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# On the Differential Operator of Curvature Type Tensors

Dedicated to Professor Shun-ichi Tachibana
on his 60th birthday

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**Introduction.** Let $M^n$ be an $n$-dimensional Riemannian space with the Riemannian metric tensor $g=(g_{ij})$. K. Nomizu [3] has shown that the Riemannian curvature tensor $R=(R_{ijkl})$ is decomposed to three orthogonal components which belong to Ker $c$ (see §2). The first component is the Weyl conformal curvature tensor and the third one is the type $g\wedge g$. Generalizing the Riemannian curvature tensor, S. Kulkarni [4] and the author [6] defined the notion of the curvature type tensors, and obtained the same orthogonal decomposition theorem as K. Nomizu.

Let $B_p$ be the space of the curvature type tensors of degree $p$ on $M^n$. Then the differential operators $\mathfrak{D}$ and $\delta$ are defined on $B_p$. The operator $\square=\mathfrak{D}\delta+\delta\mathfrak{D}$ is called the Laplace operator, and Ker $\square$ is the space of harmonic curvature type tensors. The element of Ker $\mathfrak{D}$ satisfies the analogous equation as the second Bianchi identity of the Riemannian curvature tensor. In compact case we have Ker $\square$ = Ker $\mathfrak{D}\cap$ Ker $\delta$. As for the space with the harmonic Riemannian curvature tensor some results are obtained by several authors (E. Omachi [7], M. Umehara [8]).

In general, $\square\omega$ does not belong to $B_p$ for $\omega\in B_p$, and J. P. Bourguignon [4] showed that $\square\omega\in B_2$ if and only if $\omega$ commutes with the curvature type tensor $R-\frac{1}{2}\rho\wedge g$ where $\rho=(R_{ij})$ is the Ricci tensor. K. Nomizu proved that the orthogonal decomposition of curvature type tensor $\omega$ of degree 2 are preserved by the operators $\mathfrak{D}$ and $\delta$ under the condition of the first Bianchi equation $\mathfrak{S}\omega=0$. From these facts it seems to me that the definition of the Laplacian $\square$ is not complete as the operator on $B_p$. In this paper we give a new definition of the Laplacian $\blacksquare$ on $B_p$ such that $\blacksquare\omega\in B_p$ for any $\omega\in B_p$, and show that the orthogonal decomposition of curvature type tensor $\omega$ of any degree can be preserved under $\blacksquare$ without the assumption $\mathfrak{S}\omega=0$.

**1. Notations.** Let $\Lambda_p$ be the space of differential forms of degree $p$ on $M^n$, and define $D_{p,q}=\Lambda_p\otimes\Lambda_q$. Then $D=\bigcup D_{p,q}$ is a graded algebra with the exterior product $\wedge$. Moreover it admits the inner product $\langle\ ,\ \rangle$ and hence the *-operator $*:D_{p,q}\to D_{n-p,n-q}$. The operators $g:D_{p,q}\to D_{p+1,q+1}$ and $c:D_{p,q}\to D_{p-1,q-1}$ are defined by $g\omega=g\wedge\omega$, $c\omega=(-1)^{n(p+q)}*g*\omega$ for $\omega\in D_{p,q}$. Using $g$ and $c$ we defined $conf:D_{p,q}\to D_{p,q}$ which corresponds to the Weyl curvature tensor.

A tensor $\omega=(\omega_{I_p,J_p})\in D_{p,p}$ is called a curvature type tensor of degree $p$ if it satisfies $\omega_{I_p,J_p}=\omega_{J_p,I_p}$, where we put $I_p=$the index set $i_1i_2\cdots i_p$.

It is shown that any curvature type tensor $\omega\in D_{p,q}$ can be decomposed orthogonally as

$$\omega=\sum a_r g^r \operatorname{conf} c^r\omega\,,\quad c(\operatorname{conf} c^r\omega)=0$$

where $a_r$ is a constant. This is a generalization of the results of K. Nomizu which we call the orthogonal decomposition with respect to $c$.

Corresponding to the first Bianchi identity of the Riemannian curvature tensor, we define the mapping $\mathfrak{S}:D_{p,q}\to D_{p+1,q-1}$ and $\bar{\mathfrak{S}}:D_{p,q}\to D_{p-1,q+1}$ by

$$(\mathfrak{S}\omega)_{I_{p+1},J_{q-1}}=\sum(-1)^{k-1}\omega_{I_{p+1}(\hat{k}),i_kJ_{q-1}}\,,$$
$$\bar{\mathfrak{S}}\omega=(-1)^{n(p+q)-1}*\mathfrak{S}*\omega\,.$$

Defining $\mathfrak{T}:B_p\to B_p$ by $\mathfrak{T}=\mathfrak{S}\bar{\mathfrak{S}}$ we determined all the eigenvalues of the self-adjoint operator $\mathfrak{T}$.

**2. Differential operators $\mathfrak{D}$ and $\delta$ on $D_{p,q}$.** S. Kulkarni [4] defined $\mathfrak{D}:D_{p,q}\to D_{p+1,q}$ by

$$(\mathfrak{D}\omega)_{I_p,J_q}=\sum(-1)^{k-1}\nabla_{i_k}\omega_{I_{p+1}(\hat{k}),J_q}$$

where $\nabla$ means the covariant derivative of the Riemannian connection. Next we define $\delta:D_{p,q}\to D_{p-1,q}$ by

$$(\delta\omega)_{I_{p-1},J_q}=-\nabla^k\omega_{kI_{p-1},J_q}\,.$$

In the similar way, we define $\bar{\mathfrak{D}}:D_{p,q}\to D_{p,q+1}$ and $\bar{\delta}:D_{p,q}\to D_{p,q-1}$ by

$$(\bar{\mathfrak{D}}\omega)_{I_p,J_{q+1}}=\sum(-1)^{k-1}\nabla_{jk}\omega_{I_p,J_{q+1}(\hat{k})}\,,$$
$$(\bar{\delta}\omega)_{I_p,J_{q-1}}=-\nabla^k\omega_{I_p,kJ_{q-1}}\,.$$

LEMMA 2.1. *For $\omega\in D_{p,q}$ and $\eta\in D_{r,s}$ we have*

$$\mathfrak{D}(\omega\wedge\eta)=\mathfrak{D}\omega\wedge\eta+(-1)^p\omega\wedge\mathfrak{D}\eta\,.$$

LEMMA 2.2. *For $\omega\in B_p$ we have*

$$\delta\omega=(-1)^{p} *^{-1}\mathfrak{D} * \omega\,,$$
$$\mathfrak{D}\omega=(-1)^{p-1} *^{-1}\delta * \omega\,.$$

Supposing $M^n$ to be compact and oriented, we take the global inner product on $D_{p,q}$ as

$$(\omega, \eta)=\int_M \langle\omega, \eta\rangle * 1\,.$$

Then the next lemma can be proved as usual making use of the Theorem of Stokes.

LEMMA 2.3. *The equations*

$$(\mathfrak{D}\omega, \eta)=(\omega, \delta\eta) \quad \text{and} \quad (\bar{\mathfrak{D}}\omega, \xi)=(\omega, \bar{\delta}\xi)$$

*hold for* $\omega\in D_{p,q}$, $\eta\in D_{p+1,q}$ *and* $\xi\in D_{p,q+1}$.

We calculate some local formulae of these operators.

LEMMA 2.4.

$$\mathfrak{D}g+g\mathfrak{D}=0\,, \qquad \bar{\mathfrak{D}}g+g\bar{\mathfrak{D}}=0\,,$$
$$\delta c+c\delta=0\,, \qquad \bar{\delta}c+c\bar{\delta}=0\,.$$

LEMMA 2.5.

$$\mathfrak{D}\mathfrak{S}+\mathfrak{S}\mathfrak{D}=0\,, \qquad \bar{\mathfrak{D}}\bar{\mathfrak{S}}+\bar{\mathfrak{S}}\bar{\mathfrak{D}}=0\,,$$
$$\delta\bar{\mathfrak{S}}+\bar{\mathfrak{S}}\delta=0\,, \qquad \bar{\delta}\mathfrak{S}+\mathfrak{S}\bar{\delta}=0\,.$$

LEMMA 2.6.

$$\mathfrak{D}c+c\mathfrak{D}=-\bar{\delta}\,, \qquad \bar{\mathfrak{D}}c+c\bar{\mathfrak{D}}=-\delta\,,$$
$$\delta g+g\delta=-\bar{\mathfrak{D}}\,, \qquad \bar{\delta}g+g\bar{\delta}=-\mathfrak{D}\,.$$

LEMMA 2.7.

$$\mathfrak{D}\bar{\mathfrak{S}}+\bar{\mathfrak{S}}\mathfrak{D}=\bar{\mathfrak{D}}\,, \qquad \bar{\mathfrak{D}}\mathfrak{S}+\mathfrak{S}\bar{\mathfrak{D}}=\mathfrak{D}\,,$$
$$\delta\mathfrak{S}+\mathfrak{S}\delta=\bar{\delta}\,, \qquad \bar{\delta}\bar{\mathfrak{S}}+\bar{\mathfrak{S}}\bar{\delta}=\delta\,.$$

Using these lemmas we prove the following

PROPOSITION 2.8. (K. Nomizu, S. Kulkarni, J. P. Bourguignon) *For a curvature type tensor* $\omega$ *of degree* $p$, *let* $\omega=\sum \omega_r$ *be the orthogonal decomposition with respect to* $c$. *If* $\mathfrak{S}\omega=0$, *then the following* (1), (2) *and* (3) *are equivalent*:

(1) $\mathfrak{D}\omega=0$ *and* $\delta\omega=0$,

(2) $\bar{\mathfrak{D}}\omega=0$ *and* $\bar{\delta}\omega=0$,

(3) *each* $\omega_r$ *satisfies* $\mathfrak{D}\omega_r=0$, $\delta\omega_r=0$ *and* $\mathfrak{S}\omega_r=0$.

PROOF. Since $\operatorname{Ker}\mathfrak{S}=\operatorname{Ker}\bar{\mathfrak{S}}$ on $B_p$, we have $\bar{\mathfrak{S}}\omega=0$. From Lemma 2.7 it is easily obtained that (1) and (2) are equivalent. We show that (1) and (2) imply (3). Each $\omega_r$ is written as $\omega_r=\sum a_r g^r conf\, c^r\omega$ for some constant $a_r$ and hence we have $\mathfrak{S}\omega_r=0$ because $\mathfrak{S}$ commutes with $g$, $c$ and conf. Next from Lemmas 2.4 and 2.6 we see that $g\omega$ satisfies $\mathfrak{D}g\omega=0$ and $\delta g\omega=0$ and also $c\omega$ satisfies $\mathfrak{D}c\omega=0$ and $\delta c\omega=0$. By the similar way we have $\mathfrak{S}(g^r\omega)=\mathfrak{D}(g^r\omega)=\delta(g^r\omega)=0$ and $\mathfrak{S}(c^r\omega)=\mathfrak{D}(c^r\omega)=\delta(c^r\omega)=0$ for any $r$. Since the operator conf is expressed by the operators $g^s$ and $c^s$ for $s\geqq 0$, we conclude that $\mathfrak{D}(g^r \operatorname{conf} c^r\omega)=\delta(g^r \operatorname{conf} c^r\omega)=0$. This means that $\mathfrak{D}\omega_r=\delta\omega_r=0$ for any $r$ and we see that (3) is true. The converse statement (3)→(1) is trivial.

**3. Laplace operator on $B_p$.** S. Kulkarni or J. P. Bourguignon defined the Laplacian $\square: D_{p,q}\to D_{p,q}$ by $\square\omega=(\mathfrak{D}\delta+\delta\mathfrak{D})\omega$ for $\omega\in D_{p,q}$. If $M^n$ is compact and oriented, Lemma 2.3 implies that

$$(\square\omega,\omega)=(\mathfrak{D}\omega,\mathfrak{D}\omega)+(\delta\omega,\delta\omega)$$

holds for $\omega\in D_{p,q}$. Then $\operatorname{Ker}\square$ coincides with the space $\operatorname{Ker}\mathfrak{D}\cap\operatorname{Ker}\delta$. They call a curvature type tensor $\omega$ to be harmonic if $\omega$ satisfies $\square\omega=0$. Proposition 2.8 asserts that if $\omega\in B_p$ satisfies $\mathfrak{S}\omega=0$ then each component of the orthogonal decomposition $\omega=\sum\omega_r$ is harmonic if and only if $\omega$ is harmonic. However, it is clear that for $\omega\in B_p$, $\square\omega$ is not always a curvature type tensor. We have $\square\omega\in B_p$ if and only if the tensor $X=(X_{I_p,J_p})$ is a curvature type tensor, where the tensor $X$ is defined by

$$X_{I_p,J_p}=\sum R_{i_\alpha}{}^{k}\omega_{I_p(\overset{\alpha}{\check{k}}),J_p}+\sum R_{i_\alpha}{}^{k}{}_{i_\beta}{}^{h}\omega_{I_p(\overset{\alpha}{\check{k}},\overset{\beta}{\check{h}}),J_p}.$$

Now we define a new operator $\pmb{\square}: D_{p,q}\to D_{p,q}$ by

$$\pmb{\square}=\square+\overline{\square}$$

where $\overline{\square}=\bar{\delta}\bar{\mathfrak{D}}+\bar{\mathfrak{D}}\bar{\delta}$. The local expression of $\pmb{\square}\omega$ for $\omega=(\omega_{I_n,J_n})$ is given by

$$\begin{aligned}(\pmb{\square}\omega)_{I_p,J_p}=&-2\nabla^a\nabla_a\omega_{I_p,J_p}+\sum R_{i_\alpha}{}^{k}\omega_{I_p(\overset{\alpha}{\check{k}}),J_p}\\&+\sum R_{j_\alpha}{}^{k}\omega_{I_p,J_p(\overset{\alpha}{\check{k}})}+\sum R_{i_\alpha}{}^{k}{}_{i_\beta}{}^{h}\omega_{I_p(\overset{\alpha}{\check{k}},\overset{\beta}{\check{h}}),J_p}\\&+\sum R_{j_\alpha}{}^{k}{}_{j_\beta}{}^{h}\omega_{I_p,J_p(\overset{\alpha}{\check{k}},\overset{\beta}{\check{h}})}+2\sum R_{i_\alpha}{}^{k}{}_{j_\beta}{}^{h}\omega_{I_p(\overset{\alpha}{\check{k}}),J_p(\overset{\beta}{\check{h}})}.\end{aligned}$$

It is evident that $\pmb{\square}\omega$ is a curvature type tensor for any $\omega\in B_p$.

By virtue of Proposition 2.8, we see that $\operatorname{Ker}\square=\operatorname{Ker}\overline{\square}$ on the space $\operatorname{Ker}\mathfrak{S}$. Moreover if $M^n$ is compact then the definition of $\pmb{\square}$ leads to $\operatorname{Ker}\pmb{\square}=\operatorname{Ker}\square=\operatorname{Ker}\overline{\square}$.

THEOREM 3.1. *The operator* $\Box : B_p \to B_p$ *satisfies the following equations on* $B_p$:

(1) $\Box g - g\Box = 0$,

(2) $\Box c - c\Box = 0$,

(3) $\Box\mathfrak{S} - \mathfrak{S}\Box = 0 \quad (\Box\bar{\mathfrak{S}} - \bar{\mathfrak{S}}\Box = 0)$.

PROOF. Making use of Lemmas 2.4 and 2.6, we have

$$\begin{aligned} \Box g - g\Box &= \mathfrak{D}\delta g + \delta\mathfrak{D}g - g\mathfrak{D}\delta - g\delta\mathfrak{D} \\ &= \mathfrak{D}(-\bar{\mathfrak{D}} - g\mathfrak{D}) + \delta(-g\mathfrak{D}) + \mathfrak{D}g\delta - (-\bar{\mathfrak{D}} - \delta g)\mathfrak{D} \\ &= \bar{\mathfrak{D}}\mathfrak{D} - \mathfrak{D}\bar{\mathfrak{D}}, \\ \bar{\Box} g - g\bar{\Box} &= \bar{\mathfrak{D}}(-\mathfrak{D} - g\bar{\delta}) - \bar{\delta}g\bar{\mathfrak{D}} - (-\bar{\mathfrak{D}}g\bar{\delta}) - (-\mathfrak{D} - \bar{\delta}g)\bar{\mathfrak{D}} \\ &= \mathfrak{D}\bar{\mathfrak{D}} - \bar{\mathfrak{D}}\mathfrak{D}. \end{aligned}$$

Adding side by side, we get

$$\Box g - g\Box = 0.$$

(2) and (3) can be obtained by the same way.

THEOREM 3.2. *Let* $\omega = \sum \omega_r$ *be the orthogonal decomposition with respect to* $c$ *for* $\omega \in B_p$. *Then the decomposititon*

$$\Box\omega = \sum \Box\omega_r$$

*is too the orthogonal ones, and hence if* $\Box\omega = 0$, *then* $\Box\omega_r = 0$ *holds for each* $r$.

PROOF. (1) and (2) of Theorem 3.1 imply that $\Box g^r = g^r\Box$ and $\Box c^r = c^r\Box$ hold for any integer $r$. Then we get $\Box\,\mathrm{conf} = \mathrm{conf}\,\Box$ and hence $\Box(a_r g^r \mathrm{conf}\, c^r\omega) = a_r g^r \mathrm{conf}\, c^r \Box\omega$ is valid. Thinking of the fact that $\Box\omega \in B_p$ for $\omega \in B_p$ and that $a_r$ is determined by the constant $r$, $p$ and the dimension $n$ of the manifold, we see that the theorem is true.

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