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On the Normal Integration

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Introduction. This is a continuation of our recent papers on integration theory. We shall introduce a new integration which strictly generalizes that of Denjoy and which will be termed the normal integration.

In our opinion, the most fundamental notion in the theory of the Denjoy integration is the absolute continuity (in the wide sense) of functions. The author has endeavoured for long to obtain a natural and usable generalization of this notion. It appears to us that the present paper contains an answer to this problem. The new concept, termed normal continuity, has something in common with the incremental continuity of [5] and indeed was developed from the latter.

The theory of the normal integration will be built on the basis of the normal continuity. This integration will occupy the first three sections of this paper. The following four sections will deal with further properties of normally continuous functions and with basic results on functions called normally fluctuant. These latter functions generalize the notion of the functions of bounded variation (in the wide sense), just as the normal continuity generalizes the absolute continuity. The final section will be concerned with a few open propositions on the normal continuity and with related considerations.

§1. Normally continuous functions.

As in our previous papers, a *function,* by itself, will always signify a mapping of the real line $\boldsymbol{R}$ into itself, unless another meaning is obvious from the context. We shall denote by $\boldsymbol{N}$ the set of the positive integers. A set $(\mathfrak{F}_\sigma)$ will synonymously be called *sigma-closed* set.

A linear figure W, void or not, will be said to *pertain* to a linear set E, if E contains the boundary of W, namely the set of the end-points of the component intervals of W.

We shall say that a linear set E *severs* a figure W *normally,* if every open interval contained in W and disjoint with E is shorter than every open interval contiguous to W. For example, each closed interval and the void figure are severed normally by any linear set.

Plainly, *if a set E severs a figure W normally, so does also every set containing E.* Further, *if each component of a figure W contains at most one component of a figure $Z \subset W$ and if W is severed normally by a set E, then so is also the figure Z.*

A figure W will be termed to *pertain normally* to a set E, if W pertains to E and if E severs W normally.

Given a function $\varphi(x)$ and a figure W, we shall denote by $\varphi(W)$ the *total increment* of $\varphi(x)$ on W; in other words, $\varphi(W)$ stands for $\sum \varphi(K)$, where K ranges over the components of W and where a void sum means zero. Replacing here $\varphi(K)$ by $|\varphi(K)|$, we define further a quantity $\varphi^*(W)$ named the *total absolute increment* of $\varphi(x)$ on W. We thus have

$$|\varphi(W)| = |\sum \varphi(K)| \leqq \sum |\varphi(K)| = \varphi^*(W).$$

A function $\varphi(x)$ will be called *normally continuous,* or briefly NC, on a linear set E, if $\varphi(W) \to 0$ as $|W| \to 0$, where W means a generic figure pertaining normally to E. When this is the case, the function $\varphi(x)$ is plainly continuous on E.

In the above definition of normal continuity, the total increment $\varphi(W)$ may be replaced by the total absolute increment $\varphi^*(W)$. This is immediate from the relation $|\varphi(W)| \leqq \varphi^*(W)$ and the following two simple facts.

(i) *Given any function $\varphi(x)$, each figure Z is expressible as the union of two disjoint figures Z_1 and Z_2 such that*

$$\varphi^*(Z_1) = |\varphi(Z_1)| \quad \textit{and} \quad \varphi^*(Z_2) = |\varphi(Z_2)|.$$

(ii) *If the union of two disjoint figures is severed normally by a set E, then so is also each of the two figures separately.*

The following propositions are readily verified: *Every function which is absolutely continuous on a set E, is normally continuous on this set. The converse of this also holds, provided that the set E is a closed interval.* Again, *every linear combination, with constant coefficients, of two functions which are* NC *on a set E, is itself* NC *on E.* Furthermore, *qua property of a function, the normal continuity on a set is hereditary with respect to this set.*

On the other hand, we do not know *if a function which is* NC *on a measurable set, is necessarily* AD *at almost all points of this set.*

Given a figure W and a set E, we shall write $\theta(W;E)=\sup|H|$ by definition, where H denotes a generic open interval contained in W and disjoint with E, and where the supremum vanishes if there is no such H. This quantity $\theta(W;E)$ is finite, since $\theta(W;E)\leqq|W|$. We find easily that *if a nonconnected figure W is severed normally by a set E, then $\theta(W;E)<\min|G|$, where G represents the open intervals contiguous to W.*

In the following theorem, the set $\Gamma=E_0\cap E_1\cap\cdots\cdot$ and the function $\Omega(x)$ are the same things as in Theorem 6 of [5] and its proof.

THEOREM 1. *The function $\Omega(x)$ is normally continuous on the set Γ, without being* GBV *on any portion of Γ.*

PROOF. We shall only outline the proof, since it resembles that for Theorem 6 of [5]. Let W be any figure pertaining normally to Γ. It suffices to show that if $n\in N$ and $5^{-n}\leqq\theta(W;\Gamma)<5^{1-n}$, then $|\Omega(W)|<4n^{-1}$.

Noting that W is nonvoid, let us consider a generic component J of W. Then J must be contained in a component of the figure E_{n-1}. To verify this, suppose if possible that the contrary is true. Then J contains an open interval, say H, which is contiguous to E_{n-1} and hence to Γ also. But we have $|H|\geqq5^{1-n}$ by definition of the sequence $\langle E_0, E_1, \cdots\cdot\rangle$. This contradicts $|H|\leqq\theta(W;\Gamma)<5^{1-n}$.

On the other hand, each component K of E_{n-1} can contain at most five intervals J. For otherwise K would contain at least five open intervals contiguous to the figure W which pertains normally to Γ. There would then arise the contradiction $5^{1-n}=|K|>5\theta(W;\Gamma)\geqq5^{1-n}$.

Since the figure E_{n-1} has exactly 3^{n-1} components, it follows from the above that W has at most $5\cdot3^{n-1}$ components. This fact, combined with the appraisal $\mathrm{O}(\Omega;K)<2n^{-1}3^{-n}$ valid for each K, leads at once to the relation

$$|\Omega(W)|\leqq\sum_J|\Omega(J)|<5\cdot3^{n-1}\cdot2n^{-1}3^{-n}<4n^{-1},$$

which completes the proof.

THEOREM 2. *A function $\varphi(x)$ which is normally continuous on a linear set E, necessarily maps every closed null set Q contained in E onto a null set.*

PROOF. Since every closed set is expressible as the union of a sequence of compact sets, we may assume that Q is itself compact. Moreover, we need only consider the case where Q is noncountably infinite. Then Q must contain a perfect set P such that $Q\setminus P$ is a countable set. It follows that Q may further be assumed perfect. Thus Q is a nonvoid perfect set which is bounded and null.

Let I be the minimal closed interval containing Q. Then the open set $G=I\setminus Q$ is the union of all the open intervals H contiguous to Q and we have $|G|=|I|$. Given any $\delta>0$, let us denote by $G(\delta)$ the union of all the H with $|H|>\delta$, so that $|G(\delta)|\to|I|$ as $\delta\to 0$. We find easily that if we write $W(\delta)$ for $I\setminus G(\delta)$, then $W(\delta)$ is a figure containing Q and pertaining normally to Q. Moreover, $|W(\delta)|\to 0$ as $\delta\to 0$.

The function $\varphi(x)$, which is NC on E, is NC and continuous on Q. Let K be a generic component of the figure $W(\delta)$. Then K contains a closed interval L pertaining to Q and such that $|\varphi[Q\cap K]|\leqq|\varphi(L)|$. We associate with each K such an interval L and we denote by $Z(\delta)$ the union of all the associated intervals L. The figure $Z(\delta)$ then pertains normally to Q and hence to E also, while we have $|Z(\delta)|\leqq|W(\delta)|$. Consequently it follows, in view of the relation

$$|\varphi[Q]|=|\sum_K\varphi[Q\cap K]|\leqq\sum_K|\varphi[Q\cap K]|\leqq\sum_L|\varphi(L)|=\varphi^*(Z(\delta)),$$

that $|\varphi[Q]|\to 0$ as $\delta\to 0$. This implies the nullity of $\varphi[Q]$, since the set Q is independent of the number δ. The proof is thus complete.

LEMMA 1. *If a closed interval J pertains to a subset M of the closure of a linear set E and if a closed interval K pertaining to the set E is contained in an open interval $D\supset J$, then*

$$\theta(K;E)<\theta(J;M)+|D\setminus J|.$$

PROOF. Assuming $\theta(K;E)$ positive, as we clearly may, consider any open interval $H\subset K$ disjoint with E. Then H is disjoint with the set M, since M is contained in the closure of E. Thus we have $|H\cap J|=|H\cap J^\circ|\leqq\theta(J;M)$, J° denoting the interior of J. Now plainly $|H|=|H\cap J|+|H\setminus J|$, where $|H\setminus J|\leqq|K\setminus J|$. It therefore follows that $|H|\leqq\theta(J;M)+|K\setminus J|$, whence we get $\theta(K;E)\leqq\theta(J;M)+|K\setminus J|$. On the other hand, it is obvious that the set $K\setminus J$ and the nonvoid open set $D\setminus(J\cup K)$ are disjoint and both contained in $D\setminus J$. Hence $|K\setminus J|<|D\setminus J|$, which completes the proof.

LEMMA 2. *Given a nonvoid figure Z pertaining normally to a subset M of the closure of a linear set E, let the components of Z be $J_1<\cdots<J_n$ in their natural order, where $J_i=[p_i,q_i]$ for $i=1,\cdots,n$. There then corresponds to each $\eta>0$ a figure W with exactly n components and pertaining normally to E, such that if the components of*

W are $K_i=[r_i, s_i]$, where $i=1,\cdots,n$ and $K_1<\cdots<K_n$, we have the inequalities

$$|r_i-p_i|<\eta \quad \textit{and} \quad |s_i-q_i|<\eta \quad \textit{for every } i.$$

PROOF. The assertion being trivial in the case in which $n=1$, we may suppose that $n\geqq 2$. Writing $\lambda=\min|G|$, where G represents the $n-1$ open intervals contiguous to Z, let us take a number $\rho>0$ such that

$$\rho<\eta \quad \text{and} \quad 4\rho<\lambda-\theta(Z;M),$$

where $\theta(Z;M)<\lambda$ since M severs Z normally. We then have $4\rho<\lambda$.

The set M being contained in the closure of E, we can associate with each $i=1,\cdots,n$ a closed interval $K_i=[r_i, s_i]$ pertaining to the set E and such that $|r_i-p_i|<\rho$ and $|s_i-q_i|<\rho$. Then the open interval $D_i=(p_i-\rho,\ q_i+\rho)$ contains both the intervals J_i and K_i. It thus follows from Lemma 1 that

$$\theta(K_i;E)<\theta(J_i;M)+|D_i\setminus J_i|=\theta(J_i;M)+2\rho.$$

On the other hand, we have $K_1<\cdots<K_n$ since $4\rho<\lambda$. We shall show that the figure $W=K_1\cup\cdots\cup K_n$, whose components are the intervals K_i and which pertains to the set E, is severed normally by E.

Among the intervals $K_1,\cdots,K_n$ there evidently exists one, say K_m, such that $\theta(W;E)=\theta(K_m;E)$. But $\theta(K_m;E)<\theta(J_m;M)+2\rho$, by what was already proved. Consequently we must have

$$\theta(W;E)<\theta(J_m;M)+2\rho\leqq\theta(Z;M)+2\rho<\lambda-2\rho.$$

In addition to this, each open interval contiguous to the figure W has length exceeding $\lambda-2\rho$, as we readily see from the choice of the intervals K_i. The set E thus severs W normally, and the proof is complete.

THEOREM 3. *Every function $\varphi(x)$ which is normally continuous on a set E and continuous on a set $M\supset E$ contained in the closure of E, is normally continuous on the whole set M.*

PROOF. Let W denote a generic figure pertaining normally to E. The function $\varphi(x)$ being NC on E, there corresponds to each $\varepsilon>0$ a number $\delta>0$ such that $|\varphi(W)|<\varepsilon$ whenever $|W|<\delta$. Let us keep ε and δ fixed in what follows. It suffices to show that if a figure Z with $|Z|<\delta$ pertains normally to the set M, then $|\varphi(Z)|<2\varepsilon$.

We may clearly assume that the figure Z is nonvoid. Let the components of Z be $J_1<\cdots<J_n$, where $J_i=[p_i, q_i]$ for $i=1,\cdots,n$. Given any $\eta>0$, there is by Lemma 2 a figure $W=W(\eta)$ with n components and pertaining normally to E, such that if these components are denoted by $K_i=[r_i, s_i]$, where $i=1, \cdots, n$ and $K_1<\cdots<K_n$, then

$$|r_i-p_i|<\eta \quad \text{and} \quad |s_i-q_i|<\eta \quad \text{for every } i.$$

On the other hand, the function $\varphi(x)$ is continuous on the set $M\supset E$ by hypothesis. It follows that $|W|\to|Z|$ and $\varphi(W)\to\varphi(Z)$ as $\eta\to 0$. We can therefore choose η so small that $|W|<\delta$ and $|\varphi(Z)-\varphi(W)|<\varepsilon$. Then $|\varphi(W)|<\varepsilon$ and hence we have $|\varphi(Z)|<2\varepsilon$. This completes the proof.

§ 2. Generalized normally continuous functions.

A function will be called *generalized normally continuous,* or GNC for short, on a linear set E, if the function is continuous on E and if E is expressible as the union of a sequence of sets on each of which the function is normally continuous. *This property of a function is clearly hereditary with respect to the set E.*

As readily seen, *every function which is* GAC *on a set, is* GNC *on this set* and *every linear combination of two functions which are* GNC *on a set, is itself* GNC *on this set.* Furthermore, Theorem 2 and Theorem 3 together imply that *a function which is* GNC *on a closed set S, necessarily maps every closed null set contained in S onto a null set.*

The proof of the following theorem is the same as for Theorem 11 of [1], with the help of Theorem 3.

THEOREM 4. *In order that a function which is continuous on a nonvoid closed set S, be generalized normally continuous on this set S, it is necessary and sufficient that every nonvoid closed subset of S contain a portion on which the function is normally continuous.*

THEOREM 5. *Every function $\varphi(x)$ which is generalized normally continuous on a closed interval I and has a nonnegative approximate derivative, finite or infinite, at almost every point of I, is monotone nondecreasing on I.*

THEOREM 6. *If two functions are generalized normally continuous on a closed interval I and approximately equiderivable almost everywhere on I, then the functions can differ over I only by an additive constant.*

REMARKS. As in [1], any two functions are called *approximately* *equiderivable,* or briefly AED, at a point of $\boldsymbol{R}$, if at this point the functions are AD and have coinciding approximate derivatives. We need only prove Theorem 5, from which Theorem 6 follows immediately.

PROOF OF THEOREM 5. It suffices to show that $\varphi(I)\geqq 0$. In fact, once this is established, we must have $\varphi(J)\geqq 0$ for every closed interval $J\subset I$, since the hypothesis of the theorem holds as well if we replace in it the interval I by J.

Let E be the set of the interior points x of I at which $\varphi'_{\mathrm{ap}}(x)>0$. We shall treat first the case in which we have $|I\setminus E|=0$.

Consider any closed interval $K\subset I$ such that $\varphi(K)>0$. The family of all such intervals K clearly covers the set E in the Vitali sense. Hence, by Vitali's Covering Theorem, this family contains a disjoint countable subfamily, say $\mathfrak{M}$, which covers almost all points of E. If we now replace each interval $K\in\mathfrak{M}$ by its interior, we obtain a disjoint countable family, say $\mathfrak{N}$, of open intervals. Writing D for the union of the family $\mathfrak{N}$, we find at once that $|E\setminus D|=0$. This, together with the assumption $|I\setminus E|=0$, implies the nullity of the set $Q=I\setminus D$, which is closed since D is open. But the function $\varphi(x)$ is GNC on I. It follows from Theorem 2 that $|\varphi[Q]|=0$.

This being so, consider any function $\lambda(x)$ which coincides with $\varphi(x)$ for $x\in Q$ and which is linear on each interval $K\in\mathfrak{M}$. Then $|\lambda[Q]|=0$ and further $\lambda(K)=\varphi(K)>0$ for $K\in\mathfrak{M}$. Moreover, the function $\lambda(x)$ is continuous on I together with $\varphi(x)$.

Writing $I=[a,b]$, suppose now, if possible, that

$$\varphi(a)>\varphi(b), \quad \text{i.e.} \quad \lambda(a)>\lambda(b).$$

We take a number y_0 fulfilling $\lambda(a)>y_0>\lambda(b)$ and not belonging to the null set $\lambda[Q]$. The points $x\in I$ at which $\lambda(x)=y_0$ plainly form together a nonvoid compact set, whose rightmost point we denote by x_0. Then x_0 is interior to an interval $K_0\in M$, while we evidently have $\lambda(x)<y_0=\lambda(x_0)$ for the points $x>x_0$ of the interval I. But the function $\lambda(x)$, which is linear on K_0, must increase strictly on K_0. This contradiction proves the announced inequality $\varphi(I)\geqq 0$.

We assumed in the above that $|I\setminus E|=0$. We now pass to the general case. Given any $\varepsilon>0$, consider the function $\psi(x)=\varphi(x)+\varepsilon x$. This function, being a linear combination of two functions which are GNC on I, is itself GNC on I. Further, $\psi'_{\mathrm{ap}}(x)=\varphi'_{\mathrm{ap}}(x)+\varepsilon\geqq\varepsilon$ almost everywhere on I. We therefore find, by what was already established, that $\varphi(I)+\varepsilon|I|=\psi(I)\geqq 0$. It follows that $\varphi(I)\geqq 0$, since ε is arbitrary. This completes the proof.

THEOREM 7. *Every function* $\varphi(x)$ *which is both* BV *and* GNC, *on a closed set* S, *is* AC *on* S. *Hence, every function which is both* GBV *and* GNC, *on a closed set, is* GAC *on this set.*

REMARKS ON THE PROOF. By an *interval* we mean any connected infinite set of real numbers. Thus the void set and the singletonic sets are not counted among the intervals. We call an interval *finite* or *infinite*, according as it is a bounded set or not, respectively. Closed intervals and open intervals are finite.

PROOF. We may assume the set S to be nonvoid. There then exists one and only one function $\lambda(x)$ such that

(i) we have $\lambda(x)=\varphi(x)$ for every $x\in S$,

(ii) the function $\lambda(x)$ is linear on any closed interval (if existent) contiguous to the set S,

(iii) the function $\lambda(x)$ is a constant on the closure of any infinite interval (if existent) disjoint with S.

The function $\lambda(x)$ is GNC on the whole $\boldsymbol{R}$, since it is continuous on $\boldsymbol{R}$, GNC on S, and AC on every interval disjoint with S. This function is further BV on $\boldsymbol{R}$, as easily verified, and thus has a finite derivative $\lambda'(x)$ at almost every point of $\boldsymbol{R}$. For definiteness, let us write $\lambda'(\xi)=0$ for every $\xi\in\boldsymbol{R}$ at which $\lambda(x)$ is not derivable. The function $\lambda'(x)$, thus defined over $\boldsymbol{R}$, is summable on $\boldsymbol{R}$. Let $L(x)$ be any indefinite Lebesgue integral of $\lambda'(x)$. Then $L(x)$ is AC on $\boldsymbol{R}$ and equiderivable with $\lambda(x)$ almost everywhere on $\boldsymbol{R}$.

On account of Theorem 6, the difference $\lambda(x)-L(x)$ is a constant on each closed interval. It follows that $\lambda(x)=L(x)+C$ on the whole $\boldsymbol{R}$, where C is a constant. The function $\varphi(x)$, which coincides with $\lambda(x)$ on S, is therefore AC on S. This completes the proof.

§ 3. The normal integration.

We are now in a position to state the descriptive definition of the *normal integration.* A function $f(x)$ will be termed *normally integrable* over a closed interval I, if there is a function $\varphi(x)$ which is GNC on I and which has $f(x)$ for its approximate derivative almost everywhere on I. Any such function $\varphi(x)$ is then called *indefinite normal integral* of $f(x)$ on I. By the *definite normal integral* of $f(x)$ over I we shall mean the increment $\varphi(I)$ of its indefinite integral $\varphi(x)$. The number $\varphi(I)$, which is uniquely determined by the integrand function $f(x)$ and the interval I on account of Theorem 6, will be denoted by $(\mathfrak{N})\int_I f(x)\,dx$, or simply by $\int_I f(x)\,dx$ when there is no fear of misunderstanding or confusion.

All the properties, except Theorem 19, of the powerwise integral that are stated on pp. 16–17 of [1] are shared also by the normal integral, as readily verified. On the other hand, the function $\Omega(x)$ of Theorem 1 shows that the normal integration is strictly wider than the Denjoy integration. In fact, the function $\Omega(x)$, which fails to be GAC on the interval $U=[0,1]$, is nevertheless GNC on U, since $\Omega(x)$ is NC on the compact null set $\Gamma\subset U$ and linear on each closed interval contiguous to Γ. This linearity, together with the nullity of Γ, implies further that $\Omega(x)$ is derivable almost everywhere on U. Hence any function to which $\Omega(x)$ is approximately derivable almost everywhere on U, must be normally integrable, without being Denjoy integrable, on U.

THEOREM 8. *Given any function $M(x)$ which is* BV *on a closed interval $I=[a,b]$ and given any function $F(x)$ continuous on the same interval I, let $G(x)$ be a function such that*

$$G(x)=M(x)F(x)-\int_a^x F(t)\,dM(t) \qquad \textit{for} \quad x\in I,$$

where the integral is a Riemann-Stieltjes one. Then

(i) *the function $G(x)$ is continuous on the interval I;*

(ii) *if the function $F(x)$ is normally continuous on a subset E of I, so is on E the function $G(x)$ likewise;*

(iii) *we have $G'_{\mathrm{ap}}(x)=M(x)F'_{\mathrm{ap}}(x)$ at almost every point $x\in I$ at which the function $F(x)$ is approximately derivable.*

REMARK. Parts (i) and (iii) of the theorem are implicitly contained in the Saks treatise (see [7], pp. 244–246). For convenience, however, we made them come out to the foreground, accompanied with their formal proofs. It is noteworthy that all the three parts will be deduced, in the following proof, from a common preliminary argument (virtually a quotation from p. 245 of [7]).

PROOF. We may clearly assume $M(x)$ to be monotone nondecreasing on I. Then each interval $J=[p,q]$ contained in I contains a point r such that

$$\int_J F(t)\,dM(t)=M(J)\cdot F(r),$$

as is well-known. It follows successively that

$$G(J)=M(q)[F(q)-F(p)]+[M(q)-M(p)]F(p)-\int_J F(t)\,dM(t)$$
$$=M(q)\cdot F(J)+M(J)\cdot[F(p)-F(r)],$$
$$|G(J)|\leqq A\cdot|F(J)|+M(J)\cdot \mathrm{O}(F;J),$$

where A stands for the supremum of $|M(x)|$ on the interval I.

re (i): This is immediate from the last inequality, since the function $F(x)$ is continuous on I.

re (ii): Given any $\delta>0$, let F_δ denote the supremum of $\mathrm{O}(F;J)$ for the closed intervals $J\subset I$ with $|J|<\delta$. Then the above inequality shows that

$$|G(J)|\leqq A\cdot|F(J)|+F_\delta\cdot M(J) \qquad \text{if} \quad |J|<\delta.$$

Now suppose that the function $F(x)$ is NC on a set $E\subset I$, and consider an arbitrary figure W pertaining normally to E. We find that

$$G^*(W)\leqq A\cdot F^*(W)+F_\delta\cdot M(W) \qquad \text{if} \quad |W|<\delta.$$

Given any $\varepsilon>0$, there is a $\delta>0$ such that $F_\delta<\varepsilon$ and that $F^*(W)<\varepsilon$ whenever $|W|<\delta$. Choosing such a δ, we have

$$G^*(W)\leqq A\cdot F^*(W)+2A\cdot F_\delta\leqq 3A\varepsilon \qquad \text{if} \quad |W|<\delta.$$

Hence the function $G(x)$ is NC on E.

re (iii): With the same notation as at the beginning, we have

$$G(J)=M(q)\cdot F(J)+M(J)\cdot[F(p)-F(r)]$$
$$=M(p)\cdot F(J)+M(J)\cdot[F(q)-F(r)].$$

Hence, if ξ and $\xi+h$ are any two distinct points of I, we can write

$$\frac{G(\xi+h)-G(\xi)}{h}$$
$$=M(\xi)\frac{F(\xi+h)-F(\xi)}{h}+\varepsilon(\xi;h)\frac{M(\xi+h)-M(\xi)}{h},$$

where $\varepsilon(\xi;h)\to 0$ as $h\to 0$. Consequently the function $G(x)$ is AD to $M(x)F'_{\mathrm{ap}}(x)$ at every interior point of I at which $F(x)$ is AD and $M(x)$ is derivable. But any function which is BV on I is derivable almost everywhere on I. This completes the proof.

With the aid of the foregoing theorem, we can now establish for the normal integral the integration by parts theorem and the second mean value theorem. Theorem 9 is derived at once, while the proof of Theorem 10 is the same as in the corresponding theorem of Saks [7], p. 246.

THEOREM 9. *If a function $M(x)$ is* BV *on a closed interval $I=[a,b]$ and if a function $f(x)$ is normally integrable on I, then the function $M(x)f(x)$ is also normally integrable on I, and denoting by $F(x)$ any indefinite normal integral of $f(x)$ on I, we have*

$$(\mathfrak{N})\int_I M(x)f(x)\,dx=\Big[M(x)F(x)\Big]_a^b-\int_I F(x)\,dM(x)$$

$$=\int_I M(x)\,dF(x),$$

where the second and third integrals are Riemann-Stieltjes ones.

THEOREM 10. *If a function $M(x)$ is monotone nondecreasing on a closed interval $I=[a,b]$ and if a function $f(x)$ is normally integrable on I, there necessarily exists a point $\xi\in I$ such that*

$$\int_I M(x)f(x)\,dx=M(a)\int_a^\xi f(x)\,dx+M(b)\int_\xi^b f(x)\,dx,$$

where each integral is a normal one.

§ 4. The condition (S) and the condition (S_0).

We shall say that a function $\varphi(x)$ fulfils the *condition* (S) on a linear set E, if to each number $\varepsilon>0$ there corresponds a $\delta>0$ such that, for every measurable set $M\subset E$, the inequality $|M|<\delta$ implies $|\varphi[M]|<\varepsilon$. When this is the case, we have $|\varphi[X]|<\varepsilon$ for every set $X\subset E$ with $|X|<\delta$, provided that the set E is measurable. In fact, we can enclose X in an open set G such that $|G|<\delta$. Then $E\cap G$ is a measurable set containing X and having measure $<\delta$, so that $|\varphi[X]|\leqq|\varphi[E\cap G]|<\varepsilon$.

In the particular case in which E is a closed interval, this condition is no other than the condition (S) of Banach (see Saks [7], p. 282).

If the epithet "measurable" is replaced by "compact" in the above, we obtain the definition of a condition which will be called the *weak condition* (S) on the set E. Evidently, the condition (S) on E always implies the weak condition on E. In the important case in which E is a sigma-closed set, however, the converse of this implication is also true, as asserted by the following theorem.

THEOREM 11. *A function $\varphi(x)$ which fulfils the weak condition* (S) *on a sigma-closed set A, necessarily fulfils the condition* (S) *on A.*

PROOF. By hypothesis, there corresponds to each $\varepsilon>0$ a number

$\delta>0$ such that, for every compact set $Q\subset A$, the inequality $|Q|<\delta$ implies $|\varphi[Q]|<\varepsilon$. Keeping ε and δ fixed, we shall show below that $|\varphi[M]|\leqq\varepsilon$ for each $M\subset A$ with $|M|<\delta$.

We begin with the case where the set M is sigma-closed. Then M is expressible as the limit of an ascending infinite sequence, say $Q_1\subset Q_2\subset\cdots$, of compact sets. It follows that the image $\varphi[M]$ is the limit of the ascending sequence $\varphi[Q_1]\subset\varphi[Q_2]\subset\cdots$, whence we have $|\varphi[M]|=\lim|\varphi[Q_n]|$. But $|\varphi[Q_n]|<\varepsilon$ for every $n\in N$, since Q_n is compact and contained in M. We thus have $|\varphi[M]|\leqq\varepsilon$.

We now pass on to the general case where M is any subset of A with $|M|<\delta$. Let us enclose M in an open set D with $|D|<\delta$. Then $A\cap D$ is sigma-closed and contains M. Since $|A\cap D|<\delta$, we find by what was already proved that $|\varphi[M]|\leqq|\varphi[A\cap D]|\leqq\varepsilon$. This completes the proof.

THEOREM 12. *A function $\varphi(x)$ which is normally continuous on a set E, necessarily fulfils the weak condition* (S) *on E. Consequently, if in addition the set E is sigma-closed, the function fulfils on E the condition* (S) *and hence the condition* (N).

REMARK. The first half of this theorem includes Theorem 2. Indeed, if a function fulfils the weak condition (S) on a set E, then every compact null set $\subset E$, and hence every closed null set $\subset E$, is mapped by the function onto a null set.

PROOF. We shall only outline the proof, since it resembles that of Theorem 2.

By hypothesis, there corresponds to each $\varepsilon>0$ a $\delta>0$ such that, for every figure W pertaining normally to E, the inequality $|W|<\delta$ implies $\varphi^*(W)<\varepsilon$. Keeping ε and δ fixed, we shall prove $|\varphi[Q]|<\varepsilon$ for every compact set $Q\subset E$ with $|Q|<\delta$.

We may assume Q nonvoid and perfect. Denoting by H a generic open interval contiguous to Q, and by I the minimal closed interval containing Q, let D_n be for every $n\in N$ the union of all the H with $|H|>n^{-1}$ and let us write $W_n=I\setminus D_n$. Then W_n is a figure containing Q and pertaining normally to Q. Moreover, we have $|W_n|\to|Q|$ as $n\to+\infty$. Hence there is a $p\in N$ for which $|W_p|<\delta$. The function $\varphi(x)$ being continuous on E, we can associate with each component K of W_p a closed interval $L\subset K$ pertaining to Q and such that $|\varphi[Q\cap K]|\leqq|\varphi(L)|$. If Z_p denotes the union of all the associated intervals L, the figure Z_p clearly pertains normally to E, while we have

$|Z_p| \leqq |W_p| < \delta$. Hence

$$|\varphi[Q]| = |\sum_K \varphi[Q \cap K]| \leqq \sum_K |\varphi[Q \cap K]| \leqq \sum_L |\varphi(L)| = \varphi^*(Z_p) < \varepsilon,$$

which completes the proof.

THEOREM 13. *If a function $\varphi(x)$ is normally continuous on a set E, there corresponds to each $\varepsilon > 0$ a number $\delta > 0$ such that, for any finite disjoint sequence $\langle Q_1, \cdots\cdot, Q_n \rangle$ of compact sets contained in E, the inequality*

$$|Q_1| + \cdots\cdot + |Q_n| < \delta \quad \textit{implies} \quad |\varphi[Q_1]| + \cdots\cdot + |\varphi[Q_n]| < \varepsilon.$$

PROOF. We may assume that the positive integer n, which may vary quite arbitrarily, is at least 2, since the assertion reduces for $n=1$ to the foregoing theorem. Moreover, we may confine ourselves to the case in which the sets $Q_1, \cdots\cdot, Q_n$ are nonvoid and perfect. Then so is also their union.

By hypothesis, given any $\varepsilon > 0$ there is a $\delta > 0$ such that for each figure W pertaining normally to the set E, the inequality $|W| < \delta$ implies $\varphi^*(W) < \varepsilon$. The theorem will be established if we show that

$$|\varphi[Q_1]| + \cdots\cdot + |\varphi[Q_n]| < \varepsilon \qquad \text{whenever} \quad |Q_1| + \cdots\cdot + |Q_n| < \delta.$$

Let us write $\rho = \min \operatorname{dist}(Q_i, Q_j)$, where Q_i and Q_j are two arbitrary distinct sets among $Q_1, \cdots\cdot, Q_n$. We clearly have $\rho > 0$. Let further I be the minimal closed interval containing the compact set $Q = Q_1 \cup \cdots\cdot \cup Q_n$ which is nonconnected.

We shall keep Q fixed in what follows. Let us denote by H a generic open interval contiguous to Q and by D_m the union of all the H with $|H| > m^{-1}$, where m is any positive integer so large that $m^{-1} < \rho$. Writing $W_m = I \setminus D_m$, we find easily that W_m is a figure containing Q and pertaining normally to Q. Since $|Q| < \delta$ by hypothesis and since obviously $|W_m| \to |Q|$ as $m \to +\infty$, we have $|W_m| < \delta$ for large values of m. Let us fix such an m.

This being so, consider any component interval of W_m, say L. The intersection $Q \cap L$ is then contained in one of the sets $Q_1, \cdots\cdot, Q_n$. To see this, suppose if possible that to the contrary there exist among these sets at least two each of which intersects L. We write ρ^* for the minimum of the distance between two distinct nonvoid sets taken from among the compact sets $Q_1 \cap L, \cdots\cdot, Q_n \cap L$. The interval L plainly contains a closed subinterval J with length ρ^* and whose end poins belong to Q without belonging to one and the same of the sets $Q_1, \cdots\cdot, Q_n$. It follows at once that the interior of the interval J is disjoint with Q. Accordingly J is contiguous to Q, and hence $|J|\leqq m^{-1}<\rho$. But evidently $\rho\leqq\rho^*$ and hence $|J|<\rho^*$. This contradicts the choice of J.

Let us arrange all the components L of W_m in a sequence, say $L_1<\cdots\cdot<L_p$ in their natural ordering. What was proved just now shows at once that the decomposition of the set Q into the nonvoid intersections $L_1\cap Q, \cdots\cdot, L_p\cap Q$ is a refinement of the decomposition $Q=Q_1\cup\cdots\cdot\cup Q_n$. We therefore have

$$|\varphi[Q_1]|+\cdots\cdot+|\varphi[Q_n]|\leqq|\varphi[L_1\cap Q]|+\cdots\cdot+|\varphi[L_p\cap Q]|.$$

Now let L_k be any one of the components $L_1, \cdots\cdot, L_p$. Since $\varphi(x)$ is continuous on E and hence on $L_k\cap Q$, there exists in L_k a closed interval I_k pertaining to Q and fulfilling $|\varphi[L_k\cap Q]|\leqq|\varphi(I_k)|$. The figure $Z=I_1\cup\cdots\cdot\cup I_p$ then pertains normally to Q and hence to E also, while we have

$$|Z|=|I_1|+\cdots\cdot+|I_p|\leqq|L_1|+\cdots\cdot+|L_p|=|W_m|<\delta.$$

We thus obtain the following relation, which completes the proof.

$$\sum_{i=1}^{n}|\varphi[Q_i]|\leqq\sum_{k=1}^{p}|\varphi[L_k\cap Q]|\leqq\sum_{k=1}^{p}|\varphi(I_k)|=\varphi^*(Z)<\varepsilon.$$

Theorem 12 is a special case of the present theorem and their proofs contain a certain duplication. We could have absorbed the former proposition into the latter. But we think that the separation of the two propositions has conduced to the intelligibility.

We shall say that a function $\varphi(x)$ fulfils the *condition* (S_0) on a linear set E, if there corresponds to each $\eta>0$ a number $\delta>0$ such that, for every finite disjoint sequence $\langle E_1, \cdots\cdot, E_n\rangle$ of measurable subsets of E, the inequality

$$|E_1|+\cdots\cdot+|E_n|<\delta \quad \text{implies} \quad |\varphi[E_1]|+\cdots\cdot+|\varphi[E_n]|<\eta.$$

We can easily prove that the condition (S_0), which evidently involves the condition (S), is stronger than the latter. For this purpose, let I be a closed interval. On account of a theorem on p. 288 of Saks [7], every function which is AC superposable on I, fulfils the condition (S) on I. But it is known that, in general, a function which is AC superposable on I is not AC on I (see [7], p. 286). On the other hand, we see at once that every function which is continuous on I and fulfils the condition (S_0) on I, must be AC on I. Hence the result.

THEOREM 14. *Every function $\varphi(x)$ which is normally continuous on a sigma-closed set A, fulfils the condition (S_0) on this set.*

PROOF. Given any $\eta>0$, write $\varepsilon=\eta/2$ and choose a number $\delta>0$ conforming to the assertion of Theorem 13. We shall show that this δ has the property stated in the definition of the condition (S_0).

Consider a measurable set $M\subset A$ and take in M a sigma-closed set C such that $|M\setminus C|=0$ (see Saks [7], p. 69). Since by Theorem 12 the function $\varphi(x)$ fulfils on the set A the condition (N) of Luzin, we find that

$$|\varphi[M]|=|\varphi[C]\cup\varphi[M\setminus C]|\leqq|\varphi[C]|+|\varphi[M\setminus C]|=|\varphi[C]|$$

and therefore that $|\varphi[M]|=|\varphi[C]|$. But we can express the set C as the limit of an ascending infinite sequence of compact sets. Consequently it ensues that $|\varphi[M]|$ is the supremum of $|\varphi[Q]|$, where Q is a generic compact set $\subset M$.

This being so, suppose now that $\langle E_1,\cdots,E_n\rangle$ is a finite disjoint sequence of measurable subsets of A. We attach to each $i=1,\cdots,n$ a compact set $Q_i\subset E_i$ such that $|\varphi[E_i]|<|\varphi[Q_i]|+n^{-1}\varepsilon$. On account of $|Q_1|+\cdots+|Q_n|<\delta$, we then find that

$$\sum_{i=1}^{n}|\varphi[E_i]|<\sum_{i=1}^{n}|\varphi[Q_i]|+\varepsilon<2\varepsilon=\eta,$$

which completes the proof.

§5. Fluctuation of a normally continuous function.

Given a function $\varphi(x)$ and a linear set E, let y be any real number. The number (possibly $+\infty$) of the points $x\in E$ at which $\varphi(x)$ equals y, will be denoted by $\mathrm{N}(y;\varphi;E)$ and termed *multiplicity* of y with respect to $\varphi(x)$ and E. Qua function of y, this multiplicity will be called *multiplicity function* associated with $\varphi(x)$ and E.

By a *partition* of a linear set E, we shall mean as usual any nonvoid disjoint class consisting of subsets of E and having E for its union. A partion will be called *Borel*, if all its constituents are Borel sets. To simplify the wording, by a partition, by itself, we shall always understand a countable (i. e. at most enumerable) one henceforward. This agreement will not be repeated.

Given a function $\varphi(x)$ and a set E, let $\mathfrak{S}$ be any partition of E. For each $y\in\boldsymbol{R}$ we shall denote by $\Theta(y;\varphi;\mathfrak{S})$ the number (possibly $+\infty$) of the distinct sets $X\in\mathfrak{S}$ such that $y\in\varphi[X]$. In symbols:

$$\Theta(y;\varphi;\mathfrak{S})=\sum_{X\in\mathfrak{S}}\mathrm{c}(y;\varphi[X]),$$

where $\mathrm{c}(y;\varphi[X])$ means, as in Saks [7], the characteristic function of the set $\varphi[X]$. It is obvious that $\Theta(y;\varphi;\mathfrak{S})\leqq\mathrm{N}(y;\varphi;E)$. Further, writing $\mathrm{d}(\mathfrak{S})$ for the characteristic number of the class $\mathfrak{S}$, i. e. the supremum (possibly $+\infty$) of the diameter $\mathrm{d}(X)$, where $X \in \mathfrak{S}$, we find at once that, for each $y \in \boldsymbol{R}$,

$$\Theta(y;\varphi;\mathfrak{S}) \to \mathrm{N}(y;\varphi;E) \qquad \text{as} \quad \mathrm{d}(\mathfrak{S}) \to 0.$$

THEOREM 15. *Given any function $\varphi(x)$ which is continuous on a linear Borel set M, let $\mathfrak{S}$ be a generic Borel partition of M. Then both $\Theta(y;\varphi;\mathfrak{S})$ and $\mathrm{N}(y;\varphi;M)$ are nonnegative measurable functions of y and we have*

$$\sum_{X\in\mathfrak{S}} |\varphi[X]| \to \int_{-\infty}^{+\infty} \mathrm{N}(y;\varphi;M)\,dy \qquad \textit{as} \quad \mathrm{d}(\mathfrak{S}) \to 0.$$

REMARK. This integral will be written $\Xi(\varphi;M)$ and called *fluctuation* of the function $\varphi(x)$ on the set M. We have $0 \leqq \Xi(\varphi;M) \leqq +\infty$.

PROOF. Since a continuous image of a Borel set is always a measurable set (cf. Kuratowski [6], p. 249), the characteristic function $\mathrm{c}(y;\varphi[X])$ is measurable for each $X \in \mathfrak{S}$. Hence so must also be the function $\Theta(y;\varphi;\mathfrak{S})$, which is the sum of $\mathrm{c}(y;\varphi[X])$ for all the sets $X \in \mathfrak{S}$.

This being so, let us consider any infinite sequence of Borel partitions of M, say $\langle \mathfrak{S}_1, \mathfrak{S}_2, \cdots\cdot \rangle$, such that $\lim \mathrm{d}(\mathfrak{S}_n) = 0$. (By the way, the existence of such a sequence is plain.) Then we have

$$\Theta(y;\varphi;\mathfrak{S}_n) \to \mathrm{N}(y;\varphi;M) \qquad \text{as} \quad n \to +\infty,$$

whence the measurability of the function $\mathrm{N}(y;\varphi;M)$ follows at once. Further, by Fatou's Lemma and the relation $\Theta(y;\varphi;\mathfrak{S}_n) \leqq \mathrm{N}(y;\varphi;M)$, we deduce that

$$\Xi(\varphi;M) \leqq \varliminf_n \int_{-\infty}^{+\infty} \Theta(y;\varphi;\mathfrak{S}_n)\,dy \leqq \varlimsup_n \int_{-\infty}^{+\infty} \Theta(y;\varphi;\mathfrak{S}_n)\,dy \leqq \Xi(\varphi;M),$$

which asserts no other than that

$$\int_{-\infty}^{+\infty} \Theta(y;\varphi;\mathfrak{S}_n)\,dy \to \Xi(\varphi;M) \qquad \text{as} \quad n \to +\infty.$$

Replacing here the partition $\mathfrak{S}_n$ by a generic Borel partition $\mathfrak{S}$ of the set M, we find easily that

$$\int_{-\infty}^{+\infty} \Theta(y;\varphi;\mathfrak{S})\,dy \to \Xi(\varphi;M) \qquad \text{as} \quad \mathrm{d}(\mathfrak{S}) \to 0.$$

This, together with the following relation, completes the proof.

$$\int_{-\infty}^{+\infty} \Theta(y;\varphi;\mathfrak{S})\,dy = \sum_{X\in\mathfrak{S}} \int_{-\infty}^{+\infty} \mathrm{c}(y;\varphi[X])\,dy = \sum_{X\in\mathfrak{S}} |\varphi[X]|.$$

The above theorem may be regarded as a generalization of the well-known Banach Theorem (6.4) on p. 280 of Saks [7]. In point of fact, in the particular case in which M is a closed interval I, we find easily, as in [7], that

$$\sum_{i=1}^{n}|\varphi[I_i^{(n)}]| \to \mathrm{W}(\varphi;I) \qquad \text{as} \quad n\to+\infty,$$

where $\mathrm{W}(\varphi;I)$ stands for the absolute variation of $\varphi(x)$ over I and where the class $\mathfrak{S}_n=\{I_1^{(n)},\cdots,I_n^{(n)}\}$ means, for each $n\in \boldsymbol{N}$, any partition of I into n intervals of the same length $n^{-1}|I|$. On the other hand, each set $\varphi[I_i^{(n)}]$ being sigma-closed, the function $\Theta(y;\varphi;\mathfrak{S}_n)$ is Borel measurable for each n and hence so must also be its limit $\mathrm{N}(y;\varphi;I)$.

THEOREM 16. *If a function $\varphi(x)$ is continuous on a Borel set M and if $\mathfrak{S}$ is any Borel partition of M, we have the additivity relation*

$$\Xi(\varphi;M)=\sum_{X\in\mathfrak{S}}\Xi(\varphi;X).$$

PROOF. The result follows at once if we integrate over $\boldsymbol{R}$ both sides of the following equality and use Lebesgue's theorem on term by term integration:

$$\mathrm{N}(y;\varphi;M)=\sum_{X\in\mathfrak{S}}\mathrm{N}(y;\varphi;X).$$

LEMMA 3. *If a function $\varphi(x)$ is continuous on a bounded measurable set E and fulfils the condition* (N) *on E, then the set E contains, for any $\varepsilon>0$, a measurable subset M, such that $|\varphi[E]\setminus\varphi[M]|<\varepsilon$, and on which the function $\varphi(x)$ assumes each of its values at most once.*

This proposition generalizes slightly part (i) of the Lemma on p. 283 of Saks [7] and may be established in the same way as there.

A function will be said to fulfil the *condition* (T_1) on a set E, if almost every one of its values is assumed at most a finite number of times on E. In the special case where the set E is a closed interval, this condition comes to the same thing as the Banach condition (T_1) on p. 277 of [7].

With the help of Theorem 15 and of the above lemma, we can now deduce the following extension of the Banach Theorem (7.4) on p. 284 of [7], the proof being the same as in that book.

THEOREM 17. *In order that a function which is continuous on a bounded Borel set, fulfil the condition* (S) *on this set, it is necessary* *and sufficient that the function fulfil on this set both the conditions* (N) *and* (T_1).

The boundedness of the set is essential for the validity of this theorem, as shown by a simple example. In fact, consider the function $\varphi(x)$ which is equal, for each $x \in \boldsymbol{R}$, to the distance of the point x from the set of all even numbers. Since obviously $|\varphi[X]| \leqq |X|$ for every set $X \subset \boldsymbol{R}$, the function $\varphi(x)$ fulfils the condition (S) on $\boldsymbol{R}$. But this function clearly fails to fulfil the condition (T_1) on $\boldsymbol{R}$.

A result similar to the above theorem is valid for the condition (S_0) as well. Indeed, we have the following

THEOREM 18. *In order that a function $\varphi(x)$ which is continuous on a bounded Borel set B, fulfil the condition* (S_0) *on B, it is necessary and sufficient that, on this set, the function fulfil the condition* (N) *and have finite fluctuation.*

PROOF. (i) Necessity. Since the condition (S_0) plainly implies the condition (N), we need only show that $\Xi(\varphi; B) < +\infty$.

On account of the condition (S_0) we can choose a number $\delta > 0$ such that, for each finite disjoint sequence $\langle E_1, \cdots\cdots, E_n \rangle$ of measurable subsets of B, the inequality $|E_1| + \cdots\cdots + |E_n| < \delta$ always implies $|\varphi[E_1]| + \cdots\cdots + |\varphi[E_n]| < 1$.

Now consider any Borel set $M \subset B$ with $|M| < \delta$. If $\mathfrak{S}$ denotes generically a finite Borel partition of M, Theorem 15 asserts that

$$\sum_{X \in \mathfrak{S}} |\varphi[X]| \to \Xi(\varphi; M) \qquad \text{as} \quad \mathrm{d}(\mathfrak{S}) \to 0,$$

where the sum at the head is always <1 on account of the above choice of δ. Hence we have $\Xi(\varphi; M) \leqq 1$ in the limit.

The finiteness of the fluctuation $\Xi(\varphi; B)$ is immediate from the above result. To verify this, let $\mathfrak{M}$ be a finite Borel partition of B such that $\mathrm{d}(\mathfrak{M}) < \delta$, where δ is the same number as above. If $X \in \mathfrak{M}$, then $|X| \leqq \mathrm{d}(X) \leqq \mathrm{d}(\mathfrak{M}) < \delta$, whence $\Xi(\varphi; X) \leqq 1$. Combining this with Theorem 16, we find that $\Xi(\varphi; B) = \sum_{X \in \mathfrak{M}} \Xi(\varphi; X) < +\infty$.

(ii) Sufficiency. Let $\Xi(\varphi; B) < +\infty$ and assume that $\varphi(x)$ fulfils the condition (N) on B. M will denote generically a Borel subset of B, throughout the sequel. The sets M together form an additive class, i. e. the Borel class in B, and Theorem 16 shows that the fluctuation $\Xi(\varphi; M)$ is a completely additive function of M.

This set function is absolutely continuous. In fact, if M is null and if $\mathfrak{S}$ denotes a generic Borel partition of M, the sum $\sum |\varphi[X]|$, where $X\in\mathfrak{S}$, tends to $\Xi(\varphi;M)$ as $d(\mathfrak{S})\to 0$, on account of Theorem 15. But $|\varphi[X]|$ vanishes for every X, since $X\subset M$ and $|M|=0$. We therefore have $\Xi(\varphi;M)=0$.

It follows from a well-known theorem (see Saks [7], p. 31) that to each $\varepsilon>0$ there corresponds a number $\delta>0$ such that for every M the inequality $|M|<\delta$ implies $\Xi(\varphi;M)<\varepsilon$.

This being so, let $\langle E_1,\cdots\cdot,E_n\rangle$ be any finite disjoint sequence of measurable subsets of B. It suffices to show that the inequality

$$|E_1|+\cdots\cdot+|E_n|<\delta \quad \text{implies} \quad |\varphi[E_1]|+\cdots\cdot+|\varphi[E_n]|<\varepsilon.$$

For each $i=1,\cdots\cdot,n$ there exists in E_i a Borel set M_i such that $|E_i\setminus M_i|=0$ (see Saks [7], p. 69). Then $|\varphi[E_i\setminus M_i]|=0$ and hence $|\varphi[E_i]|=|\varphi[M_i]|$. But the definition of $\Xi(\varphi;M)$ implies the relation $|\varphi[M]|\leqq\Xi(\varphi;M)$ for every M, while the union $M_0=M_1\cup\cdots\cdot\cup M_n$ clearly has measure $|M_0|<\delta$, so that $\Xi(\varphi;M_0)<\varepsilon$. It follows that

$$\sum_{i=1}^{n}|\varphi[E_i]|=\sum_{i=1}^{n}|\varphi[M_i]|\leqq\sum_{i=1}^{n}\Xi(\varphi;M_i)=\Xi(\varphi;M_0)<\varepsilon,$$

which completes the proof.

We end this section with the following proposition which is immediate from Theorem 14 and Therem 18.

THEOREM 19. *A function which is normally continuous on a sigma-closed set, necessarily has finite fluctuation on each bounded Borel set contained in this set.*

§6. ACS decomposition of a normally continuous function.

A function which maps a linear set E onto a null set, will be termed *steplike* on E. We shall say that a function $\varphi(x)$ is ACS *decomposable* on a set E, if the function is expressible on $\boldsymbol{R}$ in the from $\varphi(x)=\psi(x)+\chi(x)$, where $\psi(x)$ is an absolutely continuous function steplike on the set $\boldsymbol{R}\setminus E$ and $\chi(x)$ is a function steplike on $\boldsymbol{E}$. (In accordance with Saks [7], p. 59, we understand by an *absolutely continuous function* any function which is AC on every closed interval.) When this is the case, the above expression of $\varphi(x)$ will be called ACS *decomposition* of $\varphi(x)$ with respect to the set E. The two functions $\psi(x)$ and $\chi(x)$ will respectively be named *absolutely continuous* (or AC) *part* and *steplike part* of $\varphi(x)$ in this decomposition.

We do not know whether the following assertion is true: *If a function $\varphi(x)$ is normally continuous on a sigma-closed set A and* ACS *decomposable on A, then* (i) *the* AC *part of $\varphi(x)$ is uniquely determined on $\boldsymbol{R}$ to within an additive constant, and hence so is also the steplike part;* (ii) *the absolute variation* $\mathrm{W}(\psi;I)$ *of the* AC *part* $\psi(x)$ *on a closed interval I coincides for every I with the fluctuation* $\Xi(\varphi;A\cap I)$; *and* (iii) *the function $\varphi(x)$ is approximately equiderivable with $\psi(x)$ at almost all points of the set A.*

As we shall see later on, every function $\varphi(x)$ which is normally continuous on a sigma-closed set A and approximately derivable at almost all points of A, is ACS decomposable on A and the three conclusions of the above proposition are true.

In connection with the notion of ACS decomposition, there are still further a few open problems. Among others, we do not know if *every function normally continuous on a sigma-closed set, is* ACS *decomposable on this set.* Neither can we decide yet whether *the sum of two functions each of which is normally continuous and steplike on a sigma-closed set A, is necessarily also steplike on A.*

LEMMA 4. *Given a nonvoid metric space $\boldsymbol{M}$, let $\mathfrak{B}$ denote the class of all the Borel sets in $\boldsymbol{M}$.*

(i) *If $\Phi(X)$ is a nonnegative additive set function on the class $\mathfrak{B}$, then given any set $X\in\mathfrak{B}$ and any number $\varepsilon>0$ there exist in the space $\boldsymbol{M}$ a closed set S and an open set D such that $S\subset X\subset D$ and $\Phi(D)-\Phi(S)<\varepsilon$.*

(ii) *If $\Phi(X)$* and *$\Psi(X)$ are nonnegative additive set functions on the class $\mathfrak{B}$ and if $\Phi(G)=\Psi(G)$ for every open set G in $\boldsymbol{M}$, we have $\Phi(X)=\Psi(X)$ for every $X\in\mathfrak{B}$.*

(iii) *If $\Phi(X)$ and $\Psi(X)$ are nonnegative additive set functions on the Borel class in a closed interval I and if $\Phi(K)=\Psi(K)$ for every closed interval $K\subset I$, then the two functions coincide identically.*

REMARK. Part (ii) of the lemma is included in a theorem in small print on p. 72 of Saks [7]. He observes that this theorem may be established in the same way as for the foregoing Theorem (6.10). It does not appear to us, however, that the proof goes well in the way indicated by him.

PROOF. *re* (i): Denoting by $\mathfrak{A}$ the class of all the sets $X\in\mathfrak{B}$ with the property stated in (i), we shall show that $\mathfrak{A}=\mathfrak{B}$. For this purpose, it suffices to show that $\mathfrak{A}$ is an additive class and contains all the closed sets.

Each closed set in a metric space is expressible as the limit of a descending infinite sequence of open sets. From this fact we find at once that every closed set in $\boldsymbol{M}$ belongs to the class $\mathfrak{A}$.

In the next place, let $X\in\mathfrak{A}$ and consider the sets S and D which appear in (i). Writing $S^c=\boldsymbol{M}\setminus S$ and so on, we have $D^c\subset X^c\subset S^c$, where D^c is closed and S^c is open. Again, the evident relations $\Phi(D^c)=\Phi(\boldsymbol{M})-\Phi(D)$ and $\Phi(S^c)=\Phi(\boldsymbol{M})-\Phi(S)$ in conjunction imply that $\Phi(S^c)-\Phi(D^c)=\Phi(D)-\Phi(S)<\varepsilon$. It follows that $X^c\in\mathfrak{A}$.

Suppose finally that X is the union of an infinite sequence, say $\langle X_1, X_2, \cdots\rangle$, of sets of the class $\mathfrak{A}$. We shall verify that $X\in\mathfrak{A}$. Given an $\varepsilon>0$, we can choose for each $n\in\boldsymbol{N}$ a closed set S_n and an open set D_n such that

$$S_n\subset X_n\subset D_n \quad \text{and} \quad \Phi(D_n)-\Phi(S_n)<2^{-n}\varepsilon.$$

Writing $S=S_1\cup S_2\cdots$ and $D=D_1\cup D_2\cdots$, we find successively that

$$D\setminus S=\bigcup_{n=1}^{\infty}(D_n\setminus S)\subset\bigcup_{n=1}^{\infty}(D_n\setminus S_n),$$

$$\Phi(D)-\Phi(S)=\Phi(D\setminus S)\leqq\sum_{n=1}^{\infty}\Phi(D_n\setminus S_n)<\sum_{n=1}^{\infty}2^{-n}\varepsilon=\varepsilon.$$

On the other hand, the set S is the limit of the set $T_n=S_1\cup\cdots\cup S_n$ as $n\to+\infty$, and thus we can take n so large that $\Phi(T_n)>\Phi(S)-\varepsilon$. Then we have

$$\Phi(D)-\Phi(T_n)<\Phi(D)-\Phi(S)+\varepsilon<2\varepsilon.$$

This completes the proof of part (i), since T_n is closed, D is open, and $T_n\subset X\subset D$.

re (ii): Let X be a Borel set in $\boldsymbol{M}$. By part (i), there exists for each $\varepsilon>0$ an open set $D\supset X$ such that $\Phi(D)-\Phi(X)<\varepsilon$ as well as $\Psi(D)-\Psi(X)<\varepsilon$. Since $\Phi(D)=\Psi(D)$ by hypothesis, it follows that

$$\Phi(X)-\Psi(X)\leqq\Phi(D)-\Psi(X)=\Psi(D)-\Psi(X)<\varepsilon$$

and similarly that $\Psi(X)-\Phi(X)<\varepsilon$. We thus obtain $|\Phi(X)-\Psi(X)|<\varepsilon$, whence we get $\Phi(X)=\Psi(X)$ since ε is arbitrary.

re (iii): If we regard the interval I as a metric space $\boldsymbol{M}$, each set G open in $\boldsymbol{M}$ is expressible in the form $G=I\cap V$, where V is an open set in $\boldsymbol{R}$. Then G is expressible as the limit of an ascending infinite sequence of figures $W_1\subset W_2\subset\cdots$. Since $\Phi(W_n)=\Psi(W_n)$ for every n, it follows that

$$\Phi(G)=\lim_n\Phi(W_n)=\lim_n\Psi(W_n)=\Psi(G).$$

This, together with part (ii), shows that $\Phi(X)=\Psi(X)$ for every Borel set $X\subset I$.

THEOREM 20. *Given a function $\varphi(x)$ which is normally continuous on a bounded sigma-closed set A and approximately derivable at almost all points of A, let us write, for definiteness, $\varphi'_{\mathrm{ap}}(\xi)=0$ for every point $\xi\in\boldsymbol{R}$ at which the function $\varphi(x)$ is not approximately derivable.*

Then the function $\varphi'_{\mathrm{ap}}(x)$ is summable over A. Further, the fluctuation $\Xi(\varphi;M)$ is expressed for every Borel set $M\subset A$ by the formula

$$\Xi(\varphi;M)=\int_M|\varphi'_{\mathrm{ap}}(x)|\,dx.$$

PROOF. Consider any compact set $Q\subset A$ on which the function $\varphi(x)$ is AC. In the first place, we shall prove the summability of $\varphi'_{\mathrm{ap}}(x)$ on the set Q and the validity of the above formula for every Borel set $M\subset Q$.

For this purpose, we may obviously assume Q to contain at least two points. We denote by H a generic open interval (if existent) contiguous to Q, and by $\lambda(x)$ the linear modification of $\varphi(x)$ with respect to Q, i. e. the function which coincides with $\varphi(x)$ unless x belongs to an H and which is linear on the closure of any H. Then the function $\lambda(x)$ is AC on the minimal closed interval I containing Q (see [1], Theorem 4) and hence derivable almost everywhere on I. Writing for definiteness $\lambda'(\xi)=0$ for each $\xi\in\boldsymbol{R}$ at which $\lambda(x)$ is not derivable, we find that the function $\lambda'(x)$ is summable on I and coincides with $\varphi'_{\mathrm{ap}}(x)$ at almost all points of Q (cf. Saks [7], p. 220). The function $\varphi'_{\mathrm{ap}}(x)$ is thus summable over Q.

This being so, denote by E a generic Borel subset of I and write

$$\Phi(E)=\int_E|\lambda'(x)|\,dx.$$

Then $\Phi(E)$ is an additive set function. But so is also the function $\Xi(\lambda;E)$ by Theorem 16 and Theorem 19. Now, by what was said just after the proof of Theorem 15, we have $\Xi(\lambda;K)=\mathrm{W}(\lambda;K)$ for every closed interval $K\subset I$. On the other hand, $\mathrm{W}(\lambda;K)=\Phi(K)$ by absolute continuity of $\lambda(x)$ on I. Hence the two functions $\Xi(\lambda;E)$ and $\Phi(E)$ coincide when E is especially a closed interval. It thus follows from Lemma 4 that $\Xi(\lambda;E)=\Phi(E)$ for every E. This shows that if M is a Borel subset of Q, then

$$\Xi(\varphi;M)=\Xi(\lambda;M)=\Phi(M)=\int_M|\lambda'(x)|\,dx=\int_M|\varphi'_{\mathrm{ap}}(x)|\,dx.$$

By hypothesis, there is in A a Borel null set N such that the function $\varphi(x)$ is AD at all points of $A\setminus N$. Taking note of the continuity of $\varphi(x)$ on A, we find by a well-known theorem (see Saks

[7], p. 239) that $\varphi(x)$ is GAC on the set $A \setminus N$. On the other hand, the set A is, by hypothesis, the union of a sequence of compact sets. If R is any one of these sets, the function $\varphi(x)$ is GAC on $R \setminus N$. It follows, in view of the continuity of $\varphi(x)$ on R, that $R \setminus N$ is covered by a sequence of compact sets contained in R and on each of which $\varphi(x)$ is AC.

From what was proved already we draw two conclusions: In the first place, the function $\varphi'_{\mathrm{ap}}(x)$ is measurable on the set A. Secondly, the set $A \setminus N$ has a Borel partition, say $\mathfrak{S}$, such that the closure of each set $X \in \mathfrak{S}$ is contained in A and that the function $\varphi(x)$ is AC on this closure.

We now combine the above considerations. Suppose that M is a Borel subset of A. Using Theorem 16 and noting that $\Xi(\varphi; M \cap N)$ vanishes since $\varphi[M \cap N]$ is null on account of Theorem 12, we obtain the required expression of $\Xi(\varphi; M)$ as follows:

$$\Xi(\varphi; M) = \Xi(\varphi; M \cap N) + \sum_{X \in \mathfrak{S}} \Xi(\varphi; M \cap X) = \sum_{X \in \mathfrak{S}} \int_{M \cap X} |\varphi'_{\mathrm{ap}}(x)|\, dx$$

$$= \int_{M \setminus N} |\varphi'_{\mathrm{ap}}(x)|\, dx = \int_{M} |\varphi'_{\mathrm{ap}}(x)|\, dx.$$

Finally, specializing the set M to A in this relation, we find by Theorem 19 that the function $\varphi'_{\mathrm{ap}}(x)$ is certainly summable on the set A. This completes the proof.

The following theorem is closely connected with Theorem 5 of [1] as restricted to parts (iii) and (iv).

THEOREM 21. *Suppose that a function $\varphi(x)$ is normally (in particular, absolutely) continuous on every portion of a sigma-closed set A and approximately derivable at almost all points of A. In order that $\varphi(x)$ be steplike on A, it is necessary and sufficient that $\varphi(x)$ be* AD *to zero at almost all points of A.*

PROOF. Supposing A nonvoid as we may, let us denote by P a generic portion of A, so that P is a bounded sigma-closed set. Since plainly $|\varphi[A]|$ is the supremum of the numbers $|\varphi[P]|$, we have the relation $|\varphi[A]| = 0$ if and only if $|\varphi[P]| = 0$ for every P. But the two conditions $|\varphi[P]| = 0$ and $\Xi(\varphi; P) = 0$ are equivalent for every P. On the other hand, we find by Theorem 20 that

$$\Xi(\varphi; P) = \int_P |\varphi'_{\mathrm{ap}}(x)|\, dx.$$

Consequently, the fluctuation $\Xi(\varphi; P)$ vanishes for every P, or equivalently, we have $|\varphi[A]|=0$, if and only if the function $\varphi(x)$ is AD to zero at almost all points of A. This completes the proof.

REMARK. *Let E be a measurable set and suppose that a function $\varphi(x)$ is* AC *on every portion of E. Then the function is steplike on E, when and only when it is* AD *to zero at almost all points of E.* This proposition is easily ascribable to the above theorem, as follows. The function $\varphi(x)$ is GAC on E and hence AD at almost all points of E (see Saks [7], p. 223). Moreover, the set E contains a sigma-closed set A such that $|E\setminus A|=0$ (see [7], p. 69). We then have $|\varphi[E\setminus A]|=0$, since $\varphi(x)$ fulfils the condition (N) on E (see [7], p. 225, above). Consequently $|\varphi[E]|=|\varphi[A]|$. We may thus assume that the set E is itself sigma-closed.

THEOREM 22. *Any function $\varphi(x)$ which is normally continuous on every portion of a sigma-closed set A and approximately derivable at almost all points of A, is* ACS *decomposable on A and fulfils the conclusions* (i) *to* (iii) *of the assertion stated at the beginning of this section.*

PROOF. As in Theorem 20, we shall write $\varphi'_{\text{ap}}(\xi)=0$ for every point $\xi\in\boldsymbol{R}$ at which the function $\varphi(x)$ is not approximately derivable. Let $\psi(x)$ be an indefinite integral of the function $\varphi'_{\text{ap}}(x)\mathrm{c}(x;A)$ which is summable (on every closed interval) by Theorem 20. We shall begin by proving that $\psi(x)$ is the AC part of $\varphi(x)$ in some ACS decomposition with respect to A.

Clearly the function $\psi(x)$ is AC. Furthermore, $\psi(x)$ is AD to $\varphi'_{\text{ap}}(x)\mathrm{c}(x;A)$ almost everywhere on $\boldsymbol{R}$, and hence AD to $\varphi'_{\text{ap}}(x)$ [or to zero] at almost every point of A [or of $\boldsymbol{R}\setminus A$]. It follows from the above Remark that $\psi(x)$ is steplike on $\boldsymbol{R}\setminus A$. It remains to verify that the function $\chi(x)=\varphi(x)-\psi(x)$ is steplike on A. The function $\psi(x)$, which is AC, is NC on each finite interval. Thus the function $\chi(x)$ is NC on every portion of A. Further, $\chi(x)$ is AD to zero at every point at which $\varphi(x)$ and $\psi(x)$ are AED (approximately equiderivable), and hence at almost every point of A. It thus follows from the preceding theorem that $|\chi[A]|=0$, as required.

This being premised, we shall now proceed to ascertain the validity of the statements (i) to (iii). In what follows, $\psi(x)$ and $\chi(x)$ will mean the same functions as above.

re (i): Suppose that a function $\sigma(x)$ other than $\psi(x)$ is also an AC part of the function $\varphi(x)$ with respect to A. Then the function

$\tau(x)=\varphi(x)-\sigma(x)$ is NC on every portion of A and steplike on A. Furthermore, since both $\varphi(x)$ and $\sigma(x)$ are AD at almost all points of A, so is also the function $\tau(x)$. It follows from the foregoing theorem that $\tau(x)$ is AD to zero at almost all points of A. This implies that $\sigma(x)$ is AED with $\varphi(x)$, and hence with $\psi(x)$, at almost all points of A. On the other hand, the above Remark shows that the AC function $\sigma(x)$, which is steplike on $\boldsymbol{R}\setminus A$, must be AD to 0 at almost all points of $\boldsymbol{R}\setminus A$. But this last property is possessed by $\psi(x)$ also, as already seen. We thus find that the AC functions $\sigma(x)$ and $\psi(x)$ are AED almost everywhere on $\boldsymbol{R}$ and therefore that their difference $\sigma(x)-\tau(x)$ is a constant over the real line.

re (ii): On account of part (i), it suffices to ascertain that $\mathrm{W}(\psi;I)=\Xi(\varphi;A\cap I)$ for the function $\psi(x)$ and every closed interval I. But this is immediate from the definition of $\psi(x)$. Indeed, using Theorem 20 we have

$$\mathrm{W}(\psi;I)=\int_I|\varphi'_{\mathrm{ap}}(x)|\mathrm{c}(x;A)\,dx=\int_{A\cap I}|\varphi'_{\mathrm{ap}}(x)|\,dx=\Xi(\varphi;A\cap I).$$

re (iii): This was incidentally established already.

§7. **Normally fluctuant functions.**

A function $\varphi(x)$ will be termed *normally fluctuant*, or NF for short, or again to *fluctuate normally*, on a linear set E, if $\sup|\varphi(W)|$ is finite, where W stands for a generic figure pertaining normally to E. We see at once that *this property of* $\varphi(x)$ *is hereditary with respect to the set* E.

When this is the case, we have also $\sup\varphi^*(W)<+\infty$, as we find easily by means of the two propositions (i) and (ii) on p. 2. Moreover, *such a function* $\varphi(x)$ *is necessarily bounded on* E. To see this, assuming E infinite as we may, we need only specialize, in the above definition, the figure W to a closed interval pertaining to the set E. Again, *every function which is* BV *on a set* E, *is* NF *on* E. Finally, *every linear combination, with constant coefficients, of two functions which fluctuate normally on a set, itself does so on this set.*

The proof of the following theorem is virtually the same as for Theorem 3, with the help of Lemma 2.

THEOREM 23. *A function which fluctuates normally on a set* E, *necessarily does so on every set* $M\supset E$ *contained in the closure of* E, *provided that the function is continuous on* M.

THEOREM 24. *Any function $\varphi(x)$ which is normally continuous on a bounded set E, is normally fluctuant on this set.*

PROOF. By hypothesis we can choose a number $\delta>0$ such that, for every figure W pertaining normally to E, the inequality $|W|<\delta$ implies $|\varphi(W)|<1$. Then the function $\varphi(x)$ is bounded on the intersection $J\cap E$, whenever J is a closed interval of length $<\delta$. The set E being bounded, it follows that $\varphi(x)$ is bounded on E.

This being so, take a closed interval I whose interior contains E, and express I as the union of a finite number of non-overlapping closed intervals $I_1,\cdots,I_n$ each of which has length $<\delta$. Now let W be a figure pertaining normally to E, and let W_i denote for each $i=1,\cdots,n$ the union (possibly void) of all the component intervals of W that are contained in the interior of the interval I_i. Then the figure W_i plainly pertains normally to E and has measure $<\delta$, so that $|\varphi(W_i)|<1$. On the other hand, if we write $Z=W_1\cup\cdots\cup W_n$, the set $W\setminus Z$ is a figure with at most n components. Hence we get $|\varphi(W\setminus Z)|\leqq nL$, where L denotes the oscillation $O(\varphi;E)$ which is finite. Noting the relation $\varphi(W)=\varphi(W_1)+\cdots+\varphi(W_n)+\varphi(W\setminus Z)$ and using the above results, we conclude that

$$|\varphi(W)|\leqq|\varphi(W_1)|+\cdots+|\varphi(W_n)|+|\varphi(W\setminus Z)|<n+nL,$$

which completes the proof.

The following theorem is well-known (see Saks [7], p. 227). *In order that a function $F(x)$ which is continuous and* BV *on a compact set Q, be* AC *on Q, it is necessary and sufficient that $F(x)$ fulfil the condition* (N) *on this set.* In view of this theorem, it is natural to ask as to the validity of the assertion: *In order that a function $\varphi(x)$ which is continuous and* NF *on a compact set Q, be* NC *on Q, it is necessary and sufficient that $\varphi(x)$ fulfil the condition* (N) *on this set.* The necessity of the condition (N) is obvious by Theorem 12. The sufficiency will, however, be disproved afterward by Theorem 25 that constitutes a concrete counter example.

Let us resume the compact set $\Gamma=E_0\cap E_1\cap\cdots$ of Theorem 1, where E_0 means the unit interval $[0,1]$ and $E_{m+1}=E_m(3)$ for each integer $m\geqq0$. With each point ξ of Γ we now associate a sequence $\sigma(\xi)=\langle w_1(\xi),w_2(\xi),\cdots\rangle$ in the same way as in proving Theorem 6 of [5]. For the sake of completeness, let us repeat concisely the definition of $\sigma(\xi)$. For each integer $m\geqq0$, the figure E_m contains a component, say K_m, to which ξ belongs. Let $w_{m+1}(\xi)$ be 1 or 0, according as K_{m+1} is or is not the middle one among the three components of the figure $K_m(3)$, respectively.

Making use of the binary sequence $\sigma(\xi)$, we proceed to define on the set Γ a function $\Lambda(\xi)$ by writing

$$\Lambda(\xi)=\sum_{n=1}^{\infty} 3^{-n}w_n(\xi) \qquad \text{for} \quad \xi\in\Gamma.$$

We then extend the definition of this function to the whole $\boldsymbol{R}$, in such a manner that the extended function, still denoted by $\Lambda(x)$, becomes linear on each closed interval contiguous to Γ and vanishes outside the interval $[0,1]$.

It is easy to prove the following proposition.

LEMMA 5. *The function $\Lambda(x)$ is continuous. More precisely, if K is a generic component of the figure E_m, where $m\geqq 0$, then we have* $\mathrm{O}(\Lambda;K\cap\Gamma)=3^{-m}/2$. *Further, $\Lambda(x)$ maps the set Γ onto a null set.*

THEOREM 25. *The function $\Lambda(x)$ fluctuates normally on the set Γ and fulfils the condition* (N) *on the real line, without being normally continuous on Γ.*

PROOF. The construction of the sequence $\langle E_0, E_1, \cdots\rangle$ shows at once that the figure E_m, where $m\geqq 0$, pertains normally to Γ. Again, if K is a generic component of E_m, we have $\mathrm{O}(\Lambda;K\cap\Gamma)=3^{-m}/2$ by the above lemma. Hence there corresponds to each K a closed subinterval I pertaining to Γ and such that $|\Lambda(I)|>3^{-m-1}$. The union of all these intervals I is a figure, say A_m, which plainly pertains normally to Γ. But we have $\Lambda^*(A_m)=\sum|\Lambda(I)|>3^m\cdot 3^{-m-1}=3^{-1}$, since the figure E_m has exactly 3^m components K. On the other hand, clearly $|A_m|\leqq|E_m|=(3/5)^m$, whence $|A_m|\to 0$ as $m\to+\infty$. Accordingly the function $\Lambda(x)$ cannot be normally continuous on the set Γ.

We shall go on to show that $\Lambda(x)$ fluctuates normally on Γ. Let W be a nonvoid figure pertaining normally to Γ and let us consider the quantity $\theta(W;\Gamma)$, where the function θ means the same as on p. 2. As readily seen, $\theta(W;\Gamma)$ coincides with $\sup|H|$, where H is a generic open interval contiguous to the compact set $W\cap\Gamma$. Since $|H|\leqq 5^{-1}$ for every H, we have $0<\theta(W;\Gamma)\leqq 5^{-1}$. Hence there is an integer $m\geqq 0$ such that $5^{-m-1}\leqq\theta(W;\Gamma)<5^{-m}$.

From now on, the argument goes on as in the proof of Theorem 1. If J denotes a generic component of W, each J is contained in a component of the figure E_m. On the other hand, each component K of E_m can contain at most five intervals J, while E_m has exactly 3^m components. Hence W has at most $5\cdot 3^m$ components. This, combined with $O(\Lambda; K\cap\Gamma)=3^{-m}/2$, gives

$$\Lambda^*(W)=\sum_J|\Lambda(J)|\leqq 5\cdot 3^m\cdot 3^{-m}\cdot 2^{-1}<3,$$

which proves that $\Lambda(x)$ is NF on Γ.

Finally, the function $\Lambda(x)$ fulfils the condition (N) on $\boldsymbol{R}$, since $|\Lambda[\Gamma]|=0$ by Lemma 5 and since $\Lambda(x)$ is linear on every interval disjoint with Γ. This completes the proof.

THEOREM 26. *If a function $\varphi(x)$ fluctuates normally on a set E, this set contains at most a countable infinity of points at each of which the function is discontinuous on E. More precisely, if $\langle x_1, \cdots, x_n\rangle$ is any finite distinct sequence of such points, we have the relation*

$$\sum_{i=1}^{n} o_E(\varphi; x_i)\leqq \sup\varphi^*(W)<+\infty,$$

where W stands for a generic figure pertaining normally to the set E and where $o_E(\varphi; x_i)$ denotes the oscillation of the function $\varphi(x)$ on E at the point x_i (see Saks [7], p. 42).

PROOF. Writing $\rho=\sup\varphi^*(W)$ for short, we shall prove first the second half of the assertion. As stated already, a function normally fluctuant on E is necessarily bounded on E, so that the oscillation $o_E(\varphi; x_i)$ is finite for each x_i. On the other hand, each x_i is plainly an accumulation point of E. Hence, given any $\varepsilon>0$, we can choose a disjoint sequence of n closed intervals, $\langle K_1 \cdots, K_n\rangle$ say, such that the figure $Z=K_1\cup\cdots\cup K_n$ pertains normally to E and that

$$o_E(\varphi; x_i)<|\varphi(K_i)|+n^{-1}\varepsilon \qquad \text{for} \quad i=1, \cdots, n.$$

These inequalities imply the following relation:

$$\sum_{i=1}^{n} o_E(\varphi; x_i)<\sum_{i=1}^{n}|\varphi(K_i)|+\varepsilon=\varphi^*(Z)+\varepsilon\leqq\rho+\varepsilon.$$

Since ε is arbitrary, it follows that $o_E(\varphi; x_1)+\cdots+o_E(\varphi; x_n)\leqq\rho$.

To deduce the first half of the theorem, let ξ be a generic point of E at which $\varphi(x)$ is discontinuous on E and let T be the set of all the points ξ. Since we have $o_E(\varphi; \xi)>0$ for each ξ, the set T is expressible in the form $T=T_1\cup T_2\cup\cdots$, where T_k denotes for each $k\in N$ the set of all the ξ such that $o_E(\varphi; \xi)>k^{-1}$. Then T_k is a finite set, since it can contain at most $k\rho$ points by what we proved above. Hence the set T is at most countable, and the proof is complete.

THEOREM 27. *A function $\varphi(x)$ which fluctuates normally on a set E, necessarily maps every Borel set $M\subset E$ onto a measurable set.*

PROOF. Let T mean the same set as in the above proof. Since T is at most countable, the difference $M \setminus T$ is a Borel set. The function $\varphi(x)$ is continuous on the set $E \setminus T$, which contains $M \setminus T$. On the other hand, a continuous image of a Borel set is always a measurable set (see Kuratowski [6], p. 249). In consequence, the image $\varphi[M \setminus T]$ is measurable. Further, the set $\varphi[M \cap T]$ is at most countable and hence measurable. We therefore conclude that the set $\varphi[M] = \varphi[M \setminus T] \cup \varphi[M \cap T]$ is measurable.

THEOREM 28. *Given a function $\varphi(x)$ and a set E, suppose that $\varphi(x)$ maps each Borel set $M \subset E$ onto a measurable set. Keeping fixed such a set M, let $\mathfrak{S}$ be a generic Borel partition of M. Then both $\Theta(y;\varphi;\mathfrak{S})$ and $\mathrm{N}(y;\varphi;M)$ are nonnegative measurable functions of y and we have the relation*

$$\sum_{X \in \mathfrak{S}} |\varphi[X]| \to \int_{-\infty}^{+\infty} \mathrm{N}(y;\varphi;M)\,dy \qquad \textit{as} \quad \mathrm{d}(\mathfrak{S}) \to 0.$$

The proof is almost the same as for Theorem 15 and may be omitted. As before, the above integral will be written $\Xi(\varphi;M)$ and called *fluctuation* of $\varphi(x)$ on the set M. If in particular the function $\varphi(x)$ fluctuates normally on E, the hypothesis of the theorem is fulfilled on account of Theorem 27.

THEOREM 29. *Under the hypothesis of the foregoing theorem, we have the relation*

$$\Xi(\varphi;M) = \sum_{X \in \mathfrak{S}} \Xi(\varphi;X).$$

This theorem is easily proved (see the proof of Theorem 16).

In view of Theorem 24, the following assertion may be regarded as a generalization of Theorem 19.

THEOREM 30. *A function $\varphi(x)$ which fluctuates normally on a sigma-closed set A, necessarily has finite fluctuation on every Borel set M contained in A.*

PROOF. The fluctuation $\Xi(\varphi;M)$ exists for every Borel set $M \subset A$ in virtue of Theorem 27 and Thorem 28. On the other hand, the relation $\Theta(y;\varphi;M) \leqq \Theta(y;\varphi;A)$ shows that $\Xi(\varphi;M) \leqq \Xi(\varphi;A)$. Hence we need only prove that $\Xi(\varphi;A) < +\infty$.

Given any set $E \subset A$, denote by W a generic figure pertaining normally to E, and by $\rho(E)$ the supremum of the numbers $\varphi^*(W)$. Since every W pertains normally to the set A, we find immediately that $\rho(E) \leqq \rho(A) < +\infty$.

The theorem will be established if we show that $\Xi(\varphi; A) \leqq \rho(A)$. For this purpose, it suffices to consider the case where A is bounded. In point of fact, if we write $A_k = A \cap [-k, k]$ for each $k \in N$, the set A_k is sigma-closed together with the set A and the function $\varphi(x)$ fluctuates normally on A_k. Moreover, A is the limit of the ascending sequence $A_1 \subset A_2 \subset \cdots\cdot$ and hence is partitioned into the Borel sets $A_1, A_2 \setminus A_1, A_3 \setminus A_2, \cdots\cdot$. It therefore follows from Theorem 27 and Theorem 29 that

$$\Xi(\varphi; A) = \Xi(\varphi; A_1) + \sum_{i=1}^{\infty} \Xi(\varphi; A_{i+1} \setminus A_i) = \lim_{k \to \infty} \Xi(\varphi; A_k).$$

Consequently, if $\Xi(\varphi; A_k) \leqq \rho(A_k)$ for every k, we have $\Xi(\varphi; A) \leqq \rho(A)$ in view of $\rho(A_k) \leqq \rho(A)$. We may thus suppose A bounded in the rest of the proof.

Let D be any open interval containing A. Given an integer $n \geqq 2$, we take in D the $n-1$ points that divide D into n parts of the same length $n^{-1}|D|$. Writing S for the set of these points, let the components of the open set $D \setminus S$ be $D_1 < \cdots\cdot < D_n$ in their natural ordering. Then the set $A \setminus S$ is partitioned into the n Borel sets $D_1 \cap A, \cdots\cdot, D_n \cap A$ each of which has diameter $\leqq n^{-1}|D|$. Since S is a finite set, it ensues that the set A is itself partitioned into the sets $D_i \cap A$ plus a finite number (possibly zero) of singletonic sets. We then see by Theorem 27 and Theorem 28 that $|\varphi[D_1 \cap A]| + \cdots\cdot + |\varphi[D_n \cap A]|$ tends to $\Xi(\varphi; A)$ as $n \to +\infty$. It thus suffices to verify that this sum is $\leqq \rho(A)$ for each $n \geqq 2$. We shall keep n fixed in what follows.

The set $A \setminus S$, which is evidently sigma-closed, is the limit of an ascending infinite sequence of compact sets, say $Q_1 \subset Q_2 \subset \cdots\cdot$. For short, let us write $Q_k^{(i)} = D_i \cap Q_k$ for $k \in N$ and $i = 1, \cdots\cdot, n$. We find easily that each $Q_k^{(i)}$ is a compact set and that $D_i \cap A$ is, for each i, the limit of the sequence $Q_1^{(i)} \subset Q_2^{(i)} \subset \cdots\cdot$. Then $\varphi[D_i \cap A]$ is the limit of the sequence $\varphi[Q_1^{(i)}] \subset \varphi[Q_2^{(i)}] \subset \cdots\cdot$, and we obtain successively the relations

$$\lim_{k \to \infty} |\varphi[Q_k^{(i)}]| = |\varphi[D_i \cap A]|, \qquad \lim_{k \to \infty} \sum_{i=1}^{n} |\varphi[Q_k^{(i)}]| = \sum_{i=1}^{n} |\varphi[D_i \cap A]|.$$

Thus the proof is reduced to showing for fixed k the inequality

$$|\varphi[Q_k^{(1)}]| + \cdots\cdot + |\varphi[Q_k^{(n)}]| \leqq \rho(A).$$

To deduce this, we may plainly assume that the set Q_k is noncountably infinite. Then Q_k contains a nonvoid perfect set P such that $Q_k \setminus P$ is countable. Since $Q_k^{(i)} = D_i \cap Q_k$ by definition, we have

$$|\varphi[Q_k^{(i)}]| \leqq |\varphi[D_i \cap P]| + |\varphi[Q_k \setminus P]| = |\varphi[D_i \cap P]|$$

for $i=1, \cdots\cdot, n$. Hence it suffices to prove the inequality

$$|\varphi[D_1 \cap P]| + \cdots\cdot + |\varphi[D_n \cap P]| \leqq \rho(A)$$

on the assumption that P is a nonvoid perfect subset of $A \setminus S$.

This being so, let us denote by H a generic open interval (if existent) contiguous to P, and by G the union of all the H fulfilling $|H| \geqq \operatorname{dist}(P, S)$. If I is the minimal closed interval containing P, then the figure $F = I \setminus G$ contains P and pertains normally to P. We see further that this figure is disjoint with S. Hence F is partitioned into the figures $F_i = D_i \cap F$, where $i=1, \cdots\cdot, n$.

Keeping the index i fixed for the time being, consider a generic component K (if existent) of the figure F_i. Then the image $\varphi[K \cap P]$ is a bounded set, since the function $\varphi(x)$ is bounded on A. Given any $\varepsilon > 0$, each interval K thus contains a closed subinterval L pertaining to P and such that $\mathrm{d}(\varphi[K \cap P]) < |\varphi(L)| + \varepsilon$. Associating such an L with each K, we denote by Z_i the union of all the intervals L. The two figures Z_i and F_i have the same number of components. Writing N_i for this number, we must have $|\varphi[D_i \cap P]| \leqq \varphi^*(Z_i) + N_i \varepsilon$. In fact, the inclusion $P \subset F$ implies that $D_i \cap P = (D_i \cap F) \cap P = F_i \cap P$, whence we have the relation

$$|\varphi[D_i \cap P]| = |\varphi[F_i \cap P]| \leqq \sum_K |\varphi[K \cap P]| \leqq \sum_K \mathrm{d}(\varphi[K \cap P])$$
$$\leqq \sum_K |\varphi(L)| + N_i \varepsilon = \varphi^*(Z_i) + N_i \varepsilon.$$

Now write $Z = Z_1 \cup \cdots\cdot \cup Z_n$ and $N = N_1 + \cdots\cdot + N_n$. Then N is the number of the components of F, while the figure Z clearly pertains normally to P and hence to A also. We thus obtain the following relation, where ε may be arbitrarily small.

$$\sum_{i=1}^{n} |\varphi[D_i \cap P]| \leqq \sum_{i=1}^{n} \varphi^*(Z_i) + N\varepsilon = \varphi^*(Z) + N\varepsilon \leqq \rho(A) + N\varepsilon.$$

It follows that $|\varphi[D_i \cap P]| + \cdots\cdot + |\varphi[D_n \cap P]| \leqq \rho(A)$, which completes the proof of the theorem.

§ 8. **Two open propositions and the seminormal integration.**

We begin with a supplementary theorem on normal continuity.

THEOREM 31. *A function $\varphi(x)$ which is normally continuous on every countable subset of a set E, is necessarily normally continuous on the whole set E.*

PROOF. We shall show first that the function is continuous on E. For this purpose, suppose c to be an accumulation point for E, and extract from E any infinite sequence of points, say $\langle x_1, x_2, \cdots\rangle$, which tends to the point c. Then the function $\varphi(x)$ is NC, and hence continuous, on the countable set of the points $c, x_1, x_2, \cdots$. Thus $\varphi(x_n)\to\varphi(c)$ as $n\to+\infty$. In other words, $\varphi(x)$ is continuous at c on the set E. This proves the continuity of $\varphi(x)$ on E.

The set E plainly has a countable subset A whose closure contains E. Then the function $\varphi(x)$, which is NC on A and continuous on E, must be NC on the whole set E on account of Theorem 3. This completes the proof.

A function will be called *seminormally continuous*, or SNC for short, on a linear set E, if it is normally continuous on every closed null set contained in E. *Such a function is necessarily continuous on the set E*, as we can easily verify by the same argument as in the above proof.

The following properties of SNC functions are obvious. (i) *The seminormal continuity of a function on a set is hereditary with respect to this set.* (ii) *Every linear combination of two functions which are* SNC *on a set E, is itself* SNC *on E.* (iii) *A function which is* SNC *on a set E, necessarily maps every closed null set contained in E onto a null set* (see Theorem 2).

As Theorem 31 clearly implies, a function which is NC on every null subset of a set E, is necessarily NC on the whole set E. In contrast with this fact, we do not know if the following statement is true. From our viewpoint, its falseness is more favourable than its truth.

ASSERTION A. *Every function which is seminormally continuous on a closed set, is normally continuous on this set.*

The converse of this assertion is evidently true.

We shall say that a function is *generalized seminormally continuous*, or GSNC for short, on a set E, if the function is continuous on E and if E is expressible as the union of a sequence of closed sets on each of which the function is seminormally continuous. When this is the case, the set E is sigma-closed of itself.

The following propositions are obvious. (i) *Every function which is* GNC *on a sigma-closed set is* GSNC *on this set.* (ii) *The* GSNC *property of a function on a sigma-closed set E is hereditary with respect to E.* (iii) *Every linear combination of two functions which* *are* GSNC *on a set E, is itself* GSNC *on E.* (iv) *A function which is* GSNC *on a set E, necessarily maps every closed null set contained in E onto a null set.*

We do not know if the following statement is true. At least, however, it is an evident consequence of Assertion A.

ASSERTION B. *Every function which is generalized seminormally continuous on a closed interval I and approximately derivable almost everywhere on I, is generalized normally continuous on I.*

THEOREM 32. *Every function which is generalized seminormally continuous on a closed interval I and has a nonnegative approximate derivative at almost every point of I, is monotone nondecreasing on I.*

THEOREM 33. *If two functions are generalized seminormally continuous on a closed interval I and approximately equiderivable almost everywhere on I, then the functions differ over I only by an additive constant.*

The former of these two theorems may be established in the same way as for Theorem 5, while the latter follows immediately from the former.

The proof of the following result resembles that of Theorem 7.

THEOREM 34. *Every function which is both* BV *and* GSNC, *on a closed set S, is* AC *on S. Hence, every function which is both* GBV *and* GSNC, *on a closed set, is* GAC *on this set.*

Using the above results, we can now introduce an integration named *seminormal.* The definition and the basic properties of this integration are the same, *mutatis mutandis,* with those of the normal integration. Among others, we have the integration by parts theorem and the second mean value theorem.

Of the normal and the seminormal integration, the latter one plainly includes the former. As we readily see, the latter is strictly wider than the former, if and only if Assertion B is false.

ADDED IN PROOF. Recently the author found out that the essential part of this paper can be generalized further by replacing the normal continuity and normal fluctuancy with outwardly less restrictive properties of functions termed *sparse continuity* and *sparse fluctuancy.* We do not know if the generalization is strict. The details are to appear in the forthcoming number of this Natural Science Report.

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