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On the Existence of Semi-balayaged Measures

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§1. Introduction

Let X be a locally compact Hausdorff space with a countable base and G be a lower semicontinuous function-kernel on X, i. e., a lower semicontinuous function from $X\times X$ into $\mathbf{R}\cup\{+\infty\}$ which is finite-valued outside the digonal set $\varDelta$ of $X\times X$. Further, let F be a closed subset of X and u be a lower semicontinuous function on X. If there exist a probability measure ν and a real number k such that

$$\mathrm{supp}(\nu)\subset F, \qquad G\nu\leq u+k$$

and $G\nu=u+k$ on F with the exception of a negligible set, then ν is said to be a semi-balayaged measure of u onto F.

D. Durier has proved in [1] that, in case G and the adjoint kernel $\check{G}$ of G satisfy the continuity principle, for any non-negligible compact set F and any probability measure μ with compact support there exists a semi-balayaged measure of $G\mu$ onto F if and only G satisfies the semicomplete maximum principle, i. e., if μ, ν are probability measures with $\int G\mu d\mu<+\infty$, h is a real number and $G\mu\leq G\nu+h$ on $\mathrm{supp}(\mu)$, then $G\mu\leq G\nu+h$ on X. Furthermore, he has also proved that each compact set F has an equiliblium measure, i. e., there exists a semi-balayaged measure of the constant function 1 onto F if and only if G satisfies the maximum principle.

On the other hand, in case G is the logarithmic kernel on $\mathbf{R}^2$, N. Ninomiya proved that a closed set F has an equilibrium measure if and only if the set of logarithmic capacities of compact subsets of F is bounded from above.

In this note we shall ask necessary and sufficient conditions for a given lower semicontinuous function u to have a semi-balayaged measure of u onto F. Next, we shall consider the case where there is a probability measure λ with $\lim_{x\to\infty}\check{G}\lambda(x)=-\infty$. In this case we shall obtain more simple conditions which are extentions of the above Ninomiya's result.

Furthermore, under the additional assumption (d) in §5, we shall show that if G satisfies the semi-complete maximum principle, then for any $\mu \in M_0^1$ there exists a semi-balayaged measure of $G\mu$ onto any nonnegligible closed set.

§ 2. Preliminaries

Let X be a locally compact Hausdorff space with a countable base and G be a lower semicontinuous function-kernel on X. For a positive Radon measure μ the potential $G\mu$ is defined by

$$G\mu(x) = \int G(x, y)\, d\mu(y)$$

when it is well-defined. The adjoint kernel G of $\check{G}$ is defined by

$$\check{G}(x, y) = G(y, x).$$

By Fubini's theorem it follows that

$$\int G\mu d\nu = \int \check{G}\nu d\mu$$

for μ, $\nu \in M_0^+$. Here M_0^+ is the set of all positive Radon measures with compact support.

Now we shall consider the following families of measures:

$$\mathcal{E}(G) = \left\{\mu \in M_0^+ ;\ \int G\mu d\mu < +\infty\right\},$$

$$\mathcal{B}(G) = \{\mu \in M_0^+ ;\ G\mu \text{ is locally bounded}\},$$

$$\mathcal{F}(G) = \{\mu \in M_0^+ ;\ G\mu \text{ is finite and continuous}\},$$

$$M_0^1 = \left\{\mu \in M_0^+ ;\ \int d\mu = 1\right\},$$

$$\mathcal{E}^1(G) = \mathcal{E}(G) \cap M_0^1,$$

$$\mathcal{F}^1(G) = \mathcal{F}(G) \cap M_0^1.$$

For a closed set F we also define

$$\mathcal{E}(G; F) = \{\mu \in \mathcal{E}(G) ;\ \operatorname{supp}(\mu) \subset F\},$$

$$\mathcal{F}(G; F) = \{\mu \in \mathcal{F}(G) ;\ \operatorname{supp}(\mu) \subset F\}.$$

A lower semicontinuous function kernel G on X is said to satisfy the continuity principle, if $G\mu$ ($\mu \in M_0^+$) is finite and continuous on X whenever it is finite and continuous on $\operatorname{supp}(\mu)$. Hereafter we shall assume that G and $\check{G}$ satisfy the continuity principle.

A Borel set B is said to be G-negligible (simply negligible) if $\mu(B) = 0$ for all $\mu \in \mathcal{E}(G)$. We note that a Borel set B is G-negligible if and only if it is G-negligible. If a property holds on a subset F

of X except on a G-negligible set, it is said to hold G-n. e. on F (simply n. e. on F).

DEFINITION 1. Let F be a closed set and u be a Borel measurable function from X to $\boldsymbol{R}\cup\{+\infty\}$. A probability measure ν is said to be a semi-balayaged measure of u onto F if there exists $k\in\boldsymbol{R}$ such that

$$\operatorname{supp}(\nu)\subset F,\quad G\nu=u+k \text{ n. e. on } F,\quad G\nu\leq u+k \text{ on } X.$$

Then k is said to be a semi-balayage constant of u onto F.

DEFINITION 2. Let F be a closed set. A Borel measurable function u from X to $\boldsymbol{R}\cup\{+\infty\}$ is said to be (G, F)-semi-supermedian if

$$\nu\in\mathcal{E}^1(G;F),\qquad k\in\mathbf{R},\qquad G\nu\leq u+k \text{ on } \operatorname{supp}(\nu)$$

$\Rightarrow$

$$G\nu\leq u+k \text{ on } X.$$

A (G, X)-semi-supermedian function is simply said to be G-semi-supermedian.

Now we denote by $S(G;F)$ the set of all lower semicontinuous (G, F)-semi-supermedian functions u with $\int u d\lambda<+\infty$ for each $\lambda\in\mathcal{F}(\check{G}, F)$. Using $S(G, F)$, we shall obtain, for a compact set K, a necessary and sufficient condition for a lower semicontinuous function u to have a semi-balayaged measure onto K.

PROPOSITION 1. *Let K be a nonnegligible compact set and u be a lower semicontinuous function with $\int u d\lambda<+\infty$ for each $\lambda\in\mathcal{F}(\check{G};K)$.* Then the following assertions (i), (ii) and (iii) are equivalent:

(i) $u\in S(G;K)$,

(ii) $\sigma\in M_0^1$, $\tau\in\mathcal{E}^1(G;K)$, $k\in\boldsymbol{R}$, $\check{G}\tau\leq\check{G}\sigma+k$ on $\operatorname{supp}(\tau)$

$\Rightarrow$

$$\int u d\tau\leq\int u d\sigma+k,$$

(iii) *There exists a semi-balayaged measure of u onto K.*

PROOF. We can prove this proposition by the same method as in the proof of Proposition III. 2 in [1].

§ 3. Semi-balayaged constants

Hereafter G is a continuous function kernel on X, i. e., a continuous function from $X\times X$ into $\mathbf{R}\cup\{+\infty\}$ in the extended sense and is finite-valued outside the diagonal set.

DEFINITION 3. A kernel G is said to satisfy the semi-complete maximum principle if $G\nu$ is G-semi-supermedian for every $\nu\in M_0^1$.

Hereafter we shall assume that G satisfies the following conditions (c_1), (c_2), (c_3), (c_4) or (c_1), (c_2), (c_3), (c_4'):

(c_1) G and $\check{G}$ satisfy the continuity principle,

(c_2) No nonempty open set is G-negligible,

(c_3) G satisfies the semi-compete maximum principle,

(c_4) G is lower bounded on $X\times X$,

(c_4') For every compact set K there is a compact set F such that

$$G(x.y)\leq 0 \quad (x\in K,\ y\in X\setminus F).$$

Then, we recall that $\check{G}$ also satisfies the semi-complete maximum principle (cf. [1; Theorem III. 3.2]). Furthermore, we have the following proposition of a type of Fubini's theorem.

PROPOSITION 2. *Let μ be a bounded measure and τ be a measure in $\mathcal{B}(\check{G})$. If the potential $\check{G}\tau$ (resp. $G\mu$) is integrable with respect to μ (resp. τ), then $G\mu$ (resp. $\check{G}\tau$) is τ-integrable (resp. μ-integrable) and it follows that*

$$\int\check{G}\tau d\mu=\int G\mu d\tau. \tag{3.1}$$

PROOF. If G satisfies (c_4), we can easily show the proposition by Fubini's thorem. Next, assume that G satisfies (c_4'). Set

$$G^+(x,y)=\max\{G(x.y),0\} \quad\text{and}\quad G^-(x,y)=\max\{-G(x,y),0\}.$$

Then $G(x,y)=G^+(x,y)-G^-(x,y)$. Since $(x,y)\mapsto G^-(x,y)$ is finite and continuous on $X\times X$ and the function $y\mapsto\int G(x,y)d\tau(x)$ is locally bounded, the function $y\mapsto\int G^+(x,y)d\tau(x)$ is also locally bounded. By the assumption (c_4') there exists a compact set K such that $G(x,y)\leq 0$ for every $x\in\mathrm{supp}(\tau)$ and every $y\in X\setminus K$. Hence $\int G^+(x,y)d\tau(x)=0$ for all $y\in X\setminus K$. Since the function $y\mapsto\int G^+(x,y)d\tau(x)$ is upper bounded on K, we have

$$\iint G^+(x,y)d\tau(x)d\mu(y)=\int_K d\mu(y)\int G^+(x,y)d\tau(x)<+\infty. \tag{3.2}$$

If $\check{G}\tau$ (resp. $G\mu$) is μ-integrable (resp. τ-integrable), then, by (3.2)

$$\iint G^{-}(x, y)\, d\tau(x)\, d\mu(y) < +\infty$$

and hence

$$\iint |G(x, y)|\, d\tau(x)\, d\mu(y) < +\infty.$$

By Fubini's theorem we have the conclusion.

In the following three examples both G and $\check{G}$ satisfy the semi-complete maximum principle. Furthermore the constant functions are G-semi-supermedian.

EXAMPLE 1. $X := \{\text{natural numbers}\}$ (with the discrete topoloy).

$$G(x, y) := \begin{cases} b & \text{if } x < y \\ b+1 & \text{if } x \geq y. \end{cases}$$

Here b is a real number. Then G and $\check{G}$ have the property (c_4).

EXAMPLE 2. $X := \boldsymbol{R}$, $G(x, y) := -|x-y|$. G has the property (c_4').

EXAMPLE 3. $X := \boldsymbol{R}^2$, $G(x, y) := \begin{cases} -\log|x-y| & \text{if } x \neq y \\ +\infty & \text{if } x = y. \end{cases}$

G has the property (c_4').

Since under our assumptions $\check{G}$ also satisfies the semi-complete maximum principle, each $\check{G}\sigma$ $(\sigma \in M_0^1)$ is $\check{G}$-semi-supermebian. More generally $\check{G}\sigma$ has also the following property:

PROPOSITION 3. *Suppose that*

$\check{G}\tau \leq \check{G}\sigma + k$ n.e. on $\mathrm{supp}(\tau)$ $(\sigma \in M_0^1,\ \tau \in \mathcal{E}^1(G),\ k \in \boldsymbol{R})$.

Then

$$\check{G}\tau \leq \check{G}\sigma + k \textit{ outside } \mathrm{supp}(\tau).$$

PROOF. Let x be an arbitrary point outside $\mathrm{supp}(\tau)$. Since G satisfies the semi-complete maxlmum principle, by [1, Theorem III. 3.2] there exists $\nu \in M_0^1$ and $h \in \boldsymbol{R}$ such that

$$\mathrm{supp}(\nu) \subset \mathrm{supp}(\tau), \qquad G\nu = G\varepsilon_x + h \text{ n.e. on } \mathrm{supp}(\tau),$$
$$G\nu \leq G\varepsilon_x + h \text{ on } X.$$

Then $\nu \in \mathcal{B}(G)$ and

$$\check{G}\tau(x) = \int G\varepsilon_x\, d\tau = \int (G\nu - h)\, d\tau = \int \check{G}\tau\, d\nu - h \leq \int (\check{G}\sigma + k)\, d\nu - h$$

$$= \int (G\nu - h)\, d\sigma + k \leq \int G\varepsilon_x\, d\sigma + k = \check{G}\sigma(x) + k.$$

DEFINITION 4. We say that G satisfies the maximum principle if the constant functions are G-semi-supermedian.

PROPOSITION 4. *Let F be a nonnegligible closed set and u be a function in $S(G;F)$ locally bounded on F. Suppose that $\check{G}$ satisfies the maximum principle. Then for $k\in \boldsymbol{R}$ the following assertions* (i) *and* (ii) *are equivalent*:

(i) *If* $\tau\in\mathcal{B}(G)$, $\operatorname{supp}(\tau)\subset F$, $b\in \boldsymbol{R}$, $\check{G}\tau\leq b$ on $\operatorname{supp}(\tau)$, *then*

$$\int(u+k)\,d\tau\leq \mathrm{b},$$

(ii) *For each compact subset K of F a semi-balayage constant of u onto K is not smaller that k.*

PROOF. (i)→(ii): Let K be a compact subset of F. Since $u\in S(G;F)$ and u is bounded on K, by Proposition 1 there exists a measure $\nu\in\mathcal{E}^1(G)$ and $t\in R$ such that

(3.3) $\qquad \operatorname{supp}(\nu)\subset K,\quad G\nu=u+t \ n.\ e.$ on $K,\quad G\nu\leq u+t$ on X.

Furthermore, since $1\in S(\check{G};X)$, there exist $\tau\in\mathcal{B}^1(G)$ and $b\in\boldsymbol{R}$ such that

$$\operatorname{supp}(\tau)\subset K,\quad \check{G}\tau=b \text{ n. e. on } K,\quad \check{G}\tau\leq b \text{ on } X.$$

Then we obtain

$$\int(u+t)\,d\tau=\int G\nu\,d\tau=\int\check{G}\tau\,d\nu=b.$$

On the other hand from $\check{G}\tau\leq b$ on $\operatorname{supp}(\tau)$ and (i) it follows that

$$\int(u+k)\,d\tau\leq b.$$

Consequently we obtain $k\leq t$.

(ii)→(i): Suppose that $\tau\in\mathcal{B}(\check{G})$, $\operatorname{supp}(\tau)\subset F$, $b\in\boldsymbol{R}$ and $\check{G}\tau\leq b$ on $\operatorname{supp}(\tau)$. Then for the compact set $K=\operatorname{supp}(\tau)\subset F$ there exist $\nu\in M_0^1$ and $t\in\boldsymbol{R}$ satisfying (3.3). Since $t\geq k$ by the assumption, we obtain

$$\int(u+k)\,d\tau=\int(G\nu-t+k)\,d\tau\leq\int\check{G}\tau\,d\nu=b.$$

This completes the proof.

§ 4. The main theorem

Hereafter we shall fix an exhaustion $\{V_n\}$ of X, i. e., a sequence of relatively compaet open sets such that $\overline{V}_n\subset V_{n+1}$, $\bigcup_{n=1}^{\infty} V_n=X$.

DEFINITION 5. We say that G satisfies the unicity principle if for $\mu, \nu \in \mathcal{E}(G)$ $G\mu = G\nu$ n. e. on X implies $\mu = \nu$.

Now we shall prove our main theorem.

THEOREM 1. *Suppose that G satisfies the maximum principle and the unicity principle. Let F be a nonnegligible closed set and $u \in S(G;F)$ such that $u(x)$ is locally bounded on F. Then there exists a semi-balayaged measure ν of u onto F if and only if there exists $k \in \boldsymbol{R}$ satisfying the following assertions* (a) *and* (b):

(a) $\tau \in \mathcal{B}(G;F)$, $b \in \boldsymbol{R}$, $G\tau \leq b$ *on* $\mathrm{supp}(\tau)$ $\Rightarrow$

$$\int (u+k)\, d\tau \leq b,$$

(b) *For each $\varepsilon > 0$ there exist sequences $\{\sigma_p\}, \{\tau_p\} \subset \mathcal{B}(\check{G})$, $\{k_p\} \subset \boldsymbol{R}$ and a compact subset of K of F such that*

$$\mathrm{supp}(\tau_p) \subset K, \qquad \int d\tau_p \geq \int d\tau_p,$$

$$\check{G}\sigma_p - G\tau_p + k_p \leq \check{G}\sigma_{p+1} - \check{G}\tau_{p+1} + k_{p+1} \text{ n. e. on } F,$$

$$\lim_{p\to\infty} (\check{G}\sigma_p - \check{G}\tau_p + k_p) \geq 1 \text{ n. e. on } CK \cap F, \tag{4.1}$$

$$\lim_{p\to\infty} \left\{ \int (u+k)\, d\sigma_p - \int (u+k)\, d\tau_p + k \right\} < \varepsilon.$$

PROOF. First, suppose that there exists a semi-balayaged measure ν of u onto F. Let k be a semi-balayage constant of u onto F. To show (a), suppose that

$$\check{G}\tau \leq b \text{ on } \mathrm{supp}(\tau) \quad (\tau \in \mathcal{B}(\check{G};F),\ b \in \boldsymbol{R}).$$

Then, since $\check{G}$ satisfies the maximum principle, $\check{G}\tau \leq b$ on X and

$$\int (u+k)\, d\tau = \int G\nu\, d\tau = \int \check{G}\tau\, d\nu \leq b.$$

Next, to show (b), let ν_n be the restriction of ν to the set $\{x \in F; u(x) \leq u\} \cap \overline{V}_n$. Then $\nu_n \in \mathcal{E}(G)$ and hence $\nu(B) = 0$ for every negligible set B. Since $\check{G}$ satisfies the semi-complete maximum principle and the unicity principle, by [1, Remark of Theorem III. 6] there exist, for each n and $p \geq n+2$, sequences $\{\sigma_{np}\}, \{\tau_{np}\} \subset \mathcal{E}(G)$ and $\{k_{np}\} \subset \mathbf{R}$ such that

$$\mathrm{supp}(\sigma_{np}) \subset F \cap CV_{n+1} \cap \overline{V}_p,\ \mathrm{supp}(\tau_{np}) \subset F \cap \overline{V}_n,\ \int d\sigma_{np} = \int d\tau_{np},$$

$$\check{G}\sigma_{np} - \check{G}\tau_{np} + k_{np} = 1 \text{ n. e. on } F \cap CV_{n+1} \cap V_p,$$

$$0 \leq \check{G}\sigma_{np} - \check{G}\tau_{np} + k_{np} \leq 1 \text{ on } X.$$

Since $\mathrm{supp}(\sigma_{np})\cap\mathrm{supp}(\tau_{np})=\phi$, we note that $\sigma_{np}\in\mathcal{B}(G)$ and $\tau_{np}\in\mathcal{B}(\check{G})$. Erom Proposition 3 it follows that

$$\check{G}\sigma_{np}-\check{G}\tau_{np}+k_{np}\leq\check{G}\sigma_{n,p+1}-\check{G}\tau_{n,p+1}+k_{n,p+1}\text{ n, n. e. on } X.$$

Moreover

$$\lim_{p\to\infty}(\check{G}\sigma_{np}-\check{G}\tau_{np}+k_{np})=1\text{ n. e. on } F\cap CV_{n+1},$$

$$\lim_{p\to\infty}(\check{G}\sigma_{np}-\check{G}\tau_{np}+k_{np})=0\text{ n. e. on } F\cap\bar{V}_n.$$

Consequently

$$\lim_{n\to\infty}\lim_{p\to\infty}(\check{G}\sigma_{np}-\check{G}\tau_{np}+k_{np})=0\text{ n. e. on } F.$$

Using $\int d\nu=1$, Lebesgue's theorem and Proposition 2, we obtain

$$0=\int\lim_{n\to\infty}\lim_{p\to\infty}(\check{G}\sigma_{np}-\check{G}\tau_{np}+k_{np})\,d\nu$$

$$=\lim_{n\to\infty}\lim_{p\to\infty}\left(\int G\nu\,d\sigma_{np}-\int G\nu\,d\tau_{np}+k_{np}\right)$$

$$=\lim_{n\to\infty}\lim_{p\to\infty}\left\{\int(u+k)\,d\sigma_{np}-\int(u+k)d\tau_{np}+k_{np}\right\}.$$

Thus, for each $\varepsilon>0$, we can find a natural number m such that

$$\lim_{p\to\infty}\left\{\int(u+k)\,d\sigma_{mp}-\int(u+k)\,d\tau_{mp}+k_{mp}\right\}<\varepsilon.$$

If we set

$$K=F\cap\bar{V}_m,\quad \sigma_p=\sigma_{mp},\quad \tau_p=\tau_{mp},\quad k_p=k_{mp},$$

then $\{\sigma_p\}$, $\{\tau_p\}$ and $\{k_p\}$ satisfy (4.1).

Conversely, suppose that the assertions (a) and (b) are satisfied. Set $F_n=\{x\in F;u(x)\leq n\}\cap\bar{V}_n$. Since $u\in S(G;F)$, there exist $\mu_n\in M_0^1$ and $h_n\in\mathbf{R}$ such that

$$\mathrm{supp}(\mu_n)\subset F_n,\quad G\mu_n=u+h_n\text{ n. e. on } F_n,\quad G\mu_n\leq u+h_n\text{ on } X.$$

We note that $\mu_n\in\mathcal{E}(G)$. Since $\mu_n\in M_0^1$, we can assume that $\{\mu_n\}$ converges vaguely to a positive measure ν, by replacing a subsequence of $\{\mu_n\}$, if necessary. Then $\int d\nu\leq1$ and $\mathrm{supp}(\nu)\subset F$. For each $\tau>0$, by (b) there exist $\{\sigma_p\}$, $\{\tau_p\}\subset\mathcal{B}(\check{G})$, $\{k_p\}\subset\mathbf{R}$ and a natural number m satisfying (4.1) with $K\subset\bar{V}_m\cap F$. Then, for $n\geq m$,

$$\int_{CV_{m+1}\cap F}d\mu_n\leq\lim_{p\to\infty}\int(\check{G}\sigma_p-\check{G}\tau_p+k_p)\,d\mu_n$$

$$=\lim_{p\to\infty}\left(\int G\mu_n\,d\sigma_p-\int G\mu_n\,d\tau_p+k_p\right)$$

$$\leq \liminf_{p\to\infty}\left\{\int(u+h_n)\,d\sigma_p-\int(u+h_n)\,d\tau_p+k_p\right\}<\varepsilon.$$

Furthermore, choose a continuous function f on X with compact support such that

$$f=1 \text{ on } V_{m+1}. \qquad 0\leq f\leq 1 \text{ on } X.$$

Then

$$1=\int d\mu_n=\int f d\mu_n+\int(1-f)\,d\mu_n\leq\int f d\mu_n+\int_{CV_{m+1}\cap F}d\mu_n\leq\int f d\mu_n+\varepsilon.$$

As $\varepsilon\to 0$, we have $\int d\nu\geq 1$. Thus it follows that $\int d\nu=1$.

Moreover, let τ be a measure in $\mathscr{F}(\check{G};F)$ and choose $b\in\boldsymbol{R}$ satisfying $\check{G}\tau\leq b$ on $\operatorname{supp}(\tau)$. We shall show that $\check{G}\tau$ is ν-integrable. Since $\check{G}$ satisfies the maximum principle, $\check{G}\tau\leq b$ on X. Consequently, using that $h_n\geq k$ and $u\in S(G;F)$, we obtain

$$0\leq\int(b-\check{G}\tau)\,d\nu\leq\liminf_{n\to\infty}\int(b-\check{G}\tau)\,d\mu_n=\liminf_{n\to\infty}\int(-G\mu_n)\,d\tau+b$$

$$=\liminf_{n\to\infty}\int(-u-h_n)\,d\tau+b\leq\int(-u-k)\,d\tau+b<+\infty.$$

Thus we see that $\check{G}\tau$ is ν-integrable. Further, let $\sigma\in M_0^1$ and $\tau\in\mathscr{F}^1(\check{G};F)$ and choose $c\in\boldsymbol{R}$ satisfying $\check{G}\sigma-\check{G}\tau\geq c$ on $\operatorname{supp}(\tau)\subset F$. Since $\check{G}$ satisfies the semi-complete maximum principle, $\check{G}\sigma-\check{G}\tau\geq c$ on X. Consequently

$$\int(\check{G}\sigma-\check{G}\tau-c)\,d\nu\leq\liminf_{n\to\infty}\int(\check{G}\sigma-\check{G}\tau-c)\,d\mu_n$$

$$\leq\liminf_{n\to\infty}\left\{\int(u+h_n)\,d\sigma-\int(u+h_n)\,d\tau-c\right.$$

$$=\int u\,d\sigma-\int u\,d\tau+c.$$

Hence

$$\int(\check{G}\sigma-\check{G}\tau)\,d\nu\leq\int u\,d\sigma-\int u\,d\tau. \tag{4.2}$$

Especially, if σ is also a measure in $\mathscr{F}^1(G;F)$,

$$\int(\check{G}\sigma-\check{G}\tau)\,d\nu=\int u\,d\sigma-\int u\,d\tau. \tag{4.3}$$

Since F is nonnegligible and $\check{G}$ satisfies the continuity principle, we can choose and fix $\lambda\in\mathscr{F}^1(G;F)$. Set

$$h=\int\check{G}\lambda\,d\nu-\int u\,d\lambda.$$

Then, from (4.2) and (4.3) it follows that

$$\int \check{G}\sigma d\nu \leq \int (u+h) d\sigma \quad (\sigma \in M_0^1),$$

$$\int \check{G}\tau d\nu = \int (u+h) d\tau \quad (\tau \in \mathcal{F}^1(G; F))$$

and hence

$$G\nu(x) \leq u(x) + h \quad (x \in X),$$

$$G\nu = u + h \text{ n. e. on } F.$$

Therefore ν is a semi-balayaged measure of u onto F.

REMARK 1. The assumption that $\check{G}$ satisfies the unicity principle is used only for constructing $\{\sigma_p\}$, $\{\tau_p\}$ and $\{k_p\}$ satisfying (4.1).

§ 5. Kernels of logarithmic type

In this section we shall consider the semi-balayability in case there exists a potential $\check{G}\lambda$ such that $\lim_{x\to\infty} \check{G}\lambda(x) = -\infty$ and G satisfies the following condition (d):

(d) $\sup_{x \in X} \{G\mu(x) - G\nu(x)\} \geq 0$ for every $\mu, \nu \in M_0^1$.

We shall prepare the following propositions.

PROPOSITION 5. *Suppose that G satisfies the condition* (d). *Then, for every $\mu \in M_0^1$ and every nonnegligible compact set K, a semi-balayage constant of $G\mu$ onto K is nonnegative and $\check{G}$ satisfies the maximum principle.*

PROOF. Since $G\mu$ is G-supermedian and K is a nonnegligible compact set, by Proposition 1 there exist $\nu \in M_0^1$ and $k \in \boldsymbol{R}$ such that

(5.1) $\quad \mathrm{supp}(\nu) \subset K, \quad G\nu = G\mu + k$ n.e. on $K, \quad G\nu \leq G\mu + k$ on X.

Then, from the assumption (d), it follows that $k \geq 0$. Next, to see that $\check{G}$ satisfies the maximum principle, suppose that $\check{G}\tau \leq b$ on $\mathrm{supp}(\tau)$ for $\tau \in \mathcal{E}(G)$ and $b \in \mathbf{R}$. Let x be an arbitrary point outside $\mathrm{supp}(\tau)$ and choose $\nu \in M_0^1$ and $k \in \boldsymbol{R}$ satisfying (5.1) especially for $\mu = \varepsilon_x$ and $K = \mathrm{supp}(\tau)$, we obtain

$$\check{G}\tau(x) = \int G\varepsilon_x d\tau = \int \check{G}\tau d\nu - k \int d\tau \leq \int \check{G}\tau d\nu \leq b.$$

Hence $\check{G}\tau \leq b$ on X.

REMARK 2. All kernels of Examples 1, 2 and 3 satisfy (d).

PROPOSITION 6. *Let F be a nonnegligible closed set and $u\in S(G;F)$ such that $u(x)<+\infty$ for each $x\in F$. Furthermore, suppose that there exists $k\in\boldsymbol{R}$ satisfying* (a) *in Theorem* 1 *and that there is $\lambda\in\mathscr{T}(\check{G};F)$ with $\lim_{x\to\infty}\check{G}\lambda(x)=-\infty$. Then there exist, for each $\varepsilon>0$, $\tau\in\mathscr{T}(G;F)$, $h\in\boldsymbol{R}$ and a compact set K such that*

$$h-\check{G}\tau\geq 1 \text{ on } CK\cap F,\quad 0\leq h-\int(u+k)\,d\tau<\varepsilon.$$

PROOF. Choose $t\in\boldsymbol{R}$ satisfying

$$t-\check{G}\lambda\geq 0 \quad\text{on}\quad \mathrm{supp}(\lambda). \tag{5.2}$$

Then from Proposition 4 it follows that $t-\int(u+k)\,d\lambda\geq 0$. Moreover by $\lim_{x\to\infty}\check{G}\lambda(x)=-\infty$, there exists, for each $\varepsilon>0$, a compact set K such that

$$t-G\lambda\geq 1/\varepsilon \quad\text{on}\quad CK.$$

Set $h=\varepsilon t$ and $\tau=\varepsilon\lambda$. Then, by (5.2) and (a)

$$o\leq h-\int(u+k)\,d\tau=\varepsilon\Big(t-\int(u+k)\,d\lambda\Big).$$

Thus we have the conclusion.

THEOREM 2. *Let F be a nonnegligible closed set. Suppose that G satisfies* (d) *and there exists $\lambda\in\mathscr{T}(\check{G};F)$ with $\lim_{x\to\infty}\check{G}\lambda(x)=-\infty$. If μ is a measure in M_0^1 such that $G\mu$ is locally bounded on F, then $G\mu$ has a semi-balayaged measure onto F.*

PROOF. Since G satisfies the semi-complete maximum principle, $G\mu$ is G-semi-super median and locally bounded on F. Further, by Propositions 4 and 5, for every nonnegligible closed set F, $k=0$ and $u=G\mu$ have the property (a) in Theorem 1. On the other hand Proposition 6 implies that $k=0$ has the property (b) in Theorem 1. Therefore by Theorem 1 $G\mu$ has a semi-balayaged measure onto F.

REMARK 3. Examples 2 and 3 satisfy the assumptions of Theorem 2.

By Proposition 6 and Theorem 1 we obtain

THEOREM 3. *Let F be a nonnegligible closed set and $u\in S(G;X)$ such that $u(x)<+\infty$ for each $x\in F$. Further, suppose that $\check{G}$ satisfies the maximum principle and that there is $\lambda\in\mathscr{T}(\check{G};F)$ with $\lim_{x\to\infty}\check{G}\lambda(x)=-\infty$. Then the following assertions* (i) *and* (ii) *are equivalent:*

(i) *There exists a semi-balayaged measure* $\nu \in M^1$ *of* u *onto* F,

(ii) *There exists* $k \in \boldsymbol{R}$ *satisfying* (a) *in Theorem* 1.

Let F be a closed set and if there exists a semi-balayaged measure ν of 1 onto F, ν is said to be an equilibrium measure of F. Immediately by Theorem 3 and Proposition 4 we obtain the following corollary which is an extension of [2; Theorem 5].

COROLLARY. *Let* F *be a nonnegligible closed set and suppose that both* G *and* $\check{G}$ *satisfy the maximum principle and that there is a measure* $\lambda \in \mathcal{F}(\check{G};F)$ *with* $\lim_{x\to\infty} \check{G}\lambda(x) = -\infty$. *Then the following three assertions are equivalent:*

(i) *F has an equilibrium measure,*

(ii) *There exists* $k \in \boldsymbol{R}$ *such that*

$$\tau \in \mathcal{B}(\check{G};F),\ b \in \boldsymbol{R},\ \check{G}\tau \leq b \text{ on } \mathrm{supp}(\tau) \quad \textit{imply} \quad \int k\, d\tau \leq b,$$

(iii) *The set of all semi-balayaged constants of* 1 *onto compact subsets of* F *is lower bounded.*

References

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