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On the Sparse Integration

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This is a continuation of our recent papers on integration theory. We shall be concerned, among other things, with showing that the normal integration considered in [1] is capable of still further generalization. However, we do not know if the generalization is strict. The new integration, which we shall call the sparse integration, will be based on the notion of sparse continuity of functions.

The terminology and the notation of [1] will be used freely. We shall thus understand by a *function*, by itself, any mapping of the real line into itself, unless another meaning is obvious from the context. The sets considered in this paper will exclusively be linear.

An elementary figure W will be termed *sparse*, if every component interval of W is shorter than every open interval contiguous to W. For instance, each closed interval and the void set are sparse figures.

Plainly, *every sparse figure is severed normally by every linear set*. Further, *if each component of a sparse figure W contains at most one component of a figure $Z \subset W$, then F must itself be sparse.*

A function $\varphi(x)$ will be called *sparsely continuous*, or briefly SC, on a set E, if $\varphi(W) \to 0$ as $|W| \to 0$, where W is a generic sparse figure pertaining to E. *When this is the case, the function $\varphi(x)$ is uniformly continuous on E*, as easily seen. Clearly, *the sparse continuity on E is hereditary with respect to E.*

In the above definition, the total increment $\varphi(W)$ may be replaced by the total absolute increment $\varphi^*(W)$. This is immediate from the relation $|\varphi(W)| \leqq \varphi^*(W)$ and the following two facts:

(i) Given a function $\varphi(x)$, any figure Z is expressible as the union of two disjoint figures Z_1 and Z_2 such that $\varphi^*(Z_1) = |\varphi(Z_1)|$ and $\varphi^*(Z_2) = |\varphi(Z_2)|$.

(ii) If the union of two disjoint figures is sparse, then so is also each of the two figures separately.

The following propositions are readily verified: *Every function which is absolutely continuous, or more generally, normally continuous, on a set E, is sparsely continuous on E.* Again, *every linear combination of two* *functions which are sparsely continuous on a set E, is itself sparsely continuous on E.* Further, *every function which is sparsely continuous on a set E and continuous on a set $M \supset E$, is sparsely continuous on the whole set M, provided that M is contained in the closure of E.*

On the other hand, we do not know whether *a function which is sparsely continuous on a measurable set, is necessarily* AD *at almost all points of this set.*

THEOREM 1. *Given a function $\varphi(x)$ which is bounded on a subset M of a nonvoid figure Z, denote by J a generic component interval of Z. There then correspond to each number $\varepsilon > 0$ two disjoint sparse figures $U \subset Z$ and $V \subset Z$ pertaining to M and such that*

$$\sum_J |\varphi[M \frown J]| < \varphi^*(U) + \varphi^*(V) + \varepsilon .$$

PROOF. Let h be a positive number. A closed interval of the form $[qh, (q+1)h]$, where q is any integer, will be termed *even interval* or *odd interval, of length h,* according as q is even or odd, respectively.

For each integer $n > 0$ let A_n [or B_n] be the union, possibly void, of all the even intervals [or odd intervals] of length 2^{-n} contained in the interior Z° of the figure Z and let us write $C_n = A_n \cup B_n$. Then A_n, B_n and C_n are figures and Z° is the limit of the ascending sequence $C_1 \subset C_2 \subset \cdots$. Hence, if J is a component of Z, the set $\varphi[M \frown J^\circ] = \varphi[(M \frown J) \frown Z^\circ]$ is the limit of the ascending sequence $\langle \varphi[M \frown J \frown C_n] \rangle$. It follows that

$$|\varphi[M \frown J]| = |\varphi[M \frown J^\circ]| = \lim_{n \to \infty} |\varphi[M \frown J \frown C_n]| ,$$

where $|\varphi[M \frown J]|$ is finite since $\varphi[M]$ is a bounded set by hypothesis. Accordingly, given any number $\eta > 0$, we can choose n so large that

$$|\varphi[M \frown J]| < |\varphi[M \frown J \frown C_n]| + \eta \qquad \text{for every } J.$$

We shall keep η and this n fixed in the sequel.

We now distinguish two cases, according as the set $M \frown A_n$ is finite or infinite. We shall deal first with the latter case.

Let K be a generic component of the figure A_n such that $M \frown K$ is an infinite set. Since $|\varphi[M \frown K]| = |\varphi[M \frown K^\circ]| \leqq \mathrm{d}(\varphi[M \frown K^\circ])$, the interval K° contains a closed interval, say L, pertaining to M and fulfilling the inequality $|\varphi[M \frown K]| < |\varphi(L)| + \eta/N$, where N denotes the number of the components of A_n. We associate such an interval $L = L_K$ with each K and we denote by U the union of all the intervals L_K. Since every component of A_n is an even interval of length 2^{-n}, we find that U is a sparse figure pertaining to M. Now, the intersection $J \frown U$, where J is any component of the figure Z, is the union (which may be void) of the intervals L_K for all the $K \subset J$. It follows that

$$|\varphi[M \cap J \cap A_n]| \leqq \sum_{K \subset J} |\varphi[M \cap K]| \leqq \sum_{K \subset J} \left\{ |\varphi(L_K)| + \frac{\eta}{N} \right\} \leqq \varphi^*(J \cap U) + \eta,$$

where the two sums mean zero if there is no $K \subset J$.

In the remaining case in which $M \cap A_n$ is a finite set, we choose U to be the void figure. Then $|\varphi[M \cap J \cap A_n]| = 0 < \varphi^*(J \cap U) + \eta$.

Replacing the figure A_n by B_n in the above construction of the figure U, we obtain a sparse figure V pertaining to M and such that

$$|\varphi[M \cap J \cap B_n]| \leqq \varphi^*(J \cap V) + \eta \qquad \text{for every } J.$$

The figures U and V are disjoint, since U [or V] is contained in the interior of the figure A_n [or of B_n] and since A_n does not overlap B_n.

If m denotes the number of the components of Z, the results established in the above lead together to the following appraisal:

$$\begin{aligned}
&\sum_J |\varphi[M \cap J]| < \sum_J |\varphi[M \cap J \cap C_n]| + m\eta \\
&\leqq \sum_J |\varphi[M \cap J \cap A_n]| + \sum_J |\varphi[M \cap J \cap B_n]| + m\eta \\
&\leqq \sum_J \varphi^*(J \cap U) + \sum_J \varphi^*(J \cap V) + 3m\eta \\
&= \varphi^*(U) + \varphi^*(V) + 3m\eta .
\end{aligned}$$

This completes the proof, since we may suppose η so small that $3m\eta < \varepsilon$.

THEOREM 2. *Every function $\varphi(x)$ which is sparsely continuous on a closed interval I, is absolutely continuous on I.*

PROOF. By hypothesis, there corresponds to each $\varepsilon > 0$ a number $\delta > 0$ such that, for every sparse figure $W \subset I$, the inequality $|W| < \delta$ implies $\varphi^*(W) < \varepsilon$. It suffices to show that $\varphi^*(Z) < 3\varepsilon$ for every figure $Z \subset I$ with $|Z| < \delta$. We may plainly assume Z nonvoid.

Let J denote a generic component of Z. The function $\varphi(x)$, which is sparsely continuous on I, is continuous and hence bounded, on this interval. Accordingly Theorem 1, where we specialize the set M to Z itself, ensures the existence of two sparse figures $U \subset Z$ and $V \subset Z$ such that

$$\sum_J |\varphi[J]| < \varphi^*(U) + \varphi^*(V) + \varepsilon .$$

But $\varphi^*(U) < \varepsilon$ and $\varphi^*(V) < \varepsilon$, on account of $|U| < \delta$ and $|V| < \delta$. Thus

$$\varphi^*(Z) = \sum_J |\varphi(J)| \leqq \sum_J |\varphi[J]| < 3\varepsilon,$$

since $\varphi(x)$ is continuous on I and hence $|\varphi(J)| \leqq |\varphi[J]|$ for every J. This completes the proof.

THEOREM 3. *A function $\varphi(x)$ which is sparsely continuous on a set E, necessarily maps every closed null set $S \subset E$ onto a null set.*

PROOF. As in the proof for Theorem 2 of [1], we may assume the null set S to be nonvoid, perfect, and bounded.

By hypothesis, given any $\varepsilon > 0$ there is a number $\delta > 0$ such that for every sparse figure W pertaining to E, the inequality $|W| < \delta$ implies $\varphi^*(W) < \varepsilon$. We can easily enclose S in a figure Z with $|Z| < \delta$ and pertaining to S. The function $\varphi(x)$, being sparsely continuous on E, is continuous on E and hence bounded on S. Accordingly, by Theorem 1, there exist in Z two sparse figures U and V pertaining to S and such that

$$\sum_J |\varphi[S \cap J]| < \varphi^*(U) + \varphi^*(V) + \varepsilon,$$

where J stands for a generic component of Z. Since $|U| < \delta$ and $|V| < \delta$, we have $\varphi^*(U) < \varepsilon$ and $\varphi^*(V) < \varepsilon$. We thus obtain the following relation, which completes the proof since ε is arbitrary:

$$|\varphi[S]| \leqq \sum_J |\varphi[S \cap J]| < 3\varepsilon.$$

A function will be called *generalized sparsely continuous,* or GSC for short, on a set E, if the function is continuous on E and if E is expressible as the union of a sequence of sets on each of which the function is sparsely continuous. *This property of a function is clearly hereditary with respect to the set E.*

The following propositions are obvious. (i) *Every function which is* GAC *on a set E, or more generally,* GNC *on E, is* GSC *on this set.* (ii) *Every linear combination of two functions which are* GSC *on a set, is itself* GSC *on this set.* (iii) *A function which is* GSC *on a closed set, necessarily maps every closed null subset of this set onto a null set.*

Of the following four theorems, the first one is provable in a routine way, while each of the other three may be established in the same way as for the corresponding theorem of [1].

THEOREM 4. *In order that a function which is continuous on a nonvoid closed set S, be generalized sparsely continuous on S, it is necessary and sufficient that every nonvoid closed subset of S contain a portion on which the function is sparsely continuous.*

THEOREM 5. *Every function which is generalized sparsely continuous* *on a closed interval I and which possesses a nonnegative approximate derivative, finite or infinite, at almost every point of this interval, is monotone nondecreasing on I.*

THEOREM 6. *If two functions are generalized sparsely continuous on a closed interval I and approximately equiderivable almost everywhere on I, then the functions differ over I only by an additive constant.*

THEOREM 7. *Every function which is both* BV *and* GSC, *on a closed set, is* AC *on this set. Thus every function which is both* GBV *and* GSC, *on a closed set, is* GAC *on this set.*

We think it needless to state at great length the descriptive definition of the *sparse integration.* The basic properties of the sparse integral are the same as those of the normal integral, inclusive of the integration by parts theorem and the second mean value theorem.

The sparse integration clearly includes the normal integration (though we do not know if they are the same things) and hence generalizes strictly the Denjoy integration.

THEOREM 8. *If a function $\varphi(x)$ is sparsely continuous on a set E, there corresponds to each $\eta>0$ a number $\delta>0$ such that, for every finite disjoint sequence $\langle Q_1, \cdots, Q_n\rangle$ of compact sets which are contained in E, the inequality $|Q_1|+\cdots+|Q_n|<\delta$ implies $|\varphi[Q_1]|+\cdots+|\varphi[Q_n]|<\eta$.*

PROOF. We may assume the sets $Q_1, \cdots, Q_n$ nonvoid and perfect, without loss of generality. By hypothesis, given any $\varepsilon>0$ there is a $\delta>0$ such that for every sparse figure W pertaining to E, the inequality $|W|<\delta$ implies $\varphi^*(W)<\varepsilon$. The numbers ε and δ will be kept fixed in the sequel.

Let Z denote a generic figure pertaining to the set $Q=Q_1\cup\cdots\cup Q_n$ and containing Q. If $n\geqq 2$, then by the same argument as in the proof for Theorem 13 of [1], we can choose Z in such manner that $|Z|<\delta$ and that, for each component J of Z, the intersection $Q\cap J$ is contained in one of the sets $Q_1, \cdots, Q_n$. But such choice of Z is clearly possible for $n=1$ also. We shall keep fixed the figure Z thus chosen.

Let us arrange the components J of Z in a sequence $J_1<\cdots<J_m$. Then the partition of Q into the sets $Q\cap J_1, \cdots, Q\cap J_m$ must be a refinement of the partition $Q=Q_1\cup\cdots\cup Q_n$. We therefore have

$$|\varphi[Q_1]|+\cdots+|\varphi[Q_n]|\leqq|\varphi[Q\cap J_1]|+\cdots+|\varphi[Q\cap J_m]|.$$

By Theorem 1, on the other hand, there are in Z two sparse figures U and V pertaining to Q and such that

$$|\varphi[Q\cap J_1]|+\cdots+|\varphi[Q\cap J_m]|<\varphi^*(U)+\varphi^*(V)+\varepsilon.$$

But $|U|\leqq|Z|<\delta$ and similarly for V, so that $\varphi^*(U)+\varphi^*(V)<2\varepsilon$. We thus find finally that $|\varphi[Q_1]|+\cdots+|\varphi[Q_n]|<3\varepsilon$, which completes the proof since we may take $\varepsilon=\eta/3$ in the above.

THEOREM 9. *A function $\varphi(x)$ which is sparsely continuous on a sigma-closed set A, necessarily fulfils the condition* (S_0) *on this set. Moreover, the function has finite fluctuation on every bounded Borel set $B\subset A$.*

PROOF. Theorem 8 implies in particular that every function which is sparsely continuous on a set E, fulfils the weak condition (S) on E. This, combined with Theorem 11 of [1], shows that the function $\varphi(x)$ fulfils the condition (N) on A. Then the condition (S_0) part of the assertion may be deduced in the same way as for Theorem 14 of [1]. Finally, the finiteness of the fluctuation $\Xi(\varphi;B)$ follows from Theorem 18 of [1].

From now on, if a theorem is stated without a proof, let it be tacitly understood that the proof is the same as that of the corresponding result in the paper [1].

THEOREM 10. *Given a function $\varphi(x)$ which is sparsely continuous on a bounded sigma-closed set A and approximately derivable at almost all points of A, let us write, for definiteness, $\varphi'_{\mathrm{ap}}(\xi)=0$ for every point $\xi\in\boldsymbol{R}$ at which the function $\varphi(x)$ is not approximately derivable.*

Then the function $\varphi'_{\mathrm{ap}}(x)$ is summable over A. Further, the fluctuation $\Xi(\varphi;M)$ is expressed for every Borel set $M\subset A$ by the formula

$$\Xi(\varphi;M)=\int_M|\varphi'_{\mathrm{ap}}(x)|\,dx.$$

THEOREM 11. *Suppose that a function $\varphi(x)$ is sparsely continuous on every portion of a sigma-closed set A and approximately derivable at almost all points of A. In order that $\varphi(x)$ be steplike on A, it is necessary and sufficient that $\varphi(x)$ be approximately derivable to zero at almost all points of A.*

THEOREM 12. *Any function $\varphi(x)$ which is sparsely continuous on every portion of a sigma-closed set A and approximately derivable at almost all points of A, is* ACS *decomposable on A and fulfils the conclusions* (i) *to* (iii) *of the assertion stated at the beginning of* § 6 *of* [1].

A function $\varphi(x)$ will be termed *sparsely fluctuant*, or SF for short, or again to *fluctuate sparsely*, on a set E, if $\sup|\varphi(W)|<+\infty$, where W

stands for a generic sparse figure pertaining to E. Plainly, *this property of* $\varphi(x)$ *is hereditary with respect to* E.

When this is the case, we have also $\sup \varphi^*(W) < +\infty$, as found easily by means of the two propositions (i) and (ii) on p. 91. Moreover, *such a function* $\varphi(x)$ *is necessarily bounded on* E. We have further the following obvious propositions: (i) *A function which is* BV, *or more generally, normally fluctuant, on a set* E, *is always sparsely fluctuant on* E. (ii) *Every linear combination of two functions which fluctuate sparsely on a set, itself does so on this set.* (iii) *A function which fluctuates sparsely on a set* E, *necessarily does so on each set* $M \supset E$ *contained in the closure of* E, *provided that the function is continuous on* M.

THEOREM 13. *Every function which is sparsely continuous on a bounded set is sparsely fluctuant on this set.*

THEOREM 14. *If a function* $\varphi(x)$ *fluctuates sparsely on a set* E, *this set contains at most a countable infinity of points at each of which the function is discontinuous on* E. *More precisely, if* $\langle x_1, \cdots, x_n \rangle$ *is any finite distinct sequence of such points and if* W *denotes a generic sparse figure pertaining to* E, *we have the relation*

$$o_E(\varphi; x_1) + \cdots + o_E(\varphi; x_n) \leqq \sup \varphi^*(W) < +\infty .$$

THEOREM 15. *A function which is sparsely fluctuant on a set* E, *necessarily maps every Borel set* $M \subset E$ *onto a measurable set.*

By means of this theorem and Theorem 28 of [1], we can define the fluctuation $\Xi(\varphi; M)$ for every function $\varphi(x)$ and every Borel set M on which $\varphi(x)$ fluctuates sparsely. This fluctuation may be infinite.

THEOREM 16. *A function which fluctuates sparsely on a sigma-closed set* A, *necessarily has finite fluctuation on every Borel set* $M \subset A$.

PROOF. Writing $\rho(A) = \sup \varphi^*(W) < +\infty$, where W denotes a generic sparse figure pertaining to A, we shall show that $\Xi(\varphi; A) \leqq 2\rho(A)$. We can utilize the greater part of the ideas of the proof for Theorem 30 of [1]. We may suppose A bounded.

Taking an open interval $D \supset A$, consider any nonvoid finite set $S \subset D$ and let the components of the open set $D \setminus S$ be $D_1 < \cdots < D_n$ in their natural ordering. It suffices to prove the inequality

$$|\varphi[F \cap D_1]| + \cdots + |\varphi[F \cap D_n]| \leqq 2\rho(A)$$

for each closed set F contained in the set $A \setminus S$.

This being so, we enclose the set $F \cap D_i$ in a closed interval $J_i \subset D_i$ for $i=1, \cdots, n$ and we write $Z=J_1 \cup \cdots \cup J_n$, so that $F \subset Z$. By Theorem 1, there then correspond to each $\varepsilon>0$ two sparse figures U and V pertaining to F and such that

$$|\varphi[F \cap J_1]|+\cdots+|\varphi[F \cap J_n]|<\varphi^*(U)+\varphi^*(V)+\varepsilon,$$

where we must have both $\varphi^*(U) \leqq \rho(A)$ and $\varphi^*(V) \leqq \rho(A)$ by definition of the quantity $\rho(A)$. Consequently it follows that

$$\sum_{i=1}^{n} |\varphi[F \cap D_i]| \leqq \sum_{i=1}^{n} |\varphi[F \cap J_i]| < 2\rho(A)+\varepsilon.$$

This completes the proof, since ε is arbitrary.

THEOREM 17. *Any function which is sparsely continuous on every countable subset of a set E, is of necessity sparsely continuous on the whole set E.*

A function will be called *semisparsely continuous*, or SSC for short, on a set E, if it is sparsely continuous on every closed null set contained in E. Such a function is necessarily continuous on E, the proof being the same as in Theorem 31 of [1].

The following assertions are obvious. (i) *The semisparse continuity of a function on a set is hereditary with respect to this set.* (ii) *Every linear combination of two functions which are* SSC *on a set E, is itself* SSC *on E.* (iii) *A function which is* SSC *on a set E, necessarily maps every closed null set contained in E onto a null set.*

As Theorem 17 clearly implies, *a function which is sparsely continuous on every null subset of a set E, is necessarily sparsely continuous on the whole set E.* In contrast with this fact, we do not know if the following statement is true (though its converse is evidently true).

ASSERTION (a). *Every function which is semisparsely continuous on a closed set, is sparsely continuous on this set.*

We shall say that a function is *generalized semisparsely continuous*, or GSSC for short, on a set E, if the function is continuous on E and if E is expressible as the union of a sequence of closed sets on each of which the function is semisparsely continuous. When this is the case, the set E is sigma-closed of itself.

The following propositions are obvious. (i) *Every function which is generalized sparsely continuous on a sigma-closed set, is* GSSC *on this set.* (ii) *The* GSSC *property of a function on a sigma-closed set E is hereditary with respect to E.* (iii) *Every linear combination of two* *functions which are* GSSC *on a set E, is itself* GSSC *on E.* (iv) *A function which is* GSSC *on a set E, necessarily maps every closed null set contained in E onto a null set.*

We do not know if the following statement is true. At least, however, it is an evident consequence of Assertion (a).

ASSERTION (b). *Every function which is generalized semisparsely continuous on a closed interval I and approximately derivable almost everywhere on I, is generalized sparsely continuous on I.*

THEOREM 18. *Every function which is generalized semisparsely continuous on a closed interval I and has a nonnegative approximate derivative, finite or infinite, at almost every point of I, is monotone nondecreasing on I.*

THEOREM 19. *If two functions are generalized semisparsely continuous on a closed interval I and approximately equiderivable almost everywhere on I, then the funtions differ over I only by an additive constant.*

THEOREM 20. *Every function which is both* BV *and* GSSC, *on a closed set, is* AC *on this set. Hence, every function which is both* GBV *and* GSSC, *on a closed set, is* GAC *on this set.*

Using the above results, we can easily introduce an integration named *semisparse.* The definition and the basic properties of this integration are the same, *mutatis mutandis,* with those of the sparse integration. Among others, we have the integration by parts theorem and the second mean value theorem.

Of the sparse and the semisparse integration, the latter one plainly includes the former. As readily seen, on the other hand, the latter is strictly wider than the former, if and only if Assertion (b) is false.

Reference

[1] K. Iseki: On the Normal Integration. Nat. Sci. Rep. Ochanomizu Univ., 37 (1986), 1-34.