1$. Let k be the largest of the integers $n>0$ such that all the intervals $C_1, \cdots, C_n$ appear before I_g in the sequence $\langle I_1, I_2, \cdots\rangle$. Remembering the choice of the interval $C_{k+1}=I_m$ and noting that none of the intervals $L_1, \cdots, L_k$ contains I_g, we find at once that $m\leqq g$. But we cannot have $m<g$, since this would be incompatible with the above choice of the integer k. Consequently $g=m$, and hence $I_g=C_{k+1}\subset L_{k+1}$, which contradicts the choice of I_g. This shows that each interval of $\mathfrak{N}_2$ is contained in one of $L_1, L_2, \cdots$.

Let us establish for $\langle C_1, C_2, \cdots\rangle$ and $\langle L_1, L_2, \cdots\rangle$ the inclusion

$$\left(\overline{\lim_n}\,\square\varphi[S\cap I_n]\right)\setminus\left(\overline{\lim_i}\,\square\varphi[S\cap C_i]\right)\subset\bigcup_i\square(\varphi;S;L_i\ominus C_i)\,.$$

Suppose that a point η belongs to the left-hand side of this relation. We then have $\eta\in\square\varphi[S\cap I_n]$ for an infinity of values of n, while the relation $\eta\in\square\varphi[S\cap C_i]$ holds for at most a finite number of values of i. Now a pair $\langle n, i\rangle$ of positive integers will be called *η-permissible,* if there hold $I_n\subset L_i$ and $\eta\in\square\varphi[S\cap I_n]$. Using this notion in place of the η-admissibility, the argument proceeds quite as before. Thus there is an infinity of η-permissible pairs, while there is at most a finite number of η-permissible pairs $\langle n, i\rangle$ for which $I_n=C_i$. Accordingly we can choose an η-permissible pair $\langle r, s\rangle$ such that $I_r\neq C_s$. Then, as before, we have successively

$$I_r\subset L_s\ominus C_s \quad\text{and}\quad \eta\in\square\varphi[S\cap I_r]\subset\square(\varphi;S;L_s\ominus C_s)\,,$$

whence the desired inclusion.

Now, since each interval L_n pertains to E and since $\varphi(x)$ is continuous on S, the relation $\square\varphi[S\cap L_n]=\square\varphi[E\cap L_n]$ is verified in the same way as the previous one $\square\varphi[S\cap H_n]=\square\varphi[E\cap H_n]$. It follows immediately that

$$\overline{\lim_n}\,\square\varphi[S\cap C_n]\subset\overline{\lim_n}\,\square\varphi[S\cap L_n]=\overline{\lim_n}\,\square\varphi[E\cap L_n]\,.$$

In this relation, the last upper limit, and hence also the first, must be null since the function $\varphi(x)$ fulfils the condition (U) on E. Combining this fact with what has already been proved, we find that

$$|\overline{\lim}\,(\square\varphi;S;\mathfrak{N}_2)|=\left|\overline{\lim_n}\,\square\varphi[S\cap I_n]\right|\leqq\left|\bigcup_n\square(\varphi;S;L_n\ominus C_n)\right|$$

$$\leqq\sum_n \mathrm{d}(\varphi;S;L_n\ominus C_n)=\lim_k\sum_{n\leqq k}\mathrm{d}(\varphi;S;L_n\ominus C_n)\leqq\varepsilon\,.$$

Since the number ε is arbitrary so long as $0<\varepsilon<\varepsilon_0$, we conclude that $|\overline{\lim}\,(\square\varphi;S;\mathfrak{N}_2)|=0$, as required. This completes the proof.

THEOREM 9. *Suppose that a function $f(v)$ is* (i) *continuous on the closure S of a set L,* (ii) *subject to the condition* (U) *on L, and* (iii) *linear on every closed interval K contiguous to S. Then $f(v)$ fulfils the condition* (T_1) *on the set $\square S=\square L$.*

PROOF. On account of the hypothesis and Theorem 8, the function $f(v)$ fulfils the condition (U), and *a fortiori* the condition (F), on the set S. It follows from Theorem 4 that $f(v)$ fulfils the condition (T_1) on S. We also find that if $\mathfrak{K}$ denotes the family of all the closed intervals K contiguous to S, the set $\overline{\lim}\,(\square f;S;\mathfrak{K})$ is null. Now a number q belongs to this upper limit if, and only if, there exists an infinity of intervals $K\in\mathfrak{K}$

such that $q \in \square f[S \cap K]$. But we have $f[K] = \square f[S \cap K]$ for every K, since the function $f(v)$ is linear on K. Accordingly the numbers q for each of which there is an infinity of intervals $K \in \mathfrak{K}$ such that $q \in f[K]$, together constitute a null set. It follows that $f(v)$ fulfils the condition (T_1) on the union $[\mathfrak{K}]$ of the family $\mathfrak{K}$. We conclude that $f(v)$ fulfils the same condition on the set $\square S = S \cup [\mathfrak{K}]$, where we clearly have $\square S = \square L$. This completes the proof.

THEOREM 10. *A function $\varphi(x)$ which is bounded on a set E and which fulfils the condition* (U) *on E, necessarily has a finite right-hand limit $\varphi_E(c+)$ [or left-hand limit $\varphi_E(c-)$] relative to the set E at every point c which is a right-hand [or left-hand] accumulation point of E.*

Furthermore, there can exist at most a countable infinity of bilateral accumulation points p of E at which $\varphi_E(p+) \neq \varphi_E(p-)$.

REMARK. We have the same conclusions as above for any function $\varphi(x)$ which is BV on the set E. This well-known fact will, alongside of the present theorem, be made use of in the proof of the first main theorem.

PROOF. To establish the first half of the assertion, we may confine ourselves to the existence of the right-hand limit. Suppose that c is an accumulation point from the right for E. Let us extract from E an arbitrary decreasing sequence of points, say $c_1 > c_2 > \cdots$, which converges to the point c, so that $c_n > c$ for each n. If we write $A = \{c_1, c_2, \cdots\}$ and $I_n = [c_{n+1}, c_n]$ for $n \in \boldsymbol{N}$, the sequence $\langle I_1, I_2, \cdots \rangle$ is non-overlapping and we can write

$$\overline{\lim_n} \square \varphi[A \cap I_n] = \bigcap_n M_n, \quad \text{where} \quad M_n = \bigcup_{k \geq n} \square \varphi[A \cap I_k].$$

As we verify without difficulty, each set M_n is either a finite interval or a singletonic set. We have moreover $M_1 \supset M_2 \supset \cdots$, while by hypothesis the function $\varphi(x)$ fulfils the condition (U) on E, and hence on A. Accordingly we find that

$$\lim_n |M_n| = \left|\lim_n M_n\right| = \left|\bigcap_n M_n\right| = \left|\overline{\lim_n} \square \varphi[A \cap I_n]\right| = 0.$$

But the set M_n clearly contains the points $\varphi(c_n), \varphi(c_{n+1}), \cdots$. Accordingly the sequence $\langle \varphi(c_n); n \in \boldsymbol{N} \rangle$ converges. On the other hand, the sequence $c_1 > c_2 > \cdots$ was arbitrary so long as $\lim c_n = c$ and $c_n \in E$ for every n. Thus the limit $\varphi_E(c+)$ exists and is finite.

We pass on to the second half of the assertion. Let S be the set of the bilateral accumulation points p of E at which $\varphi_E(p+) \neq \varphi_E(p-)$. Then S is the union of the sequence $\langle S_n; n \in \boldsymbol{N} \rangle$, where S_n denotes the set of

the points $p \in S$ such that $|\varphi_E(p+)-\varphi_E(p-)|>n^{-1}$. It therefore suffices to prove that the set S_n is countable for every n. But this will follow at once if we show that S_n is an isolated set. For this purpose, consider any point p of S_n. The existence of the unilateral limits $\varphi_E(p+)$ and $\varphi_E(p-)$ necessitates that the point p has a neighbourhood $(p-\delta, p+\delta)$ in which there is exactly one point of S_n, namely the point p itself. It follows that every point of S_n is an isolated point of this set. The proof is thus complete.

DEFINITION. Given a function $\varphi(x)$, a set E, and a point $p \in \boldsymbol{R}$, take any $\delta>0$ and consider the oscillation $\mathrm{O}(\varphi; M)=\mathrm{d}(\varphi[M])$, where M is short for the set $E \cap (p-\delta, p+\delta)$. When $\delta \to 0$, this oscillation tends monotonely towards a limit (finite or infinite), which will be written $\mathrm{o}(p;\varphi;E)$ and called *oscillation* of the function $\varphi(x)$ *on the set* E *at the point* p.

This notion generalizes the oscillation $\mathrm{o}_E(\varphi;p)$ introduced on p. 42 of Saks [6]. In fact, $\mathrm{o}_E(\varphi;p)$ is considered only when $p \in E$, whereas we put no such restriction on the definition of the quantity $\mathrm{o}(p;\varphi;E)$.

Making the point p vary arbitrarily, we obtain a function $\mathrm{o}(x;\varphi;E)$ which is defined on the whole real line. Plainly, this function vanishes at every point exterior to the set E and at every point of E at which $\varphi(x)$ is continuous on E. We have $0 \leqq \mathrm{o}(x;\varphi;E) \leqq +\infty$.

DEFINITION. A function $\varphi(x)$ will be termed *fully continuous* on a set E, if we have $\mathrm{o}(x;\varphi;E)=0$ identically for every $x \in \boldsymbol{R}$.

This is certainly the case when $\varphi(x)$ is uniformly continuous on E; but the converse is clearly false. On the other hand, it is easy to prove the following proposition.

THEOREM 11. *Let $\varphi(x)$ be a function and X a set. In order that there should exist a function which coincides with $\varphi(x)$ on X and which is continuous on the closure of X, it is necessary and sufficient that the function $\varphi(x)$ be fully continuous on X.*

THEOREM 12. *Suppose that a function $\omega(x)$ is increasing on a set E and that for each set $X \subset E$ the image $\omega[X]$ is bounded or not, according as X is bounded or not, respectively. Then a function $g(v)$ fulfils the condition* (U) *on the set $P=\omega[E]$, if and only if the composite function $h(x)=g \circ \omega(x)$ fulfils the same condition on E.*

PROOF. It clearly suffices to ascertain that if a function $g(v)$ fulfils the condition (U) on the set P, then so does also the function $h(x)$ on

the set E.

For this purpose, let $\mathfrak{M}$ be any non-overlapping family of closed intervals. We are to show that the set $\overline{\lim}\,(\square h; E; \mathfrak{M})$ is null. As easily seen, we may assume that for every interval $I \in \mathfrak{M}$ the set $E \cap I$ contains at least two points.

Noting that for each $I \in \mathfrak{M}$ the image $\omega[E \cap I]$ is, together with $E \cap I$, bounded and contains at least two points, we associate with I the closed interval $J = \square\omega[E \cap I]$. Then the correspondence $I \to J$ is biunique, and we clearly have the inclusion

$$h[E \cap I] = g \circ \omega[E \cap I] \subset g[P \cap J],$$

whence $\square h[E \cap I] \subset \square g[P \cap J]$. Consequently it follows that

$$\overline{\lim}\,(\square h; E; \mathfrak{M}) \subset \overline{\lim}\,(\square g; P; \mathfrak{N}),$$

where $\mathfrak{N}$ is the family of all the intervals J. But it is obvious that the family $\mathfrak{N}$ is non-overlapping. We thus have successively

$$|\overline{\lim}\,(\square g; P; \mathfrak{N})| = 0 \quad \text{and} \quad |\overline{\lim}\,(\square h; E; \mathfrak{M})| = 0 .$$

This completes the proof.

THEOREM 13. *In order that a function $\varphi(x)$ which is continuous on an interval I be* BV *on this interval, it is necessary and sufficient that the function have finite fluctuation on I. When this is the case, we have $\Xi(\varphi; I) = \mathrm{V}(\varphi; I)$, where* V *denotes the weak variation.*

PROOF. The Banach Theorem (6.4) on p. 280 of Saks [6] shows that $\Xi(\varphi; K) = \mathrm{W}(\varphi; K)$ for every closed interval $K \subset I$, where W denotes the absolute variation. But we have the relations

$$\Xi(\varphi; I) = \sup_K \Xi(\varphi; K) \quad \text{and} \quad \mathrm{V}(\varphi; I) = \sup_K \mathrm{W}(\varphi; K),$$

of which the latter one is obvious and the former an immediate consequence of Theorem 16 of [4]. Hence $\Xi(\varphi; I) = \mathrm{V}(\varphi; I)$.

We are now in a position to establish the following proposition which generalizes Theorem (a) of the Introduction.

THEOREM 14 (first main theorem). *In order that a function $\varphi(x)$ which is continuous on a set E, be expressible on this set in the composite form $\varphi(x) = \theta \circ \psi(x)$, where the inner function $\psi(x)$ is both continuous and* BV *on the set E and where the outer function θ is* AC *on the set $\psi[E]$, it is necessary and sufficient that the function $\varphi(x)$ be bounded on E and fulfil the condition* (U) *on this set.*

PROOF. (i) Necessity. Suppose that a function $\varphi(x)$ is continuous on a set E and expressible on this set in the composite form $\varphi(x)=\theta\circ\psi(x)$, as specified in the theorem. Let M be the set of all the points $x\in\boldsymbol{R}$ at which $\mathrm{o}(x;\psi;E)\neq 0$. By hypothesis, the function $\psi(x)$ is continuous and BV on E. Consequently a point x belongs to M if, and only if, E has x as a bilateral accumulation point which does not belong to E and at which $\psi_E(x+)\neq\psi_E(x-)$. It follows that M is a countable set. Indeed, if a finite sequence of points $x_1<\cdots<x_n$ is contained in M, then we plainly have

$$\sum_{i=1}^{n}|\psi_E(x_i+)-\psi_E(x_i-)|\leqq \mathrm{V}(\psi;E)<+\infty\,,$$

so that for each $\varepsilon>0$ the number of the points $x\in M$ subject to the condition $|\psi_E(x+)-\psi_E(x-)|\geqq\varepsilon$, cannot exceed $\varepsilon^{-1}\cdot\mathrm{V}(\psi;E)$. Hence the result.

This being so, let p be a generic point of M. We can clearly associate with each p a number $\xi(p)>0$ in such a manner that $\sum\xi(p)<+\infty$, where p ranges over M and where the sum vanishes when M is void. Let us now define an increasing function $\omega(x)$ by

$$\omega(x)=x+\sum_{p<x}\xi(p)\qquad \text{for } x\in\boldsymbol{R}\,.$$

As we readily see, this function is everywhere continuous from the left and exactly has M as the set of its points of discontinuity. As to the saltus of $\omega(x)$, we have $\omega(x+)-\omega(x)=\xi(x)$ for $x\in M$. We observe further that $|I|\leqq\omega(I)$ for every closed interval I.

Let us write $\Omega(x)=[\omega(x),\omega(x+)]$ for $x\in\boldsymbol{R}$, so that $\Omega(x)$ means the closed interval $[\omega(x),\omega(x)+\xi(x)]$ for $x\in M$ and the singletonic set $\{\omega(x)\}$ for every other x. Then the family $\mathfrak{M}$ of all the sets $\Omega(x)$ is clearly disjoint. We assert that the union $[\mathfrak{M}]$ of this family is the whole real line. To prove this, suppose if possible that there is a number γ not belonging to $[\mathfrak{M}]$. Noting that each set $\Omega(x)$ is contained in one of the intervals $(-\infty,\gamma)$ and $(\gamma,+\infty)$, we define two sets P and Q by the conditions

$$x\in P \text{ if } \Omega(x)\subset(-\infty,\gamma)\,,\qquad x\in Q \text{ if } \Omega(x)\subset(\gamma,+\infty)\,.$$

The pair $\langle P,Q\rangle$ is then a Dedekind cut of the real line, as easily seen, and therefore either P contains a greatest element g, or else Q contains a least element l. If the first alternative takes place, we have

$$\omega(g+)\in\Omega(g)\subset(-\infty,\gamma)\,,\qquad \text{so that } \omega(g+)<\gamma\,.$$

Hence there is a number $h>g$ such that $\omega(h+)<\gamma$. It follows that $h\in P$ and therefore that $h\leqq g$, which contradicts $h>g$. Similarly the second alternative leads to a contradiction. We thus conclude that $[\mathfrak{M}]=\boldsymbol{R}$.

Now the inverse function, $x=\omega^{-1}(v)$, of the function $v=\omega(x)$ is defined

on the set $\omega[\boldsymbol{R}]$ and maps this set increasingly and continuously onto $\boldsymbol{R}$, the continuity being obvious from the inequality $|I| \leqq \omega(I)$ already stated. For convenience we extend the definition of the function $\omega^{-1}(v)$ to the whole $\boldsymbol{R}$, determining $\omega^{-1}(v)$ arbitrarily for v outside $\omega[\boldsymbol{R}]$.

We shall show that the function $\psi \circ \omega^{-1}(v)$ is fully continuous on the set $\omega[E]$. For this purpose, suppose if possible that there is a point v_0 at which $\mathrm{o}(v_0; \psi \circ \omega^{-1}; \omega[E]) \neq 0$. Then v_0 belongs to the closure of $\omega[E]$ and hence to that of $\omega[\boldsymbol{R} \setminus M]$. But the complement of $\omega[\boldsymbol{R} \setminus M]$ is the union of the intervals $\Omega(p)$ for $p \in M$, since $[\mathfrak{M}] = \boldsymbol{R}$ as established above. Consequently, either $v_0 \in \omega[\boldsymbol{R} \setminus M]$, or else there is a point $p_0 \in M$ such that v_0 is an end point of the interval $\Omega(p_0) = [\omega(p_0), \omega(p_0+)]$. If the first alternative takes place, then the point $q_0 = \omega^{-1}(v_0)$ belongs to $\boldsymbol{R} \setminus M$ and hence $\mathrm{o}(q_0; \psi; E) = 0$. But this contradicts $\mathrm{o}(v_0; \psi \circ \omega^{-1}; \omega[E]) \neq 0$, since the function $\omega(x)$ is increasing and bicontinuous on $\boldsymbol{R} \setminus M$. Hence we must have the second alternative, and so there exists a point $p_0 \in M$ such that either $v_0 = \omega(p_0)$ or else $v_0 = \omega(p_0+)$. This fact, together with $\mathrm{o}(p_0; \psi; E_1) = 0$ and $\mathrm{o}(p_0; \psi; E_2) = 0$, where we write

$$E_1 = E \cap (-\infty, p_0) \quad \text{and} \quad E_2 = E \cap (p_0, +\infty),$$

leads at once to $\mathrm{o}(v_0; \psi \circ \omega^{-1}; \omega[E]) = 0$, which contradicts the assumption.

The full continuity of the function $\psi \circ \omega^{-1}(v)$ on $\omega[E]$, thus established, implies by Theorem 11 the existence of a function $\chi(v)$ which is continuous on the closure S of $\omega[E]$ and which coincides on $\omega[E]$ with $\psi \circ \omega^{-1}(v)$. We may plainly assume that the function $\chi(v)$ is continuous on the set $\square S$ and linear on every closed interval contiguous to S. It is then easily seen that

$$\mathrm{V}(\chi; \square S) = \mathrm{V}(\chi; S) = \mathrm{V}(\psi \circ \omega^{-1}; \omega[E]) = \mathrm{V}(\psi; E) < +\infty .$$

We have moreover $\chi[\square S] \subset \square \chi[S] = \square \psi \circ \omega^{-1} \circ \omega[E] = \square \psi[E]$.

By hypothesis the function θ is AC on $\psi[E]$, which is a bounded set since $\mathrm{V}(\psi; E) < +\infty$. Hence this function is bounded on $\psi[E]$, and consequently the function $\varphi(x) = \theta \circ \psi(x)$ is bounded on E. On the other hand, the function θ, which is uniformly, and hence fully, continuous on $\psi[E]$, may be assumed continuous on the closure of $\psi[E]$. It follows that this function is AC on this closure. Theorem 15 of [1] then allows us to suppose finally that the function θ is AC on $\square \psi[E]$, and hence on its subset $\chi[\square S]$ also.

The function $\chi(v)$, which is BV and continuous on $\square S$, has finite fluctuation $\Xi(\chi; \square S)$ by Theorem 13 and thus fulfils the condition (T_1) on $\square S$. Hence the set $A(\chi)$ of the values assumed by the function $\chi(v)$ infinitely often on $\square S$, must be null. Again, the function θ is AC on the bounded

connected set $\chi[\square S]$ and so fulfils the condition (T_1) on this set by Theorem 5. The set $B(\theta)$ of the values assumed by the function θ infinitely often on the set $\chi[\square S]$, is therefore null. The absolute continuity of θ on $\chi[\square S]$ shows further that $|\theta[A(\chi)]|=0$. Now each value assumed infinitely often on $\square S$ by the function $\theta\circ\chi(v)$ clearly belongs to the union $B(\theta)\cup\theta[A(\chi)]$. We thus see that the set of these values is null, i. e. that this function fulfils the condition (T_1) on $\square S$.

By a remark which was made just after the definition of the condition (U), the function $\theta\circ\chi(v)$ fulfils the condition (U) on $\square S$, and *a fortiori* on its subset $\omega[E]$. Now the function $\omega(x)$ is increasing and maps each set X onto a bounded set or not, according as X is bounded or not. Theorem 12 then shows that the function $\theta\circ\chi\circ\omega(x)$ fulfils the condition (U) on E. But the function $\chi(v)$ was so chosen as to coincide with $\psi\circ\omega^{-1}(v)$ on the set $\omega[E]$. Hence we have $\theta\circ\chi\circ\omega(x)=\theta\circ\psi(x)=\varphi(x)$ for $x\in E$, and this completes the necessity proof.

(ii) Sufficiency. We shall only outline the argument, since it resembles that of part (i).

Suppose that on a set E a function $\varphi(x)$ is continuous, bounded, and subject to the condition (U). Let M be the set of the points $x\in\boldsymbol{R}$ at which $o(x;\varphi;E)\neq 0$. By Theorem 10, a point x belongs to M if, and only if, x is a bilateral accumlation point of E without belonging to E and without fulfilling $\varphi_E(x+)=\varphi_E(x-)$. It follows from the same theorem that M is a countable set.

We can associate with each point $p\in M$ a number $\xi(p)>0$ in such a manner that $\sum\xi(p)<+\infty$. We now define an increasing function $\omega(x)$ by

$$\omega(x)=x+\sum_{p<x}\xi(p) \qquad \text{for } x\in\boldsymbol{R}.$$

This function is everywhere continuous from the left and M is the set of its points of discontinuity. As we find at once, $|I|\leqq\omega(I)$ for every closed interval I and $\omega(x+)=\omega(x)+\xi(x)$ for $x\in M$. Further, if for $x\in\boldsymbol{R}$ we write $\Omega(x)=[\omega(x),\omega(x+)]$, the sets $\Omega(x)$ are mutually disjoint and their union is the real line.

The inverse function, $x=\omega^{-1}(v)$, of the function $v=\omega(x)$ is defined on the set $\omega[\boldsymbol{R}]$ and maps this set increasingly and continuously onto $\boldsymbol{R}$, the continuity being immediate from $|I|\leqq\omega(I)$ just mentioned. In the same way as in part (i), we extend the definition of the function $\omega^{-1}(v)$ to the whole $\boldsymbol{R}$. It can be shown that the function $\varphi\circ\omega^{-1}(v)$ is fully continuous on the set $\omega[E]$. This implies the existence of a function $\chi(v)$ which is continuous on the closure S of $\omega[E]$ and which coincides on $\omega[E]$ with $\varphi\circ\omega^{-1}(v)$. The function $\chi(v)$ may be assumed continuous on $\square S$ and linear

on every closed interval contiguous to S. Then $\chi(v)$ must be bounded on $\square S$, since $\varphi(x)$ is bounded on E. We thus have $|\chi[\square S]|<+\infty$. The function $\chi\circ\omega(x)$, which coincides with $\varphi(x)$ on E, fulfils the condition (U) on E. The function $\chi(v)$ therefore fulfils the same condition on $\omega[E]$ in virtue of Theorem 12. If follows from Theorem 9 that $\chi(v)$ fulfils the condition (T_1) on the set $\square S=\square\omega[E]$.

The above results, together with Theorem 7, shows that the function $\chi(v)$ is expressible on $\square S$ in the form $\chi(v)=\theta\circ\rho(v)$, where the function $\rho(v)$ is continuous on $\square S$ and has finite fluctuation $\Xi(\rho;\square S)$, and where the function θ is AC on $\rho[\square S]$. Then $\rho(v)$ must be BV on $\square S$ on account of Theorem 13. The function $\psi(x)=\rho\circ\omega(x)$, which is clearly continuous on E, is thus BV on E. But

$$\varphi(x)=\chi\circ\omega(x)=\theta\circ\rho\circ\omega(x)=\theta\circ\psi(x)$$

for $x\in E$, and we have $\psi[E]=\rho\circ\omega[E]\subset\rho[\square S]$. Hence the function θ is AC on $\psi[E]$. This completes the proof.

REMARKS. (i) The interval I underlying Theorem (a) of the Introduction is restricted to a closed one. However, a simple argument (inclusive of a change of the independent variable x) shows that the theorem is true for every underlying interval, it being assumed that the function $\varphi(x)$ is bounded on this interval. Utilization of this extended Theorem (a) enables us to avoid the recourse to Theorem 7 in the above proof.

(ii) The condition (U) is generally more restrictive than the condition (F), when the underlying set is not an interval. This may be seen by the following example.

Let $I=[0,1]$. As shown on p. 224 of Saks [6], there exist a function $\varphi(x)$ and a compact set C with $\square C=I$, such that $\varphi(x)$ is continuous on the real line, vanishes on C, and has no derivative (finite or infinite) at any point of C. Then the function $\psi(x)=\varphi(x)+x$ can possess no derivative at any point of C, while plainly $|\psi[C]|=|C|>0$. It follows from Theorem (6.2) on p. 278 of Saks [6] that $\psi(x)$ does not fulfil the condition (T_1) on the interval I. By Theorem 4, this implies that $\psi(x)$ fails to fulfil the condition (F) on I.

This being so, consider any countable set E with I for its closure. Then every function, and in particular the function $\psi(x)$, fulfils the condition (F) on E. However, $\psi(x)$ cannot fulfil the condition (U) on E. In fact, this function would otherwise be subject on the whole interval I to the condition (U) in virtue of Theorem 8, and *a fortiori* to the condition (F), contradicting what was stated above.

§ 3. Extension of Theorem (b) by means of condition (W).

Given a function $\varphi(x)$, a set E, and a countable (perhaps void) family $\mathfrak{M}$ of sets, let X be a generic set belonging to $\mathfrak{M}$. We denote by $|\mathfrak{M}|$ the outer measure of the union $[\mathfrak{M}]$ and we write further

$$\square(\varphi;E;\mathfrak{M})=\bigcup_X \square\varphi[E\cap X]\,, \qquad \mathrm{d}(\varphi;E;\mathfrak{M})=\sum_X \mathrm{d}(\varphi[E\cap X])\,.$$

It is obvious that $|\square(\varphi;E;\mathfrak{M})|\leqq \mathrm{d}(\varphi;E;\mathfrak{M})$. When $\mathfrak{M}$ is in particular the family of the components of a figure Z, we plainly have

$$\square(\varphi;E;\mathfrak{M})=\square(\varphi;E;Z) \quad \text{and} \quad \mathrm{d}(\varphi;E;\mathfrak{M})=\mathrm{d}(\varphi;E;Z)\,,$$

where the right-hand sides were defined previously.

DEFINITION. A function $\varphi(x)$ will be said to fulfil the *condition* (W) on a set E, if there corresponds to each $\varepsilon>0$ a number $\delta>0$ such that for every non-overlapping family $\mathfrak{M}$ of closed intervals, the inequality $|\mathfrak{M}|<\delta$ implies $|\square(\varphi;E;\mathfrak{M})|<\varepsilon$.

When this is the case, the function $\varphi(x)$ is uniformly continuous on E.

In the above definition, we may assume the family $\mathfrak{M}$ to be finite. In fact, each countable family $\mathfrak{K}$ of sets is the union of an ascending infinite sequence $\mathfrak{K}_1\subset\mathfrak{K}_2\subset\cdots$ of finite subfamilies of $\mathfrak{K}$. The limit of the ascending sequence $\langle\square(\varphi;E;\mathfrak{K}_n);\, n\in \boldsymbol{N}\rangle$ is the set $\square(\varphi;E;\mathfrak{K})$, and accordingly it follows that

$$|\square(\varphi;E;\mathfrak{K})|=\lim_n |\square(\varphi;E;\mathfrak{K}_n)|\,.$$

Thus, if $|\square(\varphi;E;\mathfrak{K}_n)|<\varepsilon$ for every n, we have $|\square(\varphi;E;\mathfrak{K})|\leqq\varepsilon$. Hence the result.

DEFINITION. We shall say that a function $\varphi(x)$ fulfils the *unrestricted condition* (S) on a set E, if there corresponds to each $\varepsilon>0$ a number $\delta>0$ such that for every set $X\subset E$ the inequality $|X|<\delta$ implies $|\varphi[X]|<\varepsilon$.

It we restrict here the set $X\subset E$ to a measurable one, we obtain the definition of the condition (S) on E. The two conditions are equivalent in the case where E is itself a measurable set, as shown on p. 11 of [4].

THEOREM 15. *Every function $\varphi(x)$ which fulfils the condition* (W) *on a set E, fulfils the unrestricted condition* (S) *on this set.*

The converse of this assertion also holds good, provided that the set E is an interval I and the function $\varphi(x)$ is continuous on I.

PROOF. (i) Suppose that a function $\varphi(x)$ fulfils the condition (W) on a set E. Given any $\varepsilon>0$, let δ be the positive number that appears in the definition of the condition (W).

Let us show that $|\varphi[X]|<\varepsilon$ for every set $X\subset E$ with outer measure $|X|<\delta$. Clearly X is coverable by a non-overlapping family of closed intervals, say $\mathfrak{M}$, such that $|\mathfrak{M}|<\delta$. It follows that

$$\varphi[X]=\bigcup_I \varphi[X\cap I]\subset\bigcup_I \varphi[E\cap I]\subset\square(\varphi;E;\mathfrak{M}),$$

where I ranges over the family $\mathfrak{M}$. Then $|\varphi[X]|\leqq|\square(\varphi;E;\mathfrak{M})|<\varepsilon$, which establishes the first half of the assertion.

(ii) Suppose that a function $\varphi(x)$ is continuous on an interval I and fulfils the unrestricted condition (S) on this interval. Then there corresponds to each $\varepsilon>0$ a number $\delta>0$ such that for every set $X\subset I$ the inequality $|X|<\delta$ implies $|\varphi[X]|<\varepsilon$.

Consider now any non-overlapping family $\mathfrak{M}$ of closed intervals. We shall prove that $|\square(\varphi;I;\mathfrak{M})|<\varepsilon$ whenever $|\mathfrak{M}|<\delta$. Let K be a generic interval of $\mathfrak{M}$. Then $I\cap K$ is a connected set, and hence so is also the set $\varphi[I\cap K]$ in virtue of the continuity of $\varphi(x)$ on I. It follows that $\varphi[I\cap K]$ can differ from the set $\square\varphi[I\cap K]$ at most by two points. Furthermore, $\square\varphi[I\cap K]=\varphi[I\cap K]$ if $I\cap K$ is compact. Thus we have $\square\varphi[I\cap K]\neq\varphi[I\cap K]$ only when $I\cap K$ is not compact. But it is evident that this last condition is fulfilled by at most two of the intervals K. Consequently, writing for short $M=[\mathfrak{M}]$, we see that the difference $\square(\varphi;I;\mathfrak{M})\setminus\varphi[I\cap M]$ is a finite set, so that $|\square(\varphi;I;\mathfrak{M})|=|\varphi[I\cap M]|$. If now $|M|=|\mathfrak{M}|<\delta$, then $|I\cap M|<\delta$ and hence $|\varphi[I\cap M]|<\varepsilon$. We thus find that, as desired,

$$|\square(\varphi;I;\mathfrak{M})|<\varepsilon \qquad \text{whenever } |\mathfrak{M}|<\delta.$$

The function $\varphi(x)$ therefore fulfils the condition (W) on I.

REMARK. The continuity of the function $\varphi(x)$ on the interval I is not a superfluous hypothesis in the converse part of the theorem. To see this, we need only consider the case in which $\varphi(x)$ is null on I (i. e. maps I onto a null set) without being continuous on I. It is obvious that such a function always fulfils the unrestricted condition (S) on I without fulfilling the condition (W) on this interval.

THEOREM 16. *Every function $\varphi(x)$ which is subject to the condition* (W) *on a set E and continuous on the closure Q of E, fulfils the condition* (W) *on the whole set Q.*

PROOF. The following argument consists in reducing the assertion to two inequalities. The deduction of the first one of them, being quite similar to that of $|\overline{\lim}\,(\square\varphi;S;\mathfrak{N}_1)|=0$ in the proof for Theorem 8, will be stated concisely.

Given any $\varepsilon>0$, let δ be the number which appears in the definition of the condition (W), and suppose that a non-overlapping finite family $\mathfrak{M}$ of closed intervals fulfils $|\mathfrak{M}|<\delta$. The theorem will be established if we show that $|\square(\varphi;Q;\mathfrak{M})|<3\varepsilon$. We may assume that every interval of $\mathfrak{M}$ pertains to Q.

As in the proof of Theorem 8, we introduce the temporary notion of an admissible interval (with respect to E). A closed interval I will be called *admissible,* if I pertains to the closure of the set $E\cap I$. Then any interval J of $\mathfrak{M}$ is expressible as the union of at most three closed intervals non-overlapping and each of which is either admissible or contiguous to the set Q. If we replace each $J\in\mathfrak{M}$ by these intervals, there results a new non-overlapping finite family, say $\mathfrak{N}$, of closed intervals. We find at once that

$$|\mathfrak{M}|=|\mathfrak{N}| \quad\text{and}\quad \square(\varphi;Q;\mathfrak{M})=\square(\varphi;Q;\mathfrak{N})\,.$$

Let $\mathfrak{N}_1$ be the family of the admissible intervals belonging to $\mathfrak{N}$, and let us write $\mathfrak{N}_2=\mathfrak{N}\setminus\mathfrak{N}_1$. Then obviously

$$\square(\varphi;Q;\mathfrak{N})=\square(\varphi;Q;\mathfrak{N}_1)\cup\square(\varphi;Q;\mathfrak{N}_2)\,,$$

and hence it suffices to prove that

$$|\square(\varphi;Q;\mathfrak{N}_1)|<2\varepsilon \quad\text{and}\quad |\square(\varphi;Q;\mathfrak{N}_2)|<\varepsilon\,.$$

Both $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are finite together with $\mathfrak{N}$.

Let us arrange the intervals of the family $\mathfrak{N}_1$ in a distinct sequence $\langle I_1,\cdots,I_m\rangle$, assuming $m>0$ as we may. These intervals are admissible by definition of $\mathfrak{N}_1$, and the function $\varphi(x)$ is continuous on Q by hypothesis. We can therefore attach to each $n=1,\cdots,m$ a closed interval $K_n\subset I_n$ pertaining to E and such that $\mathrm{d}(\varphi;Q;I_n\ominus K_n)<m^{-1}\varepsilon$. We then have

$$\square\varphi[Q\cap I_n]=\square\varphi[Q\cap K_n]\cup\square(\varphi;Q;I_n\ominus K_n)\,,$$

$$|\square(\varphi;Q;I_n\ominus K_n)|\leqq\mathrm{d}(\varphi;Q;I_n\ominus K_n)<m^{-1}\varepsilon\,.$$

We find moreover that $\square\varphi[Q\cap K_n]=\square\varphi[E\cap K_n]$ and $|\mathfrak{N}_1|\leqq|\mathfrak{N}|=|\mathfrak{M}|<\delta$. It thus follows successively that

$$\square(\varphi;Q;\mathfrak{N}_1)=\square(\varphi;E;\mathfrak{N}_1)\cup\bigcup_n\square(\varphi;Q;I_n\ominus K_n)\,,$$

$$|\square(\varphi;Q;\mathfrak{N}_1)|\leqq|\square(\varphi;E;\mathfrak{N}_1)|+\sum_n|\square(\varphi;Q;I_n\ominus K_n)|<2\varepsilon\,.$$

It remains to show that $|\square(\varphi;Q;\mathfrak{N}_2)|<\varepsilon$. Assuming $\mathfrak{N}_2$ nonvoid as we may, let us now arrange all the intervals of $\mathfrak{N}_2$ in a distinct sequence, say $\langle H_1, \cdots, H_k\rangle$. These intervals are contiguous to Q by definition of $\mathfrak{N}_2$. On the other hand, we have $|\mathfrak{N}_2|<\delta$. Consequently for each $n=1, \cdots, k$ we can enclose the interval H_n in a closed interval L_n pertaining to the set E, in such a way that $\langle L_1, \cdots, L_k\rangle$ is a non-overlapping sequence fulfilling the inequality $|L_1|+\cdots+|L_k|<\delta$. We then find successively the relations

$$\square\varphi[Q\cap H_n]\subset\square\varphi[Q\cap L_n]=\square\varphi[E\cap L_n] \qquad \text{for each } n,$$

$$\square(\varphi;Q;\mathfrak{N}_2)=\bigcup_n \square\varphi[Q\cap H_n]\subset\bigcup_n \square\varphi[E\cap L_n]=\square(\varphi;E;\mathfrak{L}),$$

where $\mathfrak{L}$ denotes the family of the intervals $L_1, \cdots, L_k$. But we have $|\mathfrak{L}|<\delta$, and it follows finally that

$$|\square(\varphi;Q;\mathfrak{N}_2)|\leqq|\square(\varphi;E;\mathfrak{L})|<\varepsilon,$$

which completes the proof of the theorem.

DEFINITION. A function $\varphi(x)$ will be said to fulfil the *condition* (R) on a set E, if there corresponds to each $\varepsilon>0$ a number $\delta>0$ such that for every compact set C with $|C|<\delta$, the closure of the image $\varphi[E\cap C]$ has measure $<\varepsilon$.

THEOREM 17. *Let $\varphi(x)$ be a function which is continuous over the closure Q of a set E. In order that the function fulfil the condition* (R) *on E, it is necessary and sufficient that $\varphi(x)$ fulfil the condition* (S) *on Q.*

PROOF. (i) Necessity. Supposing $\varphi(x)$ to fulfil the condition (R) on E, let ε and δ mean the same numbers as in the above definition. We shall show that $|\varphi[M]|\leqq\varepsilon$ for each set $M\subset Q$ with $|M|<\delta$. The set M is coverable by an infinite sequence of open intervals, say $\langle I_1, I_2, \cdots\rangle$, such that $|I_1|+|I_2|+\cdots<\delta$. Let C_n denote for $n\in\boldsymbol{N}$ the closure of the open set $D_n=I_1\cup\cdots\cup I_n$. We then have

$$|\overline{\varphi[E\cap D_n]}|=|\overline{\varphi[E\cap C_n]}|<\varepsilon \qquad \text{for every } n,$$

since the set $C=\overline{I}_1\cup\cdots\cup\overline{I}_n$ is compact and has measure $|C_n|<\delta$.

This being so, consider the set $M\cap D_n\subset Q$, keeping n fixed for the moment. Since Q is the closure of E, each point p of $M\cap D_n$ is expressible as the limit of an infinite point sequence chosen from the set E, say $\langle p_1, p_2, \cdots\rangle$. But p belongs to the open set D_n, so that we may assume this sequence contained in the set $E\cap D_n$. It follows that

$$\varphi(p)=\lim_i \varphi(p_i)\in\overline{\varphi[E\cap D_n]}\,.$$

Hence $\varphi[M\cap D_n]\subset\overline{\varphi[E\cap D_n]}$, which together with $|\overline{\varphi[E\cap D_n]}|<\varepsilon$ yields the inequality $|\varphi[M\cap D_n]|<\varepsilon$.

The index n, kept fixed hitherto, will now be made to vary. The sequence $\langle M\cap D_n;\, n\in \boldsymbol{N}\rangle$ is ascending and has M for its limit, since

$$\lim_n (M\cap D_n)=M\cap\lim_n D_n=M\cap(I_1\cup I_2\cup\cdots)=M\,.$$

The ascending sequence $\langle\varphi[M\cap D_n];\, n\in\boldsymbol{N}\rangle$ must therefore tend to the set $\varphi[M]$. On the the other hand, we have $|\varphi[M\cap D_n]|<\varepsilon$ as already shown. It thus follows, as desired, that $|\varphi[M]|=\lim_n|\varphi[M\cap D_n]|\leqq\varepsilon$.

We observe that the continuity of $\varphi(x)$ on Q was not used in the above argument.

(ii) Sufficiency. Supposing $\varphi(x)$ continuous on Q and to fulfil the condition (S) on Q, let us show that it fulfils the condition (R) on E. There corresponds to each $\varepsilon>0$ a number $\delta>0$ such that for every set $M\subset Q$, the condition $|M|<\delta$ implies $|\varphi[M]|<\varepsilon$. Now consider any compact set C with $|C|<\delta$ and write $T=Q\cap C$. Since $T\subset Q$ and $|T|\leqq|C|<\delta$, we have $|\varphi[T]|<\varepsilon$. But T is a compact set, and the function $\varphi(x)$ is continuous on Q. The image $\varphi[T]$ is therefore compact. This, together with the inclusion $E\subset Q$, shows that

$$\overline{\varphi[E\cap C]}\subset\overline{\varphi[Q\cap C]}=\overline{\varphi[T]}=\varphi[T]\,.$$

It follows that $|\overline{\varphi[E\cap C]}|\leqq|\varphi[T]|<\varepsilon$, which completes the proof.

We are now ready to generalize Theorem (b) of the Introduction to the following proposition.

THEOREM 18. *Let E be a set such that $\square E$ is a closed interval. In order that a function $\varphi(x)$ be* AC *superposable on E, each of the following three properties of $\varphi(x)$ is necessary and sufficient.*

(1) *The function $\varphi(x)$ coincides on E with a function which is continuous on $\square E$ and which fulfils the conditions* (S) *on $\square E$;*

(2) *the function $\varphi(x)$ fulfils the condition* (W) *on E;*

(3) *the function $\varphi(x)$ is uniformly continuous on E and fulfils both the conditions* (R) *and* (U) *on this set.*

PROOF. Let us write $I=\square E$ for short.

(i) Property (1) is necessary and sufficient. The sufficiency of

(1) being immediate from Theorem (b), we need only show its necessity. Let $\varphi(x)$ be a function which admits on the set E an AC superposition $\varphi(x)=\theta\circ\psi(x)$. Then the function $\psi(x)$, which is AC on E, is uniformly continuous on E. Hence we may, without loss of generality, assume $\psi(x)$ continuous on the closure Q of E and linear on every closed interval contiguous to Q. Then $\psi(x)$ is AC on Q and so, by Theorem 15 of [1], AC on the whole interval I.

Quite similarly the function θ, which is AC on the bounded set $\psi[E]$, may be supposed AC on the set $\square\psi[E]$.

Now $\psi[I]$ coincides with $\square\psi[Q]$, since $\psi(x)$ is linear on every closed interval contiguous to Q. On the other hand, the set $\psi[Q]$ evidently coincides with the closure of $\psi[E]$ and is therefore situated in $\square\psi[E]$. Thus we have the relation $\psi[I]=\square\psi[Q]\subset\square\psi[E]$, whence we get $\psi[I]=\square\psi[E]$ since plainly $\square\psi[E]\subset\square\psi[Q]$. It follows that the function θ is AC on $\psi[I]$. Then the function $\theta\circ\psi(x)$ is continuous on I and, by Theorem (b), fulfils the condition (S) on I. This establishes the necessity of (1).

(ii) Property (1) implies property (2). Let $\varphi(x)$ be a function which has property (1). Without loss of generality we may assume $\varphi(x)$ itself to be continuous on I and subject to the condition (S) on this interval, so that $\varphi(x)$ fulfils the unrestricted condition (S) on I. On account of Theorem 15, the function $\varphi(x)$ then fulfils the condition (W) on I, and *a fortiori* on the subset E of I. This establishes the implication (1)$\Rightarrow$(2).

(iii) Property (2) implies property (3). Let $\varphi(x)$ be a function which fulfils the condition (W) on the set E. Then $\varphi(x)$ is uniformly continuous on E. Thus we may assume $\varphi(x)$ continuous on the closure Q of E. It ensues from Theorem 16 that $\varphi(x)$ fulfils the condition (W) on the whole set Q. Then $\varphi(x)$ fulfils the condition (S) on Q on account of Theorem 15. This, together with Theorem 17, shows that $\varphi(x)$ fulfils the condition (R) on E.

It remains to verify that $\varphi(x)$ fulfils the condition (U) on E. There corresponds to each $\varepsilon>0$ a number $\delta>0$ such that for any non-overlapping family $\mathfrak{M}$ of closed intervals, we have $|\square(\varphi\,;E\,;\mathfrak{M})|<\varepsilon$ whenever $|\mathfrak{M}|<\delta$. Consider an arbitrary non-overlapping infinite sequence of closed intervals, say $\langle K_1, K_2, \cdots\rangle$. We are to show that the set

$$\overline{\lim_n}\,\square\varphi[E\cap K_n]=\bigcap_n\bigcup_{i>n}\square\varphi[E\cap K_i]$$

is null. For this purpose, we may clearly confine ourselves to the case in which $E\cap K_n$ is nonvoid for every n. Replacing, if necessary, the interval K_n by the closed interval $K_n\cap[a-1, b+1]$, where we write $\square E=[a, b]$, we

may assume that $K_n \subset [a-1, b+1]$ for every n. Now denote by $\mathfrak{M}_n$ the family of the intervals $K_{n+1}, K_{n+2}, \cdots$. Then

$$\bigcup_{i>n} \square\varphi[E \cap K_i] = \square(\varphi; E; \mathfrak{M}_n),$$

and we have $|\mathfrak{M}_p| = |K_{p+1}| + |K_{p+2}| + \cdots < \delta$ for sufficiently large p on account of $|K_1| + |K_2| + \cdots \leqq b-a+2$. Choosing such a p we get

$$\left|\overline{\lim_n} \square\varphi[E \cap K_n]\right| \leqq \left|\bigcup_{i>p} \square\varphi[E \cap K_i]\right| = |\square(\varphi; E; \mathfrak{M}_p)| < \varepsilon .$$

Since ε is arbitrary, this implies that $|\overline{\lim} \square\varphi[E \cap K_n]| = 0$.

(iv) Property (3) implies property (1). The function $\varphi(x)$, which is uniformly continuous on E, may be assumed continuous on the closure Q of E and linear on every closed interval contiguous to Q. It follows from Theorem 9 that $\varphi(x)$ fulfils the condition (T_1) on the set $\square Q$, which coincides with the interval $I = \square E$.

Now $\varphi(x)$ is subject on E to the condition (R) and hence, by Theorem 17, fulfils the condition (S) on Q. Consequently it only remains to show that $\varphi(x)$ fulfils the condition (S) on the open set $D = I \setminus Q$, or equivalently, that $\varphi(x)$ fulfils on D both the conditions (T_1) and (N). Indeed, the equivalence is ensured by Theorem 17 of [4]. But $\varphi(x)$ is linear on every component interval of D and hence evidently fulfils the condition (N) on D. On the other hand, $\varphi(x)$ fulfils the condition (T_1) on I, as already verified, and *a fortiori* on D. This completes the proof of the implication (3)$\Rightarrow$(1).

THEOREM 19. *Supposing that $\varphi(x)$ is an arbitrary function and E an arbitrary set, resume the properties* (1), (2), *and* (3) *of Theorem* 18 *and consider the implications*

$$(1) \Rightarrow (2), \quad (2) \Rightarrow (1), \quad (1) \Rightarrow (3), \quad (3) \Rightarrow (1), \quad (2) \Rightarrow (3), \quad (3) \Rightarrow (2).$$

Of these the first one is true, but the others are false.

PROOF. Concerning (1)$\Rightarrow$(2). Part (ii) of the proof of the foregoing theorem holds good, as it stands, for any function $\varphi(x)$ and any set E, irrespective as to whether the set $I = \square E$ is a closed interval or not.

Concerning (2)$\Rightarrow$(1). Let us specialize E to the set of the integers, so that $\square E$ is the real line. The function $\varphi(x) = x^3$ clearly fulfils the condition (W) on E. Now suppose, if possible, that $\varphi(x)$ has the property (1), or in other words, that there is a continuous function $f(x)$ which equals x^3 for integral values of x and which fulfils the condition (S) on the real line. There then exists a number $\delta > 0$ such that we have $|f[K]| < 1$ for

every closed interval K with $|K|\leqq\delta$. Assuming, as we evidently may, that $\delta=q^{-1}$ for an integer $q>0$, consider any interval $A=[n, n+1]$ where $n\in\boldsymbol{N}$. Then A is the union of the q intervals

$$A_i=[n+(i-1)\delta,\ n+i\delta], \qquad \text{where } i=1, \cdots, q.$$

The function $f(x)$ being continuous, it follows that

$$f(A)=\sum_{i=1}^{q} f(A_i)\leqq\sum_{i=1}^{q}|f[A_i]|<q\,,$$

where we have $f(A)=f(n+1)-f(n)=3n^2+3n+1$. Since n can vary independently of q, we have arrived at a contradiction.

Thus the function $\varphi(x)$ does not possess the property (1), and the implication (2)$\Rightarrow$(1) has been disproved.

Concerning (1)$\Rightarrow$(3). Plainly the function $\varphi(x)=\sin x$ fails to fulfil the condition (U) on the real line. But this function fulfils the condition (S) on $\boldsymbol{R}$, since Theorem (6.5) on p. 227 of Saks [6] shows that for every measurable set X we have

$$|\varphi[X]|\leqq\int_X|\varphi'(x)|\,dx=\int_X|\cos x|\,dx\leqq|X|\,.$$

The implication (1)$\Rightarrow$(3) is thus false.

Concerning (3)$\Rightarrow$(1). Resume the function $\varphi(x)=x^3$ and the set E of all the integers. It is clear that on this set $\varphi(x)$ is uniformly continuous and subject to both the conditions (R) and (U). Thus $\varphi(x)$ possesses the property (3). On the other hand, this function is devoid of the property (1), as already seen in the disproof of (2)$\Rightarrow$(1). The implication (3)$\Rightarrow$(1) is therefore false.

Concerning (2)$\Rightarrow$(3). The function $\varphi(x)=\sin x$ and the set $E=\boldsymbol{R}$ together show at once the falseness of this implication.

Concerning (3)$\Rightarrow$(2). Let m and n stand generically for an integer and a positive integer, respectively, and let E be the set of the numbers $m-n^{-1}$, so that E is a countable closed set. Consider the function $\varphi(x)$ which is linear on every closed interval contiguous to E and whose value for $x=m-n^{-1}$ is determined by

$$\varphi(m-n^{-1})=m\log 2+s_{n-1}\,,$$

where $s_0=0$ and s_n denotes the nth partial sum of the series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\log 2\,.$$

It is obvious that the function $\varphi(x)$ is uniquely determined on the real line.

As readily seen, $\varphi(x)$ is uniformly continuous on $\boldsymbol{R}$. Furthermore, since E is a countable closed set, every function which is continuous on E, and in particular the function $\varphi(x)$, fulfils the condition (R) on E. On the other hand, $\varphi(x)$ fulfils the condition (T_1) on the real line, since each value assumed infinitely often by $\varphi(x)$ is clearly of the form $m \log 2$. This, together with the continuity of $\varphi(x)$, implies that $\varphi(x)$ fulfils the condition (U) on $\boldsymbol{R}$. Thus the function $\varphi(x)$ has the property (3).

It remains to ascertain that $\varphi(x)$ is devoid of the property (2). For this purpose, let us consider the interval

$$I(k, n)=[2k-n^{-1}, 2k-(n+1)^{-1}],$$

where k and n are positive integers. The image $\varphi[I(k, n)]$ of this interval is the closed interval with the end points

$$2k \log 2+s_{n+1} \quad \text{and} \quad 2k \log 2+s_n .$$

It follows at once that $|\varphi[I(k, n)]|=n^{-1}$ and that

$$\varphi[I(k, n)] \subset [2k \log 2,\ 2(k+1) \log 2] .$$

We shall keep n fixed from now on. This inclusion then shows that the intervals $\varphi[I(k, n)]$, where k ranges over $\boldsymbol{N}$, are mutually non-overlapping. Now let $\mathfrak{M}_n$ be the family of the n intervals $I(k, n)$, where $k=1, \cdots, n$. Noting that $\square\varphi[E \cap I(k, n)]=\varphi[I(k, n)]$ for every k, we have

$$|\square(\varphi; E; \mathfrak{M}_n)|=\left|\bigcup_{k=1}^{n} \square\varphi[E \cap I(k, n)]\right|=\sum_{k=1}^{n} |\varphi[I(k, n)]|=1 .$$

As to the measure $|\mathfrak{M}_n|$, on the other hand, we find that

$$|\mathfrak{M}_n|=\sum_{k=1}^{n} |I(k, n)|=n\left(\frac{1}{n}-\frac{1}{n+1}\right)=\frac{1}{n+1} .$$

The function $\varphi(x)$ thus fails to possess the property (2). Indeed, the index n is originally arbitrary, although we have kept it fixed.

THEOREM 20. *If on a set E a function $\varphi(x)$ is the superposition of two functions of which the inner function $\psi(x)$ and the outer function θ fulfil the condition* (W) *on E and on $\psi[E]$ respectively, then the function $\varphi(x)$ itself fulfils this condition on E.*

PROOF. By hypothesis, given any $\varepsilon>0$ there is a number $\eta>0$ such that for every non-overlapping family $\mathfrak{N}$ of closed intervals the inequality $|\mathfrak{N}|<\eta$ implies $|\square(\theta; S; \mathfrak{N})|<\varepsilon$, where we write $S=\psi[E]$ for brevity. To

the number η there corresponds, by hypothesis, a $\delta>0$ such that for every non-overlapping family $\mathfrak{M}$ of closed intervals the relation $|\mathfrak{M}|<\delta$ implies $|\square(\psi;E;\mathfrak{M})|<\eta$. It is enough to show that $|\mathfrak{M}|<\delta$ implies $|\square\varphi;E;\mathfrak{M})|<\varepsilon$, where we may suppose $\mathfrak{M}$ to be a finite family.

Given $\mathfrak{M}$ as above, let I be a generic interval of $\mathfrak{M}$. Then

$$\square(\varphi;E;\mathfrak{M})=\bigcup_I \square\varphi[E\cap I]=\bigcup_I \square(\theta\circ\psi[E\cap I]).$$

If now for an I the set $\psi[E\cap I]$ consists of at most one point, the set $\square(\theta\circ\psi[E\cap I])$ clearly does so, too. We may therefore assume that $\psi[E\cap I]$ contains at least two points for every I. On the other hand, the function $\psi(x)$ is uniformly continuous on E, so that every $\psi[E\cap I]$ is a bounded set. It ensues that the set $J=\square\psi[E\cap I]$ is a closed interval for every I. $\mathfrak{M}$ being a finite family, the union of all the intervals J is a figure. We now specialize the family $\mathfrak{N}$ considered above to the family of the component intervals of this figure. Then for each I the interval J is contained in an interval $K\in\mathfrak{N}$ and hence

$$\psi[E\cap I]\subset\psi[E]\cap J=S\cap J\subset S\cap K.$$

Consequently it follows that

$$\square(\varphi;E;\mathfrak{M})=\bigcup_I \square(\theta\circ\psi[E\cap I]\subset\bigcup_I \square\theta[S\cap K]\subset\square(\theta;S;\mathfrak{N})$$

and therefore that $|\square(\varphi;E;\mathfrak{M})|\leqq|\square(\theta;S;\mathfrak{N})|$.

Since $|\mathfrak{M}|<\delta$ by hypothesis and further $[\mathfrak{N}]=\cup\square\psi[E\cap I]=\square(\psi;E;\mathfrak{M})$, we have $|\mathfrak{N}|=|\square(\psi;E;\mathfrak{M})|<\eta$, which implies that $|\square(\theta;S;\mathfrak{N})|<\varepsilon$. Hence we obtain finally $|\square\varphi;E;\mathfrak{M})|<\varepsilon$, and this completes the proof.

DEFINITION. Let α be any finite nonnegative number. We shall say that a function $\varphi(x)$ is *angular* (α) *relatively to a set* E *at a point* $p\in\boldsymbol{R}$, if there exists a number $\delta>0$ such that we have

$$|\varphi(x)-\varphi(p)|\leqq\alpha|x-p| \quad \text{whenever } x\in E \text{ and } |x-p|<\delta.$$

It is worth notice that the point p itself need not belong to E.

When we are interested only in the existence of the above number α and not in its peculiar value, we shall simply say that $\varphi(x)$ is *angular at* p *relatively to* E. Plainly, the reference to the set E is unnecessary in the particular case in which the point p is interior to E, so that in this case the function will usually be called *angular* (α) [or *angular*] *at* p. Finally, if for instance we say that $\varphi(x)$ is *angular* (α) *relatively to* E, we shall mean thereby that the function is angular (α) relatively to E at every point of E. Thus a function which is angular (α) relatively to E

need not be angular (α) at every point of E.

THEOREM 21. *If a function $\varphi(x)$ is angular (α) relatively to a set E for a number $\alpha \geqq 0$, we necessarily have $|\varphi[E]| \leqq \alpha |E|$, where the product $0 \cdot (+\infty)$ means zero.*

The proof may be omitted, since it runs almost the same as for the following lemma stated on p. 226 of Saks [6]. *If for a function $f(x)$ the inequalities $\bar{f}^{+}(x) \leqq \lambda$ and $\underline{f}^{-}(x) \geqq -\lambda$, where λ is any finite nonnegative number, hold at every point x of a set M, then $|f[M]| \leqq \lambda |M|$.*

DEFINITION. A function $\varphi(x)$ will be said to fulfil the *condition* (A) on a set E, if the outer measure $|\varphi[E(\alpha)]|$ tends to 0 as $\alpha \to +\infty$, where $E(\alpha)$ is the set of the points of E at which the function $\varphi(x)$ fails to be angular (α) relatively to E.

THEOREM 22. *Every function $\varphi(x)$ which fulfils the condition* (W) *on a set E, fulfils the condition* (A) *on this set.*

However, the converse of this assertion is false, even when the function is continuous on E.

PROOF. For each $\alpha \geqq 0$ let $E(\alpha)$ denote the set of the points of E at which the function $\varphi(x)$ fails to be angular (α) relatively to E. It is obvious that $E(\alpha)$ descends for increasing α. Hence $\varphi(x)$ fulfils the condition (A) on the set E, if and only if there is no number $\eta > 0$ such that $|\varphi[E(\alpha)]| > \eta$ for every $\alpha \geqq 0$.

Supposing, if possible, that there exists such a number η, we shall derive a contradiction. Since the function $\varphi(x)$ fulfils the condition (W) on E, there is a number $\delta > 0$ such that for every non-overlapping family $\mathfrak{M}$ of closed intervals, the inequality

$$|\mathfrak{M}| < \delta \quad \text{implies} \quad |\square(\varphi; E; \mathfrak{M})| < \eta .$$

The numbers η and δ will be kept fixed during the proof.

Let us consider the set $D = \varphi[E(\alpha_0)]$, where we write $\alpha_0 = 2\delta^{-1}\eta$. We shall construct a family $\mathfrak{S}$ of closed intervals covering D in the Vitali sense. Let p be any point of the set $E(\alpha_0)$, which is nonvoid since $|D| > \eta$. The function $\varphi(x)$ being continuous on E and not angular (α_0) at p relatively to E, the points ξ of E such that

$$\alpha_0 |\xi - p| < |\varphi(\xi) - \varphi(p)| < \eta$$

together form a set for which p is an accumulation point. For each ξ let us denote by $H(p, \xi)$ the closed interval with $\varphi(p)$ and $\varphi(\xi)$ for its end

points. Associating with p the family $\mathfrak{S}(p)$ of all the intervals $H(p, \xi)$ and defining $\mathfrak{S}$ to be the union of $\mathfrak{S}(p)$ for the points p of $E(\alpha_0)$, we see at once that the family $\mathfrak{S}$ covers the set D in the Vitali sense.

By Vitali's Covering Theorem, the family $\mathfrak{S}$ contains a disjoint subfamily $\mathfrak{N}$ which covers D almost entirely and whose union $[\mathfrak{N}]$ therefore has measure $|\mathfrak{N}| \geqq |D| = |\varphi[E(\alpha_0)]| > \eta$. For each interval $K \in \mathfrak{N}$ the set E contains two distinct points p and ξ such that

$$\alpha_0 |\xi - p| < |\varphi(\xi) - \varphi(\eta)| < \eta \quad \text{and} \quad K = H(p, \xi) .$$

We now associate with K the closed interval with p and ξ for its end points. Since $\mathfrak{N}$ is a disjoint family and since $|K| < \eta < |\mathfrak{N}|$ for every K, we can extract from $\mathfrak{N}$ a finite disjoint sequence of closed intervals, say $\langle K_1, \cdots, K_m \rangle$, such that $\eta < |K_1| + \cdots + |K_m| < 2\eta$. For each $n = 1, \cdots, m$ let I_n be the closed interval associated with K_n in the above manner, so that $\alpha_0 |I_n| < |K_n|$. Writing $Z = I_1 \cup \cdots \cup I_m$, we then have

$$|Z| \leqq |I_1| + \cdots + |I_m| < 2\eta / \alpha_0 = \delta .$$

It therefore follows from the choice of δ that the set $\square(\varphi; E; Z)$, namely the union of the sets $\square\varphi[E \cap L]$, where L is a generic component of the figure Z, has outer measure $< \eta$. Now for each $n = 1, \cdots, m$ the interval I_n pertains to E and is contained in some L, say L_n, and we thus find that

$$K_n \subset \square\varphi[E \cap I_n] \subset \square\varphi[E \cap L_n] \subset \square(\varphi; E; Z) .$$

But the intervals $K_1, \cdots, K_m$ are mutually disjoint. Accordingly

$$|K_1| + \cdots + |K_m| \leqq |\square(\varphi; E; Z)| < \eta ,$$

which contradicts $|K_1| + \cdots + |K_m| > \eta$. This proves that $\varphi(x)$ fulfils the condition (A) on E.

It remains to show that the condition (A) on E does not always imply the condition (W) on E, even if the function is continuous on E. Given a closed interval I and a countable set E whose closure is I, let $\varphi(x)$ be any function which is continuous on I without fulfilling the condition (N) on I. Such a function evidently fulfils the condition (A) on E. Now, if $\varphi(x)$ fulfils the condition (W) on E, then $\varphi(x)$ does so too on I by Theorem 16 and must therefore fulfil the condition (S) on I by Theorem 15. It follows that $\varphi(x)$ fulfils the condition (N) on I, contradicting the choice of $\varphi(x)$. Hence the result.

THEOREM 23. *Given a function $\varphi(x)$ which is continuous on a Borel set B, let $B(\alpha)$ denote for $\alpha \geqq 0$ the set of the points of B at which the function $\varphi(x)$ fails to be angular (α) relatively to B. Then $B(\alpha)$ is a Borel*

set and hence $\varphi[B(\alpha)]$ *is a measurable set, for every* $\alpha \geqq 0$.

PROOF. Keeping α fixed, let us denote for each $n \in \boldsymbol{N}$ by B_n the set of the points p of B such that for every $x \in B$ we have

$$|\varphi(x)-\varphi(p)| \leqq \alpha |x-p| \qquad \text{whenever } |x-p|<n^{-1}.$$

We shall prove first that B_n is a Borel set. For this purpose, it suffices to show that every point q of the set $B \setminus B_n$ is exterior to B_n. In fact, once this is established, then $B \setminus B_n$ is contained in an open set G disjoint with B_n, so that B_n, which coincides with $B \setminus G$, will turn out to be a Borel set.

Suppose if possible that a point q of $B \setminus B_n$ is not exterior to B_n. We then can extract from B_n an infinite sequence of points, say $\langle q_1, q_2, \cdots \rangle$, which converges to q. But the definition of the sets B_n implies that for every $x \in B$ and every $i \in \boldsymbol{N}$ we have

$$|\varphi(x)-\varphi(q_i)| \leqq \alpha |x-q_i| \qquad \text{whenever } |x-q_i|<n^{-1}.$$

Now consider any point $\xi \in B$ such that $|\xi-q|<n^{-1}$. Since the sequence $\langle q_i \rangle$ converges to q, the inequality $|q_i-q|<n^{-1}-|\xi-q|$ holds for large values of i. Since this inequality implies $|\xi-q_i|<n^{-1}$, it follows that $|\varphi(\xi)-\varphi(q_i)| \leqq \alpha |\xi-q_i|$ for large i. Making $i \to +\infty$ here and using the continuity on E of the function $\varphi(x)$, we get in the limit the inequality $|\varphi(\xi)-\varphi(q)| \leqq \alpha |\xi-q|$. Accordingly we have $q \in B_n$, which contradicts that $q \in B \setminus B_n$. We thus find that B_n is a Borel set.

By definition of the set $B(\alpha)$, the difference $M=B \setminus B(\alpha)$ is the set of the points of B at which $\varphi(x)$ is angular (α) relatively to B. Hence M is the limit of the ascending sequence $B_1 \subset B_2 \subset \cdots$, which consists exclusively of Borel sets. Thus M is itself a Borel set, and so is also the set $B(\alpha)=B \setminus M$.

THEOREM 24. *Given a nnmber* $h>0$ *and a family* $\mathfrak{S}$ *of closed intervals, suppose that* $|\mathfrak{T}| \leqq h$ *for every disjoint subfamily* $\mathfrak{T}$ *of* $\mathfrak{S}$. *Then necessarily* $|\mathfrak{S}| \leqq 5h$.

REMARK. To prevent any ambiguity, let us mention once more that for each family $\mathfrak{M}$ of linear sets we denote by $|\mathfrak{M}|$ the outer measure of the union $[\mathfrak{M}]$ of $\mathfrak{M}$. In particular, if $\mathfrak{M}$ is void, then so is $[\mathfrak{M}]$ and we have $|\mathfrak{M}|=0$.

PROOF. Given a subfamily $\mathfrak{N}$ of $\mathfrak{S}$, let $\mathfrak{T}$ denote generically a disjoint subfamily of $\mathfrak{N}$. Then the collection of all the families $\mathfrak{T}$ is inductively ordered by family inclusion. Hence, by Zorn's Lemma, this collec-

tion contains a maximal family, say $f(\mathfrak{N})$, namely a maximal disjoint subfamily of $\mathfrak{N}$. We shall keep fixed this mapping f. The family $f(\mathfrak{N})$ is evidently countable and for each interval K of $\mathfrak{N}$ there exists in $f(\mathfrak{N})$ an interval which intersects K.

Let $\mathfrak{L}$ be a disjoint subfamily of a family $\mathfrak{V}\subset\mathfrak{S}$ and suppose that for each interval $V\in\mathfrak{V}$ there exists in $\mathfrak{L}$ an interval L intersecting V and fulfilling $|V|\leqq 2|L|$. We shall show that $|\mathfrak{V}|\leqq 5h$. For this purpose, we attach to each interval $J=[a, b]$ of $\mathfrak{L}$ the interval

$$J^*=[a-2|J|,\ b+2|J|]=[3a-2b,\ 3b-2a]\,.$$

Then $|J^*|=5|J|$ and consequently the union U of all the intervals J^* has measure $|U|\leqq 5|\mathfrak{L}|\leqq 5h$. In fact, the disjointness of $\mathfrak{L}$ implies $|\mathfrak{L}|\leqq h$ by hypothesis. On the other hand, if V and L have the same meanings as above, then plainly $V\subset L^*\subset U$. This implies $[\mathfrak{V}]\subset U$, and it follows that $|\mathfrak{V}|\leqq 5h$, as desired.

This being so, let $\mathfrak{S}_n$ be for each $n\in\boldsymbol{N}$ the family of all the intervals $I\in\mathfrak{S}$ such that $2^{-n}h<|I|\leqq 2^{1-n}h$. Then, since by hypothesis every interval of $\mathfrak{S}$ has length $\leqq h$, we have $\mathfrak{S}=\mathfrak{S}_1\cup\mathfrak{S}_2\cup\cdots\cdot$. We shall associate with each n a disjoint subfamily $\mathfrak{T}_n$ of $\mathfrak{S}_n$ by induction. Let $\mathfrak{T}_1=f(\mathfrak{S}_1)$ in the first place and suppose that the first n families $\mathfrak{T}_1, \cdots, \mathfrak{T}_n$ have already been constructed. Writing $T_n=[\mathfrak{T}_1\cup\cdots\cup\mathfrak{T}_n]$, let $\mathfrak{R}_{n+1}$ be the family of all the intervals of $\mathfrak{S}_{n+1}$ each of which is disjoint with the set T_n. We then define $\mathfrak{T}_{n+1}=f(\mathfrak{R}_{n+1})$.

Since for each n the set $[\mathfrak{R}_{n+1}]$ contains $[\mathfrak{T}_{n+1}]$ and is disjoint with T_n, it follows that the sequence $\langle[\mathfrak{T}_1], [\mathfrak{T}_2], \cdots\cdot\rangle$ is disjoint. Consequently, if we write for short

$$\mathfrak{V}_n=\mathfrak{S}_1\cup\cdots\cdot\cup\mathfrak{S}_n \quad\text{and}\quad \mathfrak{L}_n=\mathfrak{T}_1\cup\cdots\cdot\cup\mathfrak{T}_n\,,$$

then $\mathfrak{L}_n$ is a disjoint subfamily of $\mathfrak{V}_n$. We shall show, by induction, that for each $V\in\mathfrak{V}_n$ there exists in $\mathfrak{L}_n$ an interval L intersecting V and fulfilling $|V|\leqq 2|L|$. Let us denote this assertion by P_n. Then P_1 is obvious, since $\mathfrak{T}_1=f(\mathfrak{S}_1)$ by definition and since we have $2^{-1}h<|I|\leqq h$ for every $I\in\mathfrak{S}_1$. Let us proceed to verify P_{n+1}, assuming P_n true. For this purpose, consider any interval V of $\mathfrak{V}_{n+1}$ (we need only deal with the case where $\mathfrak{V}_{n+1}$ is nonvoid). Then V belongs either to $\mathfrak{V}_n$ or to $\mathfrak{S}_{n+1}$, and we may confine ourselves to the second alternative since P_n is true. We now distinguish two cases, according as V belongs to $\mathfrak{R}_{n+1}$ or to $\mathfrak{S}_{n+1}\setminus\mathfrak{R}_{n+1}$. In the first case, the family $f(\mathfrak{R}_{n+1})=\mathfrak{T}_{n+1}$ must contain an interval L which intersects V. But both L and V belong to $\mathfrak{S}_{n+1}$, so that

$$|V|\leqq 2^{-n}h=2\cdot 2^{-n-1}h<2|L|\,.$$

In the second case, on the other hand, the interval V intersects the set T_n, and hence there exists in $\mathfrak{L}_n$ an interval L which intersects V. Since $V \in \mathfrak{S}_{n+1}$ and $L \in \mathfrak{B}_n$, we have $|V| \leqq 2^{-n}h < |L|$, whence $|V| < 2|L|$. The assertion P_n is thus inductively established for every n.

This result, together with what we have already proved, leads to the inequality $|\mathfrak{B}_n| \leqq 5h$. On the other hand, the ascending infinite sequence $[\mathfrak{B}_1] \subset [\mathfrak{B}_2] \subset \cdots$ has the set $[\mathfrak{S}]$ for its limit, since $\mathfrak{S} = \mathfrak{S}_1 \cup \mathfrak{S}_2 \cup \cdots$. It thus follows that $|\mathfrak{S}| = \lim |\mathfrak{B}_n| \leqq 5h$, which completes the proof.

NOTATION. For each function $\varphi(x)$ and each family $\mathfrak{K}$ of linear sets, we shall denote by $\varphi[\mathfrak{K}]$ the image of the union $[\mathfrak{K}]$ under the mapping $\varphi(x)$.

THEOREM 25. *Given a function $\varphi(x)$ and a non-overlapping family $\mathfrak{M}$ of closed intervals I, suppose that this function is uniformly continuous on $[\mathfrak{M}]$, linear on every I, and further subject to the condition $|\varphi[I]| < h$ for every I, where h is a positive constant. If there corresponds to each $\varepsilon > 0$ a number $\delta > 0$ such that for every subfamily $\mathfrak{N}$ of $\mathfrak{M}$ the inequality $|\mathfrak{N}| < \delta$ implies $|\varphi[\mathfrak{N}]| < \varepsilon$, then the function $\varphi(x)$ necessarily fulfils the unrestricted condition* (S) *on* $[\mathfrak{M}]$.

PROOF. For each $\lambda > 0$ let $\mathfrak{M}(\lambda)$ be the family of the intervals $I \in \mathfrak{M}$ fulfilling $\lambda|I| < |\varphi[I]|$, so that for every $I \in \mathfrak{M}(\lambda)$ the image $\varphi[I]$ is a closed interval and we have $|I| < \lambda^{-1}h$. The image $\varphi[\mathfrak{M}(\lambda)]$ is plainly a sigma-compact set.

By hypothesis, there is a number $\rho > 0$ such that for every subfamily $\mathfrak{N}$ of $\mathfrak{M}$ the inequality $|\mathfrak{N}| < \rho$ implies $|\varphi[\mathfrak{N}]| < h$. Let us show first that $\varphi[\mathfrak{M}(\lambda)]$ has finite measure for every $\lambda > 2\rho^{-1}h$. For this purpose, suppose if possible that there exists a number $\alpha > 2\rho^{-1}h$ for which $|\varphi[\mathfrak{M}(\alpha)]| = +\infty$. If $\mathfrak{S}$ denotes the family of the intervals $\varphi[I]$, where $I \in \mathfrak{M}(\alpha)$, we have $|\mathfrak{S}| = +\infty$. Hence, by Theorem 24, the family $\mathfrak{S}$ contains a disjoint subfamily $\mathfrak{T}$ such that $|\mathfrak{T}| > h$. Then $\mathfrak{T}$ clearly contains a finite subfamily $\mathfrak{T}_0$ fulfilling $|\mathfrak{T}_0| > h$. We may assume that, among all the finite subfamilies of this kind, $\mathfrak{T}_0$ has the smallest cardinality, say m. Now let $\varphi[I_1], \cdots, \varphi[I_m]$ be the m intervals that constitute $\mathfrak{T}_0$, where $I_1, \cdots, I_m$ are m distinct intervals of $\mathfrak{M}(\alpha)$. Then $I_1, \cdots, I_m$ are mutually non-overlapping by hypothesis. Since $\alpha|I| < |\varphi[I]| < h$ for every $I \in \mathfrak{M}(\alpha)$, and since the index m was chosen minimal, we find that

$$h < |\mathfrak{T}_0| = |\varphi[I_1]| + \cdots + |\varphi[I_m]| < 2h$$

and that $\alpha(|I_1| + \cdots + |I_m|) < |\varphi[I_1]| + \cdots + |\varphi[I_m]| < 2h$. Therefore, if we write $\mathfrak{N} = \{I_1, \cdots, I_m\}$, we have $|\mathfrak{N}| = |I_1| + \cdots + |I_m| < \rho$, which implies that

$|\varphi[\mathfrak{N}]|<h$. But $\varphi[\mathfrak{N}]=[\mathfrak{T}_0]$, so that $|\varphi[\mathfrak{N}]|=|\mathfrak{T}_0|>h$. This contradiction proves that $|\varphi[\mathfrak{M}(\lambda)]|<+\infty$ for $\lambda>2\rho^{-1}h$, as announced above.

This being so, let us now consider the intersection D of all the sets $\varphi[\mathfrak{M}(\lambda)]$, where $\lambda>0$. Since $\varphi[\mathfrak{M}(\lambda)]$ descends together with $\mathfrak{M}(\lambda)$ for increasing λ and is a Borel set of finite measure when $\lambda>2\rho^{-1}h$, we find at once that D is also a Borel set of finite measure and that $|\varphi[\mathfrak{M}(\lambda)]|$ converges to $|D|$ as $\lambda\to+\infty$. Let us proceed to show that D is a null set. For this purpose, suppose the contrary true, if possible, and consider any point p of D. Then for each $\lambda>0$ the family $\mathfrak{M}(\lambda)$ contains an interval $I(p)$ such that $p\in\varphi[I(p)]$. On the other hand, every interval $I\in\mathfrak{M}(\lambda)$ has length $<\lambda^{-1}h$, while the function $\varphi(x)$ is uniformly continuous on $[\mathfrak{M}]$ by hypothesis. Moreover, $\mathfrak{M}(\lambda)$ descends for increasing λ, as already stated. Hence, for each $\lambda>0$, the family $\mathfrak{F}(\lambda)$ of all the intervals $\varphi[I]$, where $I\in\mathfrak{M}(\lambda)$, covers the set D in the Vitali sense. Accordingly, by Vitali's Covering Theorem, the family $\mathfrak{F}(\lambda)$ contains for each $\varepsilon>0$ a disjoint subfamily, say $\mathfrak{G}(\lambda)$, which covers D almost entirely and which consists exclusively of intervals of lengths less than ε. We notice that λ and ε can vary independently.

Now let $0<\varepsilon<|D|$ and write $\beta=2\delta^{-1}\varepsilon$, where δ is the number associated with ε in the theorem. Noting that $|\mathfrak{G}(\beta)|\geqq|D|>\varepsilon$, we take from the non-void family $\mathfrak{G}(\beta)$ a finite disjoint sequence of intervals, say $\langle K_1,\cdots,K_q\rangle$, such that $\varepsilon<|K_1|+\cdots+|K_q|<2\varepsilon$. Since $\mathfrak{G}(\beta)\subset\mathfrak{F}(\beta)$, there corresponds to each $n=1,\cdots,q$ an interval $J_n\in\mathfrak{M}(\beta)$ such that $K_n=\varphi[J_n]$. Then $\beta|J_n|$ is less than $|K_n|$ and hence we have

$$|J_1|+\cdots+|J_q|<\beta^{-1}(|K_1|+\cdots+|K_n|)<2\beta^{-1}\varepsilon=\delta\,.$$

But the sequence $\langle K_1,\cdots,K_n\rangle$ is disjoint, and consequently the sequence $\langle J_1,\cdots,J_q\rangle$ is distinct. It thus follows from the choice of the number δ that $|K_1|+\cdots+|K_q|=|K_1\cup\cdots\cup K_q|<\varepsilon$. This contradicts the choice of $\langle K_1,\cdots,K_q\rangle$ and establishes the nullity of D.

There thus exists for each $\eta>0$ a $\lambda>0$ such that $|\varphi[\mathfrak{M}(\lambda)]|<\eta$. Keeping η and λ fixed, let us write for brevity

$$M=[\mathfrak{M}],\quad A=[\mathfrak{M}(\lambda)],\quad\text{and}\quad B=M\setminus A\,.$$

Now, by hypothesis, the family $\mathfrak{M}$ is non-overlapping and the function $\varphi(x)$ is linear on every interval of $\mathfrak{M}$. Hence $|\varphi[E]|\leqq\lambda|E|$ for every set $E\subset U$, where U denotes the union of all the intervals $I\in\mathfrak{M}$ such that $|\varphi[I]|\leqq\lambda|I|$. This fact, combined with $B\subset U$ and $|\varphi[A]|<\eta$, shows that for every set $X\subset M$ with finite $|X|$ we have

$$|\varphi[X]|\leqq|\varphi[X\cap A]|+|\varphi[X\cap B]|<\eta+\lambda|X|\,.$$

It follows that $|\varphi[X]|<2\eta$ whenever $X\subset M$ and $|X|<\lambda^{-1}\eta$. The function $\varphi(x)$ therefore fulfils the unrestricted condition (S) on M.

THEOREM 26. *Every function $\varphi(x)$ which is* AC *superposable (in particular, absolutely continuous) on a set E, fulfils the condition* (W) *on this set.*

PROOF. On account of Theorem 20, we may confine ourselves to the case in which the function $\varphi(x)$ is AC on E.

As readily seen, there exists for each $\varepsilon>0$ a number $\delta>0$ such that for every non-overlapping family $\mathfrak{M}$ of closed intervals, the inequality $|\mathfrak{M}|<\delta$ implies $\mathrm{d}(\varphi;E;\mathfrak{M})<\varepsilon$. This, together with the obvious relation $|\square(\varphi;E;\mathfrak{M})|\leqq\mathrm{d}(\varphi;E;\mathfrak{M})$, shows that the function $\varphi(x)$ fulfils the condition (W) on E.

THEOREM 27 (second main theorem). *Let $\varphi(x)$ be a function and E a set. In order that $\varphi(x)$ be* AC *superposable on E, it is necessary and sufficient that the function fulfil the condition* (W) *on this set.*

PROOF. In view of the foregoing theorem, we may restrict ourselves to the sufficiency part of the assertion.

Suppose that the function $\varphi(x)$ fulfils the condition (W) on E. Then $\varphi(x)$ is uniformly continuous on E. We may therefore assume $\varphi(x)$ continuous on the closure Q of E and linear on every closed interval contiguous to Q. Such a function must of itself be continuous on the whole set $\square Q$. Since $\varphi(x)$ then fulfils the condition (W) on Q by Theorem 16, there corresponds to each $\varepsilon>0$ a number $\rho>0$ such that $|\varphi(I)|<\varepsilon$ for any closed interval I with length $|I|<\rho$ and pertaining to Q.

Now let C be a generic closed interval contiguous to Q and such that $|\varphi[C]|<1$. Then $|\varphi[K]|<\varepsilon$ for every closed interval $K\subset C$ with length $|K|<\rho\varepsilon$, where ρ is the same number as above. In fact, this is obvious if $|C|<\rho$, since $\varphi(x)$ is linear on C; while in the opposite case the same linearity shows that

$$|\varphi[K]|=\frac{|K|}{|C|}|\varphi[C]|<\frac{\rho\varepsilon}{\rho}\cdot 1=\varepsilon\,.$$

Accordingly, denoting by M the union of all the intervals C, we obtain at once the appraisal $|\varphi(I)|<3\varepsilon$ for every closed interval I with $|I|<\rho\varepsilon$ and pertaining to M, provided however that $\varepsilon<1$. Hence $\varphi(x)$ is uniformly continuous on M. It thus follows from Theorem 25 that the function $\varphi(x)$ fulfils the unrestricted condition (S) on M. On the other hand, this function fulfils the same condition on Q also, in virtue of Theorem 15.

Hence $\varphi(x)$ fulfils the unrestricted condition (S) on the union $R=Q\cup M$.

The set R is contained in the closed connected set $\square Q$ and the difference $\square Q\setminus R$ is the union of all the open intervals G that are contiguous to Q and subject to the condition $|\varphi[G]|\geqq 1$. We thus see that R is a closed set and that the intervals G are no other than the open intervals contiguous to R. Moreover, the function $\varphi(x)$ is continuous on R, since it is so on the set $\square Q\supset R$.

The condition (W) being fulfilled on Q by $\varphi(x)$, there exists a number $\delta_0>0$ such that $|\square\varphi[Q\cap I]|<1$ for each closed interval I with $|I|<\delta_0$. But the compact set $R\cap I$ is connected for such I. Indeed, if this is false, there exists an open interval H which is contiguous to $R\cap I$ and hence to R. Then H must coincide with one of the intervals G considered just now, so that H is contiguous to Q and fulfils $|\varphi[H]|\geqq 1$. We then have obviously $\varphi[H]\subset\square\varphi[Q\cap I]$, whence we find $|\varphi[H]|\leqq|\square\varphi[Q\cap I]|<1$. This contradiction proves the connectedness of $R\cap I$. The number δ_0 will be kept fixed in the sequel.

As already shown, the function $\varphi(x)$ fulfils the unrestricted condition (S) on the set R. Accordingly there is for each $\varepsilon>0$ a positive number $\delta<\delta_0$ such that for every non-overlapping family $\mathfrak{M}$ of closed intervals J, the inequality $|\mathfrak{M}|<\delta$ implies $|\varphi[R\cap\mathfrak{M}]|<\varepsilon$, where $R\cap\mathfrak{M}$ denotes the family of all the sets $R\cap J$. Let us show that $\square(\varphi;R;\mathfrak{M})=\varphi[R\cap\mathfrak{M}]$ for such family $\mathfrak{M}$ with $|\mathfrak{M}|<\delta$. Every interval $J\in\mathfrak{M}$ has length $|J|<\delta<\delta_0$, and this, combined with what we have already proved, requires that the compact set $R\cap J$ is connected. Then its continuous image $\varphi[R\cap J]$ must also be a compact connected set, so that $\square\varphi[R\cap J]=\varphi[R\cap J]$. We thus have $\square(\varphi;R;\mathfrak{M})=\varphi[R\cap\mathfrak{M}]$. It follows that $|\square(\varphi;R;\mathfrak{M})|<\varepsilon$. Consequently the function $\varphi(x)$ fulfils the condition (W) on the set R. Then Theorem 22 shows that $\varphi(x)$ fulfils the condition (A) on R.

For each $n\in\boldsymbol{N}$ let R_n be the set of the points of R at which the function $\varphi(x)$ fails to be angular (n) relatively to R. Then $R_1\supset R_2\supset\cdots$, and $|\varphi[R_n]|$ tends to zero as $n\to+\infty$. The set $D=\varphi[R_1]\cap\varphi[R_2]\cap\cdots$ is therefore null. Furthermore, since R is a closed set and $\varphi(x)$ is continuous on R, we find by Theorem 23 that every $\varphi[R_n]$ is a measurable set and accordingly that so is also the set $D_n=\varphi[R_{n-1}]\setminus\varphi[R_n]$ for every n, where we put $R_0=R$ and where the set $\varphi[R_0]=\varphi[R]$ is clearly sigma-compact. The sequence $\langle D_1, D_2, \cdots\rangle$ is disjoint and has $\varphi[R]\setminus D$ for its union.

Let us now consider the sets $P=R_1\cap R_2\cap\cdots$ and $P_n=R_{n-1}\setminus R_n$, where $n\in\boldsymbol{N}$, so that $\varphi[P]$ is a null set and $\langle P, P_1, P_2, \cdots\rangle$ is a disjoint sequence with union R. For each point x of $R\setminus P$ there thus exists an $n\in\boldsymbol{N}$ such that $x\in P_n$. Then $x\in R_{n-1}$, and hence, unless $\varphi(x)\in D$, there is

an integer $k \geqq n$ such that $\varphi(x) \in D_k$. This last fact will be used later on.

This being so, we proceed to construct a function $f(y)$ by means of the sets D_n, as follows. Let y be any real number. We write $f(y)=n^{-1}$ by definition, if there is an $n \in \boldsymbol{N}$ for which $y \in D_n$, while we simply set $f(y)=1$ if not so. Noting that the function $f(y)$ is positive, measurable, and bounded, we choose an indefinite integral $\omega(y)$ of this function. The function $\omega(y)$ is both increasing and absolutely continuous, on the real line, and therefore maps the real line biuniquely onto an interval, say Ω. Let $y=\theta(t)$ be the inverse mapping of the function $t=\omega(y)$, so that the domain of $\theta(t)$ is Ω and its range is the real line. Clearly $\theta(t)$ is increasing and continuous on Ω.

We shall show further that $\theta(t)$ is absolutely continuous on Ω. The proof amounts to associating with each $\varepsilon>0$ a number $\delta>0$ such that for any non-overlapping family $\mathfrak{S}$ of closed intervals, the inequality $|\omega[\mathfrak{S}]|<\delta$ implies $|\mathfrak{S}|<2\varepsilon$. For this purpose, recalling that $|\varphi[R_n]| \to 0$ as $n \to +\infty$, let us choose an integer $q>0$ such that $|\varphi[R_q]|<\varepsilon$. If y belongs to $\varphi[R] \setminus \varphi[R_q]$, there is a positive integer $r \leqq q$ such that $y \in D_r$ and hence that $f(y)=r^{-1} \geqq q^{-1}$. On the other hand, $f(y)=1 \geqq q^{-1}$ for y not belonging to $\varphi[R]$. We thus have $f(y) \geqq q^{-1}$ unless $y \in \varphi[R_q]$. We now designate by V the complementary set of $\varphi[R_q]$ and by K a generic interval of the family $\mathfrak{S}$. Writing further $S=[\mathfrak{S}]$ for short, so that $\omega[\mathfrak{S}]=\omega[S]$, we find at once that

$$|\omega[S]|=\sum_K |\omega[K]|=\sum_K \int_K f(y)\,dy$$

$$\geqq \sum_K \int_{K \cap V} f(y)\,dy=\int_{S \cap V} f(y)\,dy \geqq q^{-1}|S \cap V|,$$

whence $|S \cap V| \leqq q|\omega[S]|$. Consequently, if we put $\delta=q^{-1}\varepsilon$, the condition $|\omega[\mathfrak{S}]|=|\omega[S]|<\delta$ necessarily implies that

$$|\mathfrak{S}|=|S|=|S \cap V|+|S \cap \varphi[R_q]| \leqq q|\omega[S]|+|\varphi[R_q]|<2\varepsilon\,.$$

This exhibits that the mapping $\theta(t)$ is AC on Ω.

We shall proceed to verify that the composite function $\psi(x)=\omega \circ \varphi(x)$ is AC on the set R.

Since the function $\omega(y)$ is an indefinite integral of $f(y)$, there is a null set T such that we have $\omega'(y)=f(y)$ unless $y \in T$. We may assume that T contains the set $D=\varphi[R_1] \cap \varphi[R_2] \cap \cdots$ which is null as already mentioned. Let L be the set of the points x of R such that $\varphi(x) \in T$. Then $\varphi[L]$ is a null set and, since the set $P=R_1 \cap R_2 \cap \cdots$ is mapped by $\varphi(x)$ into D, we have $P \subset L$. Let us now consider any point ξ of $R \setminus L$. Since ξ belongs to the set $R \setminus P=P_1 \cup P_2 \cup \cdots$, there is an index n such

that $\xi \in P_n$. Further, $\varphi(\xi)$ does not belong to the set T and hence neither to D. It thus follows from what has already been stated that there is an integer $k \geqq n$ for which $\varphi(\xi) \in D_k$. Moreover, the function $\omega(y)$, which is derivable at the point $\eta = \varphi(\xi)$ to the derivative $\omega'(\eta) = f \circ \varphi(\xi) = k^{-1} \leqq n^{-1}$, is evidently angular $(2n^{-1})$ at η. On the other hand, as $\xi \in P_n = R_{n-1} \setminus R_n$, the function $\varphi(x)$ is angular (n) at ξ relatively to R. Accordingly, the function $\psi(x) = \omega \circ \varphi(x)$ must be angular (2) at ξ relatively to R.

This being so, given any non-overlapping family $\mathfrak{N}$ of closed intervals pertaining to R, write $N = [\mathfrak{N}]$ and let I be a generic interval of $\mathfrak{N}$. The nullity of the set $\varphi[L]$, together with the absolute continuity of $\omega(y)$ on the real line, shows $\psi[L]$ to be null (see the theorem at the top of p. 225 of Saks [6]). On the other hand, the number $\delta_0 > 0$ already taken has the property that, for every closed interval J with $|J| < \delta_0$, the compact set $R \cap J$ is connected. Thus every $I \in \mathfrak{N}$, and hence the union $N = [\mathfrak{N}]$ itself, is contained in R, on condition that $|N| < \delta_0$. Let us assume this condition fulfilled henceforth. Noting that $\psi(x)$ is continuous on R and making use of Theorem 21, we find that

$$|\psi(I)| \leqq |\psi[I]| \leqq |\psi[I \cap L]| + |\psi[I \setminus L]| \leqq 2|I \setminus L| \leqq 2|I|$$

for every $I \in \mathfrak{N}$. In point of fact, the function $\psi(x)$, which is angular (2) relatively to R at all points of $R \setminus L$, is *a fortiori* angular (2) relatively to $I \setminus L$. It follows that, given any $\varepsilon > 0$, we have

$$\sum_I |\psi(I)| \leqq 2 \sum_I |I| = 2|N| < \varepsilon \qquad \text{if } |N| < \min(2^{-1}\varepsilon, \delta_0).$$

This establishes the absolute continuity of the function $\psi(x)$ on R.

Consider again the inverse mapping $y = \theta(t)$ of the function $t = \omega(y)$. Remembering that $\theta(t)$ is AC on the set Ω as already proved, we express $\varphi(x)$ in the form $\varphi(x) = \theta \circ \omega \circ \varphi(x) = \theta \circ \psi(x)$ and we find that $\varphi(x)$ is AC superposable on the set R. Then $\varphi(x)$ is *a fortiori* so on the subset E of R, and this completes the proof.

The rest of this section will be concerned with a few supplementary propositions on the condition (A).

THEOREM 28. *Every function $\varphi(x)$ which fulfils the condition* (A) *on a set E, fulfils the unrestricted condition* (S) *on this set.*

However, the converse of this assertion is false, even when the function is continuous on E.

PROOF. By hypothesis, there exists for any $\varepsilon > 0$ a number $\alpha > 0$ such that $|\varphi[E(\alpha)]| < \varepsilon$, where $E(\alpha)$ is the set of the points of E at which $\varphi(x)$

is not angular (α) relatively to E. If we write for short $M=E\setminus E(\alpha)$, the function $\varphi(x)$ is angular (α) relatively to every subset of M. Then Theorem 21 shows that

$$|\varphi[X\cap M]\leqq\alpha|X\cap M|\leqq\alpha|X| \qquad \text{for every } X\subset E.$$

On the other hand, there holds $|\varphi[X\setminus M]|\leqq\varphi[E(\alpha)]|<\varepsilon$ for such X, since $E(\alpha)$ contains $X\setminus M$. We therefore find that

$$|\varphi[X]|\leqq|\varphi[X\cap M]|+|\varphi[X\setminus M]|<2\varepsilon$$

for every $X\subset E$ with $|X|<\alpha^{-1}\varepsilon$. This establishes the first half of the theorem.

We shall go on to disprove the converse by a counter-example. Let there be contained in a compact set Q a compact nonnull set C which differs from Q at most by a countable set. Suppose that a nonnegative function $\psi(x)$ is continuous on Q, vanishes identically on C, and further, at every point of a set $L\subset C$ which contains almost all points of C, fails to be angular (1) relatively to Q. As a concrete example of such a function, we may quote the function $F(x)$ constructed on p. 224 of Saks [6].

Let us show that the function $\sqrt{\psi(x)}$ cannot be angular relatively to Q at any point of the set L. To see this, suppose if possible that there are to the contrary a point $p\in L$ and a number $\alpha>0$ such that $\sqrt{\psi(x)}$ is angular (α) relatively to Q at p. By hypothesis we can extract from Q an infinite sequence of points, say $\langle x_1, x_2, \cdots\rangle$, converging to the point p and such that

$$|x_n-p|<|\psi(x_n)-\psi(p)| \qquad \text{for } n\in\boldsymbol{N}.$$

Without loss of generality we may assume that

$$|\sqrt{\psi(x_n)}-\sqrt{\psi(p)}|\leqq\alpha|x_n-p| \qquad \text{for every } n.$$

But we must have $\psi(p)=0$, since $\psi(x)$ vanishes on C and hence on L. It follows at once that, for every n,

$$|x_n-p|<\psi(x_n)\leqq\alpha^2|x_n-p|^2, \qquad \text{whence} \quad |x_n-p|>\alpha^{-2}.$$

But this evidently contradicts the choice of the sequence $\langle x_1, x_2, \cdots\rangle$.

Consider now the function $\omega(x)=x+\sqrt{\psi(x)}$. Since $\sqrt{\psi(x)}$ is angular relatively to Q at no point of L, the same is true of $\omega(x)$ also. On the other hand, we have $\omega[L]=L$ because $\omega(x)=x$ for $x\in C$, while $|L|=|C|$ on account of $|C\setminus L|=0$. Thus $|\omega[L]|=|C|>0$, and this shows that the function $\omega(x)$ cannot fulfil the condition (A) on Q. Nevertheless, $\omega(x)$ certainly fulfils the unrestricted condition (S) the set on Q, since $\omega(x)=x$ for $x\in C$ and since $\omega[Q\setminus C]$ is a countable set together with $Q\setminus C$. This establishes

the second half of the theorem.

REMARK. As an addendum to Theorem 22 let us observe that a function $\varphi(x)$ which is continuous on an interval I (finite or infinite) and which fulfils the condition (A) on this interval, necessarily fulfils the condition (W) on I. Indeed, the function $\varphi(x)$ then fulfils the unrestricted condition (S) on I by the above theorem, and the result follows at once from the second half of Theorem 15.

THEOREM 29. *Suppose that a function $\varphi(x)$ is expressible on a set E in the form $\varphi(x)=\theta\circ\psi(x)$, where the functions $\psi(x)$ and θ fulfil the condition* (A) *on E and on $\psi[E]$, respectively. Then the function $\varphi(x)$ itself fulfils this condition on E.*

PROOF. Given any number $\alpha\geqq 0$ let S be the set of the points of E at which the function $\varphi(x)$ fails to be angular (α^2) relatively to E. Writing $M=\psi[E]$ for short, let further T [or U] be the set of the points of E [or of M] at which the function $\psi(x)$ [or θ] fails to be angular (α) relatively to E [or to M].

In virtue of Theorem 28 there corresponds to each $\varepsilon>0$ two numbers $\eta>0$ and $\delta>0$ such that for any pair of sets $Y\subset M$ and $X\subset E$ the conditions $|Y|<\eta$ and $|X|<\delta$ respectively imply $|\theta[Y]|<\varepsilon$ and $|\psi[X]|<\eta$. But there is a number $\alpha_0>0$ such that both $|T|<\delta$ and $|U|<\eta$ hold whenever $\alpha>\alpha_0$. For any such α we have

$$|\psi[T]|<\eta,\quad |\varphi[T]|=|\theta\circ\psi[T]|<\varepsilon,\quad |\theta[U]|<\varepsilon\,.$$

This being so, let us show that $\psi[S\setminus T]\subset U$. Suppose, if possible, that this is false. Then $S\setminus T$ contains a point p such that $\psi(p)$ does not belong to U. It follows that the functions $\psi(x)$ and $\theta(t)$ are respectively angular (α) relatively to E at $x=p$ and relatively to M at $t=\psi(p)$. We thus find that

$$|\varphi(x)-\varphi(p)|=|\theta\circ\psi(x)-\theta\circ\psi(p)|\leqq\alpha|\psi(x)-\psi(p)|\leqq\alpha^2|x-p|$$

for every point $x\in E$ sufficiently near p. But this contradicts that $p\in S$. Hence the inclusion $\psi[S\setminus T]\subset U$.

From this relation we deduce at once that

$$\varphi[S]=\theta\circ\psi[S]\subset\theta\circ\psi[S\setminus T]\cup\theta\circ\psi[T]\subset\theta[U]\cup\varphi[T]\,,$$

whence $|\varphi[S]|<2\varepsilon$ for $\alpha>\alpha_0$ on account of $|\theta[U]|<\varepsilon$ and $|\varphi[T]|<\varepsilon$. This shows that the function $\varphi(x)$ fulfils the condition (A) on E. The theorem is thus established.

THEOREM 30. *Every function $\varphi(x)$ which fulfils the condition* (A) *on a closed set Q and which is linear on each closed interval contiguous to Q, is derivable at all points of the set $\square Q$, except perhaps at those of a set $M \subset Q$ with null image $\varphi[M]$.*

PROOF. We may clearly assume that the set $\square Q$ is an interval, say I. Let S be the set of all the points of Q that are interior to I and at each of which the function $\varphi(x)$ is angular relatively to Q. Then the set $\varphi[Q \setminus S]$ is null, since $\varphi(x)$ fulfils the condition (A) on Q. Furthermore, as easily seen, this function is angular (relatively to the real line) at every point of S. Consequently, by Theorem (4.2) on p. 270 of Saks [6], the function $\varphi(x)$ is derivable at all points of a set $T \subset S$ such that $\varphi[S \setminus T]$ is null. If we write $M = Q \setminus T$, then $M = (Q \setminus S) \cup (S \setminus T)$ and it follows that $|\varphi[M]| = 0$. On the other hand, we have

$$I \setminus M = (I \setminus Q) \cup (Q \setminus M) = (I \setminus Q) \cup T,$$

where $I \setminus Q$ is the union of all the open intervals contiguous to Q. We infer that $\varphi(x)$ is derivable at all points of the set $I \setminus M$.

THEOREM 31. *Every function $\varphi(x)$ which is continuous on a compact set C and derivable at all points of C, except perhaps at those of a set $M \subset C$ with null image $\varphi[M]$, fulfils on C the condition* (W) *and hence the condition* (A) *also.*

PROOF. Without loss of generality we may suppose that the set $\square C$ is a closed interval, say I, and further that every point of the set $C \setminus M$ is a bilateral point of accumulation for C.

Let $\lambda(x)$ be the linear modification of the function $\varphi(x)$ with respect to the set C. In other words, the function $\lambda(x)$ is linear on every closed interval contiguous to C and we have $\lambda(x) = \varphi(x)$ unless x belongs to the set $I \setminus C$. Then, by Lemma 18 of [3], the function $\lambda(x)$ is derivable to $\varphi'(x)$ at every point of $C \setminus M$. On the other, hand, $\lambda(x)$ is derivable at all points of $I \setminus C$, since this set is open and since $\lambda(x)$ is linear on every component of this set. Hence $\lambda(x)$ is derivable at all points of $I \setminus M$.

We now quote the following theorem from p. 289 of Saks [6]. *In order that a function $F(x)$ which is continuous on a closed interval I be* AC *superposable on this interval, it is necessary and sufficient that the set of the points of I at which $F(x)$ is not derivable, be mapped by $F(x)$ onto a null set.* We combine this proposition with Theorem (b) of the Introduction. It thus follows that the function $\lambda(x)$, which is plainly continuous on the closed interval $I = \square C$, fulfils the unrestricted condition (S)

on this interval. The second half of Theorem 15 then requires that $\lambda(x)$ fulfil the condition (W) on I, and all the more on C. By Theorem 22, this implies that $\lambda(x)$, and hence $\varphi(x)$ also, fulfils the condition (A) on C.

REMARKS. (i) In the present theorem, the compactness of the set C cannot be replaced by its closedness. To see this, consider the function $\varphi(x)=\sin \pi x^2$, so that $\varphi'(x)=2\pi x \cos \pi x^2$. Writing for short $\delta=4^{-1}$ and $I_n=[\sqrt{2n-\delta},\ \sqrt{2n+\delta}]$, where $n\in \boldsymbol{N}$, we have

$$\varphi'(x)>6\sqrt{2n-\delta}\cos \pi\delta>\sqrt{n} \qquad \text{for } x\in I_n\,,$$

and so $\varphi(x)$ is angular $(\sqrt{n})$ at no point of I_n. On the other hand,

$$|\varphi[I_n]|=2\sin \pi\delta=\sqrt{2} \quad \text{since} \quad \varphi[I_n]=[-\sin \pi\delta,\ \sin \pi\delta]\,.$$

Consequently the function $\varphi(x)$, although everywhere derivable, fails to fulfil the condition (A) on the real line.

(ii) It is evident that every function which fulfils the condition (W) on a set is continuous on this set. In the hypothesis of Theorem 31, however, the continuity of $\varphi(x)$ on C is not redundant. As a couter-example, we may propound the function $\varphi(x)$ vanishing for $x=0$ and defined for $x\neq 0$ by $\varphi(x)=\cos \pi x^{-1}$. This function, although everywhere derivable on the real line except at $x=0$, fails to fulfil the condition (W) on the interval $C=[-1, 1]$, since $\varphi(x)$ is discontinuous at $x=0$ on C. Moreover, it is easy to show that $\varphi(x)$ cannot fulfil the condition (A) on C, either.

THEOREM 32. *Every function $\varphi(x)$ which is continuous on a compact set C and subject to the condition* (A) *ont his set, fulfils the condition* (W) *on C.*

PROOF. Let $\lambda(x)$ be the linear modification of $\varphi(x)$ with respect to C, so that $\lambda(x)$ fulfils the condition (A) on C. Then, by Theorem 30, the function $\lambda(x)$ is derivable at all points of C, except perhaps at those of a set $M\subset C$ with null image $\varphi[M]$. It follows from Theorem 31 that $\lambda(x)$, and hence $\varphi(x)$ also, fulfils the condition (W) on C.

REMARKS. Let us show that, in the above theorem, the compactness of the set C cannot be replaced by its closedness and that the continuity of $\varphi(x)$ on C is not superfluous.

(i) Let Q be the closed set $\{\log n;\ n\in \boldsymbol{N}\}$. This set being countable, any function, and in particular the function $\varphi(x)=e^x$, fulfils the condition (A) on Q. But this function is not uniformly continuous on Q; indeed, its increment on the interval $[\log n, \log(n+1)]$ is 1 for every n, although the

length of this interval tends to 0 as $n \to +\infty$. Hence $\varphi(x)$ fails to fulfil the condition (W) on Q.

(ii) Let C be the countable compact set consisting of 0 and of the numbers n^{-1}, where $n \in \boldsymbol{N}$. Then every function which is continuous on C and discontinuous at $x=0$ on this set, fulfils the condition (A) on C, without fulfilling the condition (W) on this set. In brief, the condition (A) on a compact set does not always imply the continuity of the relevant function on the same set.

§ 4. Continuation of the first section.

DEFINITION. A function $\varphi(x)$ will be said to fulfil the *condition* (H) on a set E, if for each fixed open set D we have $\inf|\varphi[(E \cap D) \setminus C]|=0$, where C is a generic compact set contained in $E \cap D$.

As we find easily, if a function $\varphi(x)$ is continuous on a set E and fulfils the condition (H) on this set, then $|\varphi[E]|$ is finite.

THEOREM 33. *Every function $\varphi(x)$ which fulfils the condition* (H) *on a set E, fulfils the condition* (G) *on this set.*

PROOF. In conformity to the definition of the condition (G), we shall show that for each compact set Q and each $\varepsilon>0$ there exists an open set $U \supset Q$ with the property $|\varphi[E \cap (U \setminus Q)]|<\varepsilon$. The set $D=\boldsymbol{R} \setminus Q$ is open and therefore, by hypothesis, the set $E \cap D$ cotains a compact set C such that $|\varphi[(E \cap D) \setminus C]|<\varepsilon$. Then the open set $U=\boldsymbol{R} \setminus C$ contains Q and

$$(E \cap D) \setminus C=(E \setminus Q) \cap U=E \cap (U \setminus Q),$$

which completes the proof.

THEOREM 34. *Every function $\varphi(x)$ which is continuous on an analytic set A and subject to the condition* (T_1) *on this set, fulfils the condition* (H) *on A, provided that $|\varphi[A]|$ is finite.*

PROOF. We shall first treat the case in which the open set D of the above definition is the real line.

Writing $f(y)=P(y;\varphi;A)$, where the notation is the same as in § 1, we have $0 \leqq f(y) \leqq 1$ for each $y \in \boldsymbol{R}$ and the function $f(y)$ is measurable on the real line. Moreover, $f(y)$ is positive almot everywhere, since the function $\varphi(x)$ fulfils the condition (T_1) on A.

This being so, let $F(y)$ be an indefinite Lebesgue integral of the function $f(y)$. Then $F(y)$ is an increasing function which is absolutely con-

tinuous on the whole real line. We shall show that the set $F[Q]$ is nonnull for every compact nonnull set Q. For this purpose, we may clearly assume that Q is not an interval. Then the difference $\Delta = I \setminus Q$ is a nonvoid open set, where we write $I = \square Q$. If P denotes a generic component interval of Δ, the image $F[P]$ is evidently an open interval for each P, and $F[\Delta]$ is the union of all the intervals $F[P]$, which are mutually disjoint. It follows that

$$|F[\Delta]| = \sum_P |F[P]| = \sum_P \int_P f(y)\,dy = \int_\Delta f(y)\,dy\,.$$

On the other hand, we have $F[Q] = F[I \setminus \Delta] = F[I] \setminus F[\Delta]$. Thus

$$|F[Q]| = |F[I]| - |F[\Delta]| = \int_I f(y)\,dy - \int_\Delta f(y)\,dy = \int_Q f(y)\,dy.$$

But this last integral is positive, since Q is nonnull and since $f(y) > 0$ for almost every y. Hence $|F[Q]| > 0$, as desired.

Consider now the composite function $\psi(x) = F \circ \varphi(x)$, where $x \in \boldsymbol{R}$, and let ρ be the infimum of $\Xi(\psi; A \setminus \Gamma)$, where Γ is an arbitrary compact set contained in A. This fluctuation makes sense, since $\psi(x)$ is continuous on A and since the set $A \setminus \Gamma$ is evidently analytic. The number ρ is finite, for we have $\Xi(\psi; A \setminus \Gamma) \leqq |\varphi[A \setminus \Gamma]| \leqq |\varphi[A]|$ by Theorem 6.

By definition of ρ, there exists in A an ascending infinite sequence of compact sets, say $C_1 \subset C_2 \subset \cdots$, such that

$$\Xi(\psi; A \setminus C_n) \text{ tends to } \rho \qquad \text{as } n \to +\infty\,.$$

Writing $S = C_1 \cup C_2 \cup \cdots$ and $S_n = S \setminus C_n$ for each n, we shall show that the intersection $L = \varphi[S_1] \cap \varphi[S_2] \cap \cdots$ is null. For this purpose, we may assume L nonvoid. Taking any point η of L, let W be the set of the points $x \in S$ at which $\varphi(x) = \eta$. If W is a finite set, there exists an index k such that C_k contains W. Then the set $S_k = S \setminus C_k$ is disjoint with W, so that its image $\varphi[S_k]$, and *a fortiori* the set L, cannot contain the point η. It follows that W is an infinite set and therefore that

$$\mathrm{N}(\eta; \varphi; A) \geqq \mathrm{N}(\eta; \varphi; S) = +\infty\,.$$

But the function $\varphi(x)$ fulfils the condition (T_1) on A. We thus conclude that $|L| = 0$, as announced above.

Since $S_1 \supset S_2 \supset \cdots$, we have $\varphi[S_1] \supset \varphi[S_2] \supset \cdots$, where each set $\varphi[S_n]$ is measurable since it is a continuous image of a Borel set. We have further

$$L = \lim_n \varphi[S_n] \quad \text{and} \quad |\varphi[S_1]| \leqq |\varphi[A]| < +\infty\,.$$

It follows that $\lim |\varphi[S_n]| = |\lim \varphi[S_n]| = |L| = 0$. On the other hand, from

the relation $A\setminus S\subset A\setminus C_n=(A\setminus S)\cup S_n$ we deduce at once that

$$\Xi(\psi;A\setminus S)\leqq\Xi(\psi;A\setminus C_n)=\Xi(\psi;A\setminus S)+\Xi(\psi;S_n)\,.$$

Making $n\to+\infty$ here and using the facts

$$\lim_n\Xi(\psi;A\setminus C_n)=\rho\quad\text{and}\quad\lim_n\Xi(\psi;S_n)\leqq\lim_n|\varphi[S_n]|=0\,,$$

we find that $\Xi(\psi;A\setminus S)=\rho$.

From this last relation we shall now derive the nullity of the set $\varphi[A\setminus S]$. Suppose if possible that $|\varphi[A\setminus S]|>0$. On account of Theorem 2 the set $A\setminus S$, which is an analytic set contained in A, contains a compact set C_0 such that $|\varphi[C_0]|>0$. Then $\varphi[C_0]$ is a compact nonnull set, and hence, by what has already been proved, we have $|\psi[C_0]|=|F\circ\varphi[C_0]|>0$. Accordingly there exists an index $q>0$ such that $|\varphi[S_q]|<|\psi[C_0]|$. This, together with $|\psi[C_0]|\leqq\Xi(\psi;C_0)$ and Theorem 6, gives $\Xi(\psi;S_q)<\Xi(\psi;C_0)$. Writing $C^*=C_q\cup C_0$, we thus find that

$$\begin{aligned}\Xi(\psi;A\setminus C^*)&=\Xi(\psi;A\setminus S)+\Xi(\psi;S)-\Xi(\psi;C_q)-\Xi(\psi;C_0)\\&=\Xi(\psi;A\setminus S)+\Xi(\psi;S_q)-\Xi(\psi;C_0)<\rho\,.\end{aligned}$$

This contradicts the definition of the number ρ, since C^* is a compact set contained in A.

The set $\varphi[A\setminus S]$ being null as proved just now, we have

$$|\varphi[A\setminus C_n]|=|\varphi[A\setminus S]\cup\varphi[S\setminus C_n]|\leqq|\varphi[S_n]|$$

for every n. This, together with $\lim|\varphi[S_n]|=0$ already proved, leads at once to $\lim|\varphi[A\setminus C_n]|=0$. Therefore $\inf|\varphi[A\setminus C]|=0$, where C denotes a generic compact set contained in A. This establishes the assertion in the case in which D is the real line, where D is the set which appears in the definition of the condition (H).

The general case is reduced at once to this particular case, and the proof is complete.

THEOREM 35. *Let $\varphi(x)$ be a function which is continuous on an analytic set A. In order that this function fulfil the condition* (H) *on A, it is necessary and sufficient that the function fulfil on A the condition* (F) *and the condition $|\varphi[A]|<+\infty$.*

PROOF. The sufficiency part of the assertion is direct from Theorem 4 and the foregoing theorem. Hence we may confine ourselves to the necessity part in what follows.

Suppose that the function $\varphi(x)$ fulfils the condition (H) on A. We shall show that for every non-overlapping infinite sequence $\langle I_1, I_2, \cdots\rangle$ of

closed intervals, the sequence $\langle\varphi[A\cap I_n];\, n\in \boldsymbol{N}\rangle$ has a null set as its upper limit. For this purpose, consider the open set $D=I_1^\circ\cup I_2^\circ\cup\cdots$, where I_n° means for each n the interior of the interval I_n. By hypothesis, there corresponds to each $\varepsilon>0$ a compact set C contained in $A\cap D$ and such that $|\varphi[(A\cap D)\setminus C]|<\varepsilon$. Then there is an index k such that C is covered by the k intervals $I_1^\circ,\cdots,I_k^\circ$. Writing $R_k=I_{k+1}\cup I_{k+2}\cup\cdots$, we have

$$\overline{\lim_n}\,\varphi[A\cap I_n]\subset\bigcup_{n>k}\varphi[A\cap I_n]=\varphi\Big[\bigcup_{n>k}(A\cap I_n)\Big]=\varphi[A\cap R_k]\,.$$

We thus come to appraise the measure $|\varphi[A\cap R_k]|$. Now the set

$$D_k=I_{k+1}^\circ\cup I_{k+2}^\circ\cup\cdots=D\setminus(I_1^\circ\cup\cdots\cup I_k^\circ)$$

differs from R_k at most by a countable set. On the other hand, we must have $D_k\subset D\setminus C$. It thus follows that

$$|\varphi[A\cap R_k]|=|\varphi[A\cap D_k]|\leqq|\varphi[A\cap(D\setminus C)]|=|\varphi[(A\cap D)\setminus C]|<\varepsilon\,,$$

which implies that $\overline{\lim}\,\varphi[A\cap I_n]$ is null.

As already observed, on the other hand, every function which is continuous on a set E and subject on this set to the condition (H), maps E onto a set of finite outer measure. This completes the proof.

EXAMPLE. In the above theorem, the hypothesis that A is an analytic set, cannot be replaced by its measurability. Let us confirm this fact by an example in what follows.

We shall begin with the following well-known construction (stated at short) of a nonmeasurable subset of the interval $I=(0,1)$. A subset of I will temporarily be called *admissible*, if two distinct points of this set always have an irrational difference. By Zorn's Lemma, there exists in I a maximal admissible subset, say W. Let now the rational numbers r with $|r|<1$ be arranged, without repetitions, in an infinite sequence $\langle r_1, r_2,\cdots\rangle$. Supposing S to be any measurable subset of W, we denote for each $n\in\boldsymbol{N}$ by S_n the set of the numbers $x+r_n$, where x ranges over S. Then plainly the sequence $\langle S_1, S_2,\cdots\rangle$ is disjoint and its union $U=S_1\cup S_2\cup\cdots$ is bounded. On the other hand, each S_n is a measurable set and we have $|S_n|=|S|$. It follows that $|S|=0$, for otherwise the following relation would contradict the boundedness of U:

$$|U|=|S_1|+|S_2|+\cdots=|S|+|S|+\cdots.$$

We shall now show that the set W is nonmeasurable. For this purpose, suppose the contrary true, if possible, and specialize the set S to W. Then $|W|=|S|=0$, whence $|U|=0$. But this contradicts the relation $U\subset I$, which we verify as follows. Since W is a maximal admissible subset of

I, there is for each $x \in I$ a point $w \in W = S$ such that $x - w$ is rational. Noting that $|x-w|<1$, we have $x = w + r_n \in S_n$ for some n. Hence $I \subset U$.

Thus W is a nonmeasurable subset of $I=(0,1)$ and contains no measurable set of positive measure.

Let $\langle a_1, a_2, \cdots\rangle$ be a generic infinite sequence such that a_n equals 0 or 2 for each $n \in \boldsymbol{N}$ and further that we have $a_n \neq a_{n+1}$ for an infinity of values of n. We make correspond to the sequence $\sigma = \langle a_1, a_2, \cdots\rangle$ the ternary decimal σ^T with a_n for its nth digit, so that

$$\sigma_T = (0.a_1a_2\cdots)_3 = \sum_{n=1}^{\infty} 3^{-n} a_n .$$

It is obvious that this correspondence is biunique. Clearly $0<\sigma^T<1$.

Let us now write L for the set of all the decimals σ^T. Then L is a subset of the Cantor ternary set and hence we have $|L|=0$.

As well as the sequence $\sigma = \langle a_1, a_2, \cdots\rangle$ we consider another sequence $\langle b_1, b_2, \cdots\rangle$, where $b_n = 2^{-1}a_n$ for each n, and we denote by σ^B the binary decimal with b_n for its nth digit:

$$\sigma^B = (0.b_1b_2\cdots)_2 = \sum_{n=1}^{\infty} 2^{-n} b_n = \sum_{n=1}^{\infty} 2^{-n-1} a_n .$$

The set M of all the decimals σ^B is plainly contained in the interval $(0, 1)$. On the other hand, every irrational point of this interval belongs to M. Thus M is a Borel set of measure 1.

This being so, let $\varphi(x)$ be the function which equals σ^B for each point $x = \sigma^T$ of the set L and which vanishes for all the x outside L. As we find easily, the function $\varphi(x)$ is both continuous and increasing, on the set L.

We now resume the nonmeasurable set $W \subset (0,1)$ constructed above. The set M introduced just now differs from the interval $(0, 1)$ only by a countable set, and so the set $W_0 = W \cap M$ is nonmeasurable. Let L_0 be the set of the points x of L such that $\varphi(x) \in W_0$. We then have $\varphi[L_0] = W_0$ on account of $\varphi[L] = M$. Further, L_0 is a null set and the function $\varphi(x)$ is both continuous and increasing, on this set. Thus $\varphi(x)$ is subject on L_0 to the condition (F) and the condition $|\varphi[L_0]| < +\infty$.

We shall show that the function $\varphi(x)$ fails to fulfil the condition (H) on L_0. For this purpose, consider any compact set $C \subset L_0$. Then the set $\varphi[C]$ is compact and contained in W, so that we must have $|\varphi[C]|=0$. It follows that $|\varphi[L_0 \setminus C]| = |\varphi[L_0]| = |W_0| > 0$. This leads to the inequality $\inf |\varphi[L_0 \setminus C]| > 0$, whence the result.

Thus Theorem 35 will cease to hold if the set A is merely assumed measurable. By the way, the same is true of Theorem 2 also.

DEFINITION. By an *admissible measure* we shall understand any finite nonnegative set-function $\mu(M)$ which is

(1) defined for the analytic sets M in the real line,

(2) completely additive on the class of these sets, and

(3) monotone nondecreasing for ascending M.

DEFINITIONS. A function $\varphi(x)$ will be called *measure-controllable* on a set E, if there is an admissible measure μ such that for every analytic set $M\subset E$ the relation $\mu(M)=0$ implies $|\varphi[M]|=0$. When this is the case, we shall say that the function $\varphi(x)$ *is controlled* on E by the measure μ.

For instance, if $\mathrm{Tan}^{-1}x$ means for $x\in\boldsymbol{R}$ the principal value of $\tan^{-1}x$, then the set-function $\mu(M)=|\mathrm{Tan}^{-1}[M]|$, where M is any analytic set, is clearly an admissible measure. We find easily that every function which fulfils the condition (N) on a set E, is controlled on E by this measure.

DEFINITIONS. A function $\varphi(x)$ will be termed *analytically* [or *Borel*] *partitionable* (T_1) on a set E, if this set is expressible as the union of a nonvoid, disjoint, countable family of analytic [or Borel] sets on each of which the function $\varphi(x)$ fulfils the condition (T_1). When this is the case, the above expression of E will be called *analytic* [or *Borel*] *partition* (T_1) of the set E with respect to the function $\varphi(x)$.

THEOREM 36. *Every function $\varphi(x)$ which is continuous on an analytic set A and measure-controllable over this set, is analytically partitionable* (T_1) *on A. More precisely, there is in the set A a Borel set B for which the set $\varphi[A\setminus B]$ is null and on which $\varphi(x)$ is Borel partitionable* (T_1).

PROOF. Let μ be an admissible measure by which the function $\varphi(x)$ is controlled on the set A. Let us write $\rho=\inf\mu(A\setminus L)$, where L is a generic Borel set contained in A and on which $\varphi(x)$ is Borel partitionable (T_1). From the family $\mathfrak{M}$ of all such sets L, we can evidently choose an infinite sequence of sets, say $\langle B_1, B_2, \cdots\rangle$, such that $\lim\mu(A\setminus B_n)=\rho$. If we write $B=B_1\cup B_2\cup\cdots$, then the set B itself belongs to the class $\mathfrak{M}$ and further $B_n\subset B$ for every n. It thus follows that $\mu(A\setminus B)=\rho$.

We shall verify that $\varphi[A\setminus B]$ is null. For this purpose, suppose if possible that the contrary is true. The set $A\setminus B$ being analytic and the function $\varphi(x)$ being continuous on A, Theorem 2 ensures the existence of a compact set $C\subset A\setminus B$ such that $|\varphi[C]|>0$. In virtue of Theorem 3, the set C contains a Borel set B_0 which fulfils the relation $\varphi[B_0]=\varphi[C]$ and on which $\varphi(x)$ is biunique. Then $\varphi(x)$ is obviously Borel partitionable (T_1) on the Borel set $M=B\cup B_0$. But the set B_0 is disjoint with B. Hence

$$\mu(A\setminus M)=\mu(A\setminus B)-\mu(B_0)=\rho-\mu(B_0)\,,$$

where we have $\mu(B_0)>0$ as $|\varphi[B_0]|=|\varphi[C]|>0$. This evidently contradicts the definition of the number ρ, and the proof is complete.

THEOREM 37. *Let a function $\varphi(x)$ be continuous over an analytic set A, subject to the condition* (G) *on A, and further such that $|\varphi[A]|<+\infty$. If we write by definition*

$$\psi(x)=|\varphi[A\cap(-\infty, x)]| \qquad \textit{for } x\in \boldsymbol{R}\,,$$

then $\psi(x)$ is a bounded, continuous, nondecreasing function and for every set X we have the relation

$$|\psi[X]|=|\psi[A\cap X]|\leqq|\varphi[A\cap X]|\,.$$

If in particular the set A is bounded, then $|\psi[A]|=|\varphi[A]|$.

Further, the set-function $|\psi[X]|$ coincides with $\psi^*(X)$ which denotes the outer Carathéodory measure determined by the function $\psi(x)$ in the usual way.

PROOF. It is convenient to express $\psi(x)=|\varphi[A(x)]|$, where we write $A(x)=A\cap(-\infty, x)$. The function $\psi(x)$ is evidently nondecreasing and we have $0\leqq\psi(x)\leqq|\varphi[A]|$. The set $\varphi[A(x)]$ is measurable, since it is analytic together with the set $A(x)$. It thus follows that

$$\psi(q)-\psi(p)=|\varphi[A(q)]|-|\varphi[A(p)]|=|\varphi[A(q)]\setminus\varphi[A(p)]|$$

whenever $p<q$. But it is obvious that

$$\varphi[A(q)]\setminus\varphi[A(P)]\subset\varphi[A\cap I]\,, \qquad \text{where } I=[p, q]\,.$$

Hence $\psi(I)\leqq|\varphi[A\cap I]|$ for every closed interval I.

We shall show that the function $\psi(x)$ is continuous. Let ξ be any point of $\boldsymbol{R}$. Since the function $\varphi(x)$ fulfils the condition (G) on A, there corresponds to each $\varepsilon>0$ a figure Z for which ξ is an interior point and which fulfils $|\varphi[A\cap Z]|<\varepsilon$. The point ξ is interior to a component interval, say P, of the figure Z and we clearly have $|\varphi[A\cap P]|<\varepsilon$. Hence, if we write $K(h)=[\xi-h, \xi+h]$ for $h>0$ and if we take h so small as to ensure $K(h)\subset P$, then

$$\psi(\xi+h)-\psi(\xi-h)\leqq|\varphi[A\cap K(h)]|\leqq|\varphi[A\cap P]|<\varepsilon\,,$$

which implies the continuity of $\psi(x)$ at $x=\xi$.

Let us prove in the next place that $|\psi[N]|\leqq|\varphi[N]|$ for every subset N of A. The set $\varphi[N]$ is clearly the limit of the ascending infinite sequence $\langle\varphi[N\cap I_n]; n\in\boldsymbol{N}\rangle$, where $I_n=[-n, n]$; and the same holds good of

the set $\psi[N]$ as well. It follows that

$$|\varphi[N]|=\lim_n|\varphi[N\cap I_n]| \quad \text{and} \quad |\psi[N]|=\lim_n|\psi[N\cap I_n]|\,.$$

Therefore we need only consider the case where N is bounded. We may further assume that N is nonvoid.

We have $|\varphi[N]|<+\infty$ since $N\subset A$, and there corresponds to each $\varepsilon>0$ an open set $D\supset\varphi[N]$ such that $|D|<|\varphi[N]|+\varepsilon$. The function $\varphi(x)$ being continuous on A, we can enclose each point $\xi\in N$ in an open interval $U(\xi)$ with length $|U(\xi)|<1$ and such that $\varphi[A\cap U(\xi)]\subset D$. The union, say U, of all the intervals $U(\xi)$ is a bounded open set containing N and we have $\varphi[A\cap U]\subset D$. If we write $H=[p,q]$, where (p,q) is a generic component interval of the set U, it is obvious that

$$\psi[N]\subset\psi[U]\subset\bigcup_H\psi[H]\,,\qquad |\psi[N]|\leqq\sum_H|\psi[H]|=\sum_H\psi(H)\,.$$

Let us denote by V_H the measurable set $\varphi[A(q)]\setminus\varphi[A(p)]$. Then the family of all the sets V_H is evidently countable and disjoint. Accordingly

$$|\psi[N]|\leqq\sum_H\psi(H)=\sum_H|\varphi[A(q)]\setminus\varphi[A(p)]|=\sum_H|V_H|$$

$$=\left|\bigcup_H V_H\right|\leqq\left|\bigcup_H\varphi[A\cap H]\right|=\left|\bigcup_H\varphi[A\cap H^\circ]\right|=|\varphi[A\cap U]|\,,$$

the last equality being clear from the fact that the set $\varphi[A\cap H^\circ]$, where H° denotes the interior of the interval H, differs from $\varphi[A\cap H]$ at most by two points. We thus obtain the appraisal

$$|\psi[N]|\leqq|\varphi[A\cap U]|\leqq|D|<|\varphi[N]|+\varepsilon\,.$$

Since ε is arbitrary, this gives $|\psi[N]|\leqq|\varphi[N]|$, Q. E. D.

Let us now consider the outer Carathéodory measure $\psi^*(X)$ defined for every set X in the manner described on p. 64 of Saks [6]. We then have $\psi^*(X)=|\psi[X]|$ identically for every X, on account of a theorem on p. 100 of [6]. Moreover, for any set E measurable $(\mathfrak{L}_\psi)$, there is a sigma-compact set $L\subset E$ such that $|\psi[E\setminus L]|=0$ (see p. 69 of [6]). On the other hand, every analytic set is measurable $(\mathfrak{L}_\psi)$, as stated in small print on p. 48 of [6].

We shall proceed to establish that the function $\psi(x)$ is null on the set $S=\boldsymbol{R}\setminus A$, or in other words, that $\psi[S]$ is a null set. As S is measurable $(\mathfrak{L}_\psi)$ together with A, there exists in S a sigma-compact set L such that $|\psi[S\setminus L]|=0$. It thus suffices to verify that $\psi(x)$ is null on every compact set $Q\subset S$. Taking such set Q and assuming Q nonvoid as we may, we express Q as the limit of a descending infinite sequence of figures, say

$Z_1 \supset Z_2 \supset \cdots$. Since the function $\psi(x)$ is continuous and nondecreasing, we find at once that

$$|\psi[Q]| \leqq |\psi[Z_n]| = \psi(Z_n) \qquad \text{for } n \in \boldsymbol{N},$$

where $\psi(Z_n)$ denotes the total increment of $\psi(x)$ on the figure Z_n. But $\psi(Z_n) \leqq |\varphi[A \cap Z_n]|$, so that $|\psi[Q]| \leqq |\varphi[A \cap Z_n]|$. Indeed, if we put $J=[r, s]$, where J ranges over the component intervals of Z_n, then the measurable sets $W_J = \varphi[A(s)] \setminus \varphi[A(r)]$ are mutually disjoint and we have

$$\psi(Z_n) = \sum_J \psi(J) = \sum_J |W_J| = \left| \bigcup_J W_J \right| \leqq \left| \bigcup_J \varphi[A \cap J] \right| = |\varphi[A \cap Z_n]|,$$

whence the result follows. Now the function $\varphi(x)$ fulfils the condition (G) on A, and hence given any $\varepsilon > 0$ there exists an open set T containing Q and such that $|\varphi[A \cap (T \setminus Q)]| < \varepsilon$. But this inequality means the same as $|\varphi[A \cap T]| < \varepsilon$, since $A \cap Q$ is void on account of $Q \subset S = \boldsymbol{R} \setminus A$. On the other hand, the relation $Q \subset T$ and the choice of the sequence $Z_1 \supset Z_2 \supset \cdots$ together show that $Z_n \subset T$ for large n. Hence

$$|\psi[Q]| \leqq |\varphi[A \cap Z_n]| \leqq |\varphi[A \cap T]| < \varepsilon$$

for large n. Since ε is arbitrary, we obtain $|\psi[Q]| = 0$. This proves that $|\psi[S]| = 0$, as desired.

We have already seen that $|\psi[N]| \leqq |\varphi[N]|$ for every set $N \subset A$. This inequality can now be improved to the following relation, where X is an arbitrary set:

$$|\psi[X]| = |\psi[A \cap X]| \leqq |\varphi[A \cap X]|.$$

In point of fact, if we write $N = A \cap X$ and if S means the set $\boldsymbol{R} \setminus A$ as above, we have successively

$$X \subset N \cup S, \quad \psi[X] \subset \psi[N] \cup \psi[S], \quad |\psi[X]| \leqq |\psi[N]| + |\psi[S]|.$$

But $|\psi[S]| = 0$, $|\psi[N]| \leqq |\psi[X]|$, and $|\psi[N]| \leqq |\varphi[N]|$. Hence the result.

Suppose finally that the set A is bounded. We have to show that then $|\psi[A]| = |\varphi[A]|$. Choosing a $k \in \boldsymbol{N}$ such that $A \subset [-k, k]$, we have

$$\psi(x) = 0 \text{ for } x < -k \quad \text{and} \quad \psi(x) = |\varphi[A]| \text{ for } x > k.$$

But $\psi(x)$ is a nondecreasing continuous function, and thus $|\psi[\boldsymbol{R}]| = |\varphi[A]|$. This, combined with $|\psi[X]| = |\psi[A \cap X]| \leqq |\varphi[A \cap X]|$ established just now, leads to the following relation which implies $|\psi[A]| = |\varphi[A]|$:

$$|\varphi[A]| = |\psi[\boldsymbol{R}]| = |\psi[A]| \leqq |\varphi[A]|.$$

REMARK. The boundedness of the set A is essential for the validity of $|\psi[A]| = |\varphi[A]|$, as seen at once by considering the function $\varphi(x) = \sin x$

and the set $A=\boldsymbol{R}$ on which $\varphi(x)$ clearly fulfils the condition (G).

THEOREM 38. *Every function $\varphi(x)$ which is continuous on an analytic set A and subject to the condition* (G) *on this set, is measure-controllable on A.*

More precisely, there exists an admissible measure μ by which the function $\varphi(x)$ is controlled over A and which satisfies the inequality $\mu(M)\leqq|\varphi[A\cap M]|$ for every analytic set M.

PROOF. The function $\tan t$ fulfils the condition (N) on the interval $|t|<2^{-1}\pi$, while we have $|F[X]|\leqq|\varphi[X]|$ for the function $F(x)=\mathrm{Tan}^{-1}\varphi(x)$ and for each set X. Hence it suffices to establish the assertion for $F(x)$ in place of $\varphi(x)$.

Since the function $\varphi(x)$ is continuous on A, so is also $F(x)$ on A. Moreover, $\varphi(x)$ fulfils the condition (G) on A, and the same property is inherited by the function $F(x)$ on account of $|F[X]|\leqq|\varphi[X]|$ mentioned above. Again, $F(x)$ is bounded since $|F(x)|<2^{-1}\pi$.

We now apply the preceding theorem to the function $F(x)$ and any bounded analytic set $E\subset A$. Thus there exists for each E an admissible measure $\mu(M;E)$ such that $\mu(M;E)\leqq|F[E\cap M[|$ for any analytic set M and further that $\mu(E;E)=|F[E]|$.

An elementary figure will be called *rational,* if its boundary points are rational numbers. As readily seen, the family of all the rational figures is countable. Hence this family can be arranged in a distinct infinite sequence, say $\langle Z_1, Z_2, \cdots\rangle$. We then write by definition

$$\mu_n(M)=\mu(M;A\cap Z_n) \quad \text{and} \quad \mu(M)=\sum_n 2^{-n}\mu_n(M)\,,$$

where M is a generic analytic set and $n\in\boldsymbol{N}$. The convergence of this series is evident from the appraisal $|\mu_n(M)|\leqq|F[A\cap Z_n]|\leqq\pi$. We find at once that the set-function $\mu(M)$, thus defined, is an admissible measure. Moreover, we have $\mu_n(M)\leqq|F[M\cap A\cap Z_n]|\leqq|F[A\cap M]|$, so that

$$\mu(M)\leqq\sum_n 2^{-n}|F[A\cap M]|=|F[A\cap M]|\,.$$

Let us proceed to show that the function $F(x)$ is controlled on A by the measure μ, or in other words, that for each analytic set $S\subset A$ the relation $\mu(S)=0$ implies $|F[S]|=0$. On account of Theorem 2, there corresponds to such a set S and any number $\varepsilon>0$ a compact set $C\subset S$ such that $|F[C]|>|F[S]|-\varepsilon$. We then can enclose C in the interior of a figure Z such that $|F[A\cap(Z\setminus C)]|<\varepsilon$. We do not lose generality by assuming that Z is a rational figure, so that $Z=Z_k$ for some $k\in\boldsymbol{N}$. It follows that

$$|F[S]|-\varepsilon<|F[C]|\leqq|F[A\cap Z]|=\mu(A\cap Z;A\cap Z)$$
$$\leqq\mu(Z;A\cap Z)=\mu(C;A\cap Z)+\mu(Z\setminus C;A\cap Z)\,.$$

But we have on one hand the relation

$$\mu(S)\geqq\mu(C)=\sum_n 2^{-n}\mu(C;A\cap Z_n)\geqq 2^{-k}\mu(C;A\cap Z)\,,$$

while on the other hand the above inequality $\mu(M;E)\leqq|F[E\cap M]|$ shows that $\mu(Z\setminus C;A\cap Z)\leqq|F[A\cap(Z\setminus C)]|<\varepsilon$. We therefore obtain the estimation $|F[S]|<2^k\mu(S)+2\varepsilon$. Consequently, if $\mu(S)=0$, we have $|F[S]|<2\varepsilon$ for every $\varepsilon>0$, so that $|F[S]|=0$. This completes the proof.

THEOREM 39. *Every function $\varphi(x)$ which is continuous on an analytic set A and subject to the condition* (T_1) *on this set, is measure-controllable on A.*

PROOF. It is enough to show that the function $F(x)=\mathrm{Tan}^{-1}\varphi(x)$ is measure-controllable on the set A. This function is continuous on A and we have $|F[A]|\leqq\pi$. Moreover, $F(x)$ fulfils the condition (T_1) on A, since $\varphi(x)$ does so by hypothesis and since the function $\mathrm{Tan}^{-1}y$ fulfils the condition (N) on the real line. It follows that $F(x)$ fulfils the condition (F) on A. Theorem 1 then shows that this function fulfils the condition (G) on A. We can thus apply the foregoing theorem to $F(x)$ and A, and we conclude that $F(x)$ is measure-controllable on A. This completes the proof.

THEOREM 40. *Every function $\varphi(x)$ which is continuous on an analytic set A and analytically partitionable* (T_1) *on this set, is measure-controllable on A.*

PROOF. By hypothesis the set A is expressible as the union of a disjoint infinite sequence $\langle A_1, A_2, \cdots\rangle$ of analytic sets on each of which the function $\varphi(x)$ fulfils the condition (T_1). It follows from the foregoing theorem that there exists for each $n\in\boldsymbol{N}$ an admissible measure $\mu_n(X)$ by which $\varphi(x)$ is controlled on the set A_n. We can associate with each n a number $a_n>0$ such that $a_n\cdot\mu_n(\boldsymbol{R})<2^{-n}$, since $\mu_n(\boldsymbol{R})$ is finite for each n. Writing now by definition

$$\mu(M)=a_1\cdot\mu_1(M)+a_2\cdot\mu_2(M)+\cdots\,,$$

where M is any analytic set, we find immediately that $\mu(M)$ is an admissible measure.

If $\mu(M)=0$ for an analytic set $M\subset A$, then for each n we have successively the relations:

$$\mu_n(M)=0, \quad \mu_n(A_n \cap M)=0, \quad |\varphi[A_n \cap M]|=0 .$$

Consequently the set $\varphi[M]$, which is the union of the sets $\varphi[A_n \cap M]$, is null. The function $\varphi(x)$ is thus controlled on A by the measure μ.

DEFINITION. A function $\varphi(x)$ will be said to fulfil the *countability condition* on a set E, if this set contains no disjoint noncountable family of compact sets on each of which the function is nonnull (namely, each of which is mapped by the function onto a nonnull set).

THEOREM 41. *Every function $\varphi(x)$ which is continuous on an analytic set A and measure-controllable on this set, fulfils the countability condition on A.*

PROOF. Let μ be an admissible measure by which the function $\varphi(x)$ is controlled on the set A. Suppose given any nonvoid disjoint family $\mathfrak{M}$ of compact sets $C \subset A$ on each of which $\varphi(x)$ is nonnull. Then $\mu(C)>0$ for every $C \in \mathfrak{M}$, since $\mu(C)=0$ would imply $|\varphi[C]|=0$.

We shall show that the family $\mathfrak{M}$ is necessarily countable. For this purpose, let $\mathfrak{M}_n$ denote for each $n \in \boldsymbol{N}$ the family of the sets $C \in \mathfrak{M}$ such that $\mu(C)>n^{-1}$. It is evident that $\mathfrak{M}=\mathfrak{M}_1 \cup \mathfrak{M}_2 \cup \cdots$. If now $\mathfrak{M}_n$ is infinite for an n, then for each $k \in \boldsymbol{N}$ we can extract from $\mathfrak{M}_n$ a disjoint sequence $\langle C_1, \cdots, C_k \rangle$ of sets, and the following relation will contradict the finiteness of $\mu([\mathfrak{M}])$:

$$kn^{-1}<\mu(C_1)+\cdots+\mu(C_k)=\mu(C_1 \cup \cdots \cup C_k) \leqq \mu([\mathfrak{M}]) .$$

The family $\mathfrak{M}_n$ is therefore finite for every n. It ensues that the family $\mathfrak{M}$ must be countable. The function $\varphi(x)$ thus fulfils the countability condition on A.

THEOREM 42. *Every function $\varphi(x)$ which is continuous on an analytic set A and which fulfils the countability condition on this set, is analytically partitionable* (T_1) *on A.*

More precisely, the set A contains a set M with null image $\varphi[M]$ and such that the set $A \setminus M$ is expressible as the union of a disjoint countable family of compact sets on each of which the function $\varphi(x)$ is biunique and nonnull.

PROOF. By an *admissible family* let us temporarily understand any disjoint family of compact sets contained in A and on each of which the function $\varphi(x)$ is both biunique and nonnull. For instance, the void family is admissible. Every admissible family must be countable, since $\varphi(x)$ fulfils the countability condition on A.

Let Γ be the collection of all the admissible families. It is obvious that Γ is partially ordered by family inclusion and that every linearly ordered subcollection of Γ is bounded from above. Accordingly, by Zorn's Lemma, the collection Γ has at least one maximal element, say $\mathfrak{S}$.

Since $\mathfrak{S}$ is a countable family of compact sets, its union $[\mathfrak{S}]$ is a Borel set, and hence the set $M=A\setminus[\mathfrak{S}]$ is analytic. It remains to prove that $|\varphi[M]|=0$. For this purpose suppose, if possible, that this is false. Then Theorem 2 and Theorem 3 together show that the set M contains a compact set C on which $\varphi(x)$ is both biunique and nonnull. But C is plainly disjoint with $[\mathfrak{S}]$. The existence of such a set C contradicts that the family $\mathfrak{S}$ is maximal in the collection Γ. We therefore have $|\varphi[M]|=0$, which completes the proof.

THEOREM 43. *For every function $\varphi(x)$ which is continuous over an analytic set A, the following three properties are mutually equivalent.*

(1) *The function is measure-controllable on A;*

(2) *the function fulfils the countability condition on A;*

(3) *the function is analytically partitionable* (T_1) *on A.*

This follows immediately from Theorems 41, 42, and 40. By the way, Theorems 41 and 42 together yield a more particular result than Theorem 36.

THEOREM 44. *Every function $\varphi(x)$ which is continuous on a bounded analytic set A and subject to the condition* (G) *on this set, fulfils the condition* (T_1) *on A.*

PROOF. Let I be a closed interval which contains the set A. This interval will be kept fixed during the proof.

By Theorems 38, 41, and 42 there exist in A a set M and a disjoint infinite sequence $\langle C_1, C_2, \cdots\rangle$ of compact sets, in such a manner that

$$|\varphi[M]|=0\,, \quad A\setminus M=C_1\cup C_2\cup\cdots\,,$$

and that the function $\varphi(x)$ is biunique on C_n for every n.

Since $\varphi(x)$ fulfils the condition (G) on A, given any $\varepsilon>0$ there corresponds to each n an open set D_n containing the set C_n and such that $|\varphi[A\cap(D_n\setminus C_n)]|<2^{-n}\varepsilon$. Writing $S=D_1\cup D_2\cup\cdots$ and $Q=I\setminus S$, we find that Q is a compact set. Hence there is an open set $D\supset Q$ such that $|\varphi[A\cap(D\setminus Q)]|<\varepsilon$. Then the sequence $\langle D, D_1, D_2, \cdots\rangle$, which consists of open sets, covers the interval I. Consequently I is already covered by a partial sequence $\langle D, D_1, \cdots, D_n\rangle$ of this sequence.

This being so, let us write for simplicity

$$B=A\cap D \quad \text{and} \quad B_i=A\cap D_i \quad \text{for } i=1,\cdots,n.$$

Then $A=B\cup B_1\cup\cdots\cup B_n$, since $A\subset I\subset D\cup D_1\cup\cdots\cup D_n$. But

$$B=(A\cap Q)\cup[A\cap(D\setminus Q)], \qquad \text{where } A\cap Q=A\setminus S\subset M.$$

Accordingly $|\varphi[B]|\leqq|\varphi[M]|+|\varphi[A\setminus(D\cap Q)]|<\varepsilon$. On the other hand

$$B_i=A\cap D_i=C_i\cup[A\cap(D_i\setminus C_i)] \qquad \text{for } i=1,\cdots,n.$$

Hence, writing $S_n=C_1\cup\cdots\cup C_n$, we find successively that

$$\begin{aligned} A\setminus S_n &\subset B\cup(B_1\setminus C_1)\cup\cdots\cup(B_n\setminus C_n) \\ &\subset B\cup[A\cap(D_1\setminus C_1)]\cup\cdots\cup[A\cap(D_n\setminus C_n)], \end{aligned}$$

$$|\varphi[A\setminus S_n]|\leqq|\varphi[B]|+\sum_{i=1}^{n}|\varphi[A\cap(D_i\setminus C_i)]|<\varepsilon+\sum_{i=1}^{n}2^{-n}\varepsilon<2\varepsilon.$$

Consider now the set Y of all the values of the function $\varphi(x)$ each of which is assumed by $\varphi(x)$ an infinity of times on A. If a number y belongs to the set $Y\setminus\varphi[A\setminus S_n]$, then $\varphi(x)$ must take the value y infinitely often on the set $S_n=C_1\cup\cdots\cup C_n$. But this is impossible, since $\varphi(x)$ is biunique on each of the sets $C_1,\cdots,C_n$. Consequently we have the inclusion $Y\subset\varphi[A\setminus S_n]$, whence $|Y|\leqq|\varphi[A\setminus S_n]|<2\varepsilon$. Since ε is arbitrary, this implies the nullity of Y. The theorem is thus established.

REMARK. In the abve theorem, the boundedness of the set A is not a superfluous hypothesis. This may be seen by considering the function $\sin x$, which is, on the real line, subject to the condition (G), but not to the condition (T_1).

EXAMPLES. (i) A function will temporarily be called *piecewise linear* on a closed interval, if this interval is expressible as the union of a finite number of non-overlapping closed intervals on each of which the function is linear.

Let us construct a function which is both continuous and Borel partitionable (T_1), on a closed interval, without fulfilling the condition (T_1) on this interval.

For this purpose, let P be a bounded, perfect, nowhere dense set of positive measure. As shown on p. 224 of Saks [6], we can construct a function $\varphi(x)$ which vanishes for $x\in P$, is piecewise linear on each closed interval contiguous to P, is continuous on the closed interval $I=\square P$, and has no derivative (finite or infinite) at any point of P. Together with $\varphi(x)$, the function $\psi(x)=\varphi(x)+x$ is continuous on I and has no derivative at any point of P. But the image $\psi[P]$, which coincides with P, has

positive measure. It follows from Theorem (6.2) on p. 278 of Saks [6] that the function $\psi(x)$ cannot fulfil the condition (T_1) on I. On the other hand, $\psi(x)$ is biunique on P and piecewise linear on every closed interval contiguous to P. Hence $\psi(x)$ is Borel partitionable (T_1) on I.

(ii) Let us show by an example that there is a function $\varphi(t)$ which is continuous on the unit interval $I=[0,1]$ without being analytically partitionable (T_1) on this interval.

Let S be the unit square, namely the set of the points $\langle x, y\rangle$, where $x\in I$ and $y\in I$, and let $f(t)$ be a Peano curve defined on I and which maps I continuously onto S. Writing $f(t)=\langle x(t), y(t)\rangle$, consider any function $\varphi(t)$ which coincides with $x(t)$ for $t\in I$. The function $\varphi(t)$ is plainly continuous on I. On the other hand, there exists for each $\xi\in I$ a noncountable infinity of points $t\in I$ such that $\varphi(t)=\xi$. Hence $\varphi(t)$ does not fulfil the condition (T_2) of Banach on I (see p. 277 of Saks [6]). It is obvious, however, that a function which is analytically partitionable (T_1) on a set, necessarily fulfils the condition (T_2) on this set. Accordingly the function $\varphi(x)$ cannot be analytically partitionable (T_1) on I.

We shall end this paper with a theorem which supplements the definition of an admissible measure and which clarifies the nature of such a measure.

THEOREM 45. *Given any admissible measure μ, there always exists a nondecreasing function $\psi(x)$ such that $\mu(M)=\psi^*(M)$ for every analytic set M, where ψ^* denotes as hitherto the outer Carathéodory measure determined by the function $\psi(x)$.*

PROOF. The measure μ, which is completely additive on the class of the analytic sets, is *a fortiori* so on the Borel class. Therefore, by Theorem (6.10) on p. 71 of Saks [6], there is a nondecreasing function $\psi(x)$ such that $\mu(B)=\psi^*(B)$ for every bounded Borel set B, and hence for every Borel set B.

We shall prove that $\mu(M)=\psi^*(M)$ for every analytic set M. Since M is measurable $(\mathfrak{L}_\psi)$, Theorem (6.6) on p. 69 of Saks [6] shows that there exists in M a Borel set B for which we have $\psi^*(M\setminus B)=0$. Since the set $M\setminus B$ is also measurable $(\mathfrak{L}_\psi)$, the same theorem ensures the existence of a Borel set $L\supset M\setminus B$ such that $\psi^*(L)=0$. This, together with the relation

$$0\leqq\mu(M\setminus B)\leqq\mu(L)=\psi^*(L)\,,$$

requires that $\mu(M\setminus B)=0$. It follows that

$$\mu(M)=\mu(B)+\mu(M\setminus B)=\mu(B)$$
$$=\psi^*(B)=\psi^*(B)+\psi^*(M\setminus B)=\psi^*(M)\,,$$

which completes the proof.

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