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On the Peripheral Point Spectrum of a Simplex Homomorphism.

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§1. Introduction.

It is well known that the peripheral spectrum of a positive operator in a finite dimensional space is cyclic (Perron-Frobenius Theorem) [2]. As an extension of this theorem, it has been studied that the spectrum and the point spectrum of a lattice homomorphism of a Banach lattice is cyclic [5, 10]. As for a simplex homomorphism, whose second adjoint operator is a lattice homomorphism, it has been studied by F. Jellett [3], A. W. Wickstead [9] et al. We are interested in investigating the peripheral point spectrum $P_\sigma(T)\cap\Gamma$ of a simplex homomorphism T. In [8], we obtained the result that it is cyclic under some conditions and we gave an example which shows that it is not necessarily cyclic without any conditions.

In this paper, we shall investigate the property of a peripheral point spectrum of a simplex homomorphism. In §2, we give notations and terminologies. §3 is devoted to the study of some properties of a simplex space E and of a simplex homomorphism T. If $Tf=\alpha f$ holds for some $f\in E$ and $|\alpha|=1$, put $f(x)=|f(x)|\exp(i\theta(x))$ for $x\in\overline{\partial_e X}$ and $g_j(x)=|f(x)|\exp(ij\theta(x))$ for $j\in N$. Then we have $g_j\in F$ ($\simeq$minimal sublattice of E'' containing E) and $\tilde{T}g_j=\alpha^j g_j$ ($\tilde{T}$ is the induced operator by T in F). If g_j belongs to E, then α^j belongs to $P_\sigma(T)$. In general, g_j does not belong to E. So in §3, we examine the condition that g_j belongs to E (Theorem 1). Even if g_j does not belong to E, α^j may belong to a point spectrum of T. Under the condition (C 4) denoted at §5, it can be reduced to the case of a finite dimensional space. §4 is devoted to the study of an operator in a finite dimensional space. In §5, by using the result of §4, we investigate the condition that α^j belongs to $P_\sigma(T)$ and the condition that $P_\sigma(T)\cap\Gamma$ is cyclic (Theorem 2, 3 and 4).

§ 2. Notations and Terminologies.

Let E be a simplex space and X be the set $\{x\in E';x\geqq 0, \|x\|\leqq 1\}$ endowed with the weak*-topology. Then X is a simplex and for any $x\in X$, there exists a unique maximal probability measure μ_x on X with resultant x [1, § 28]. Then E can be expressed as a space of functions on $\overline{\partial_e X}$ (the weak*-closure of the set $\partial_e X$ of all extreme points of X), namely

$$\left\{f\in C(\overline{\partial_e X});f(x)=\int f d\mu_x \text{ for all } x\in\overline{\partial_e X} \text{ and } f(0)=0\right\}.$$

We assume the condition

(C1) $\inf\{\|x\|;x\in\overline{\partial_e X}\setminus\{0\}\}=a>0.$

For any subset K of $\overline{\partial_e X}$, put $\tilde{\theta}(K):=[a,\infty)K\cap\overline{\partial_e X}$ and for any relatively open subset U of $\overline{\partial_e X}$, put $\theta(U):=\overline{\partial_e X}\setminus\tilde{\theta}(\overline{\partial_e X}\setminus U)$. Then we can consider θ-topology generated by $\{\theta(U);U\in\mathfrak{U}\}$, where $\mathfrak{U}$ is the set of all relatively open subsets of $\overline{\partial_e X}$ [7, Lemma 4].

Let T be a simplex homomorphism of E [7, 8]. Then there is a function γ on $\overline{\partial_e X}$ and a mapping $k:\overline{\partial_e X}\to\overline{\partial_e X}$ satisfying;

(i) $0\leqq\gamma(x)\leqq\|T\|$ for all $x\in\overline{\partial_e X}$ and $\gamma(x)=0$ if and only if $T'x=0$.

(ii) $k(\partial_e X)\subset\partial_e X$, k is θ-continuous on $\{x\in\overline{\partial_e X};\gamma(x)\neq 0\}$ and $k(x_\alpha)=k(x'_\alpha)$ if $x_\alpha=c_\alpha x'_\alpha$ for some $c_\alpha\in \boldsymbol{R}_+$.

(iii) $Tf(x)=\gamma(x)\cdot f\circ k(x)$ for any $f\in E$ and any $x\in\overline{\partial_e X}$.

Put $\gamma(n,x):=\prod_{j=0}^{n-1}\gamma(k^j(x))$ for any $x\in\overline{\partial_e X}$ and any $n\in \boldsymbol{N}$. Then we have $T^n f(x)=\gamma(n,x)\cdot f\circ k^n(x)$ for any $f\in E$. Put $N_0:=\{x\in\overline{\partial_e X};\lim_{n\to\infty}\|T'^n x\|=0\}$ and $N_\infty:=\{x\in\overline{\partial_e X};k^j(x)\neq k^m(x) \text{ if } j\neq m\}$. We assume the conditions

(C2) $\sup_n\|T^n\|<\infty$ and

(C3) $\sigma(T)\cap\Gamma\neq\Gamma$,

where Γ is the unit circle $\{\lambda\in \boldsymbol{C};|\lambda|=1\}$. Then we have $N_\infty\subset N_0$ [8. Lemma 3] and for any $x\in\overline{\partial_e X}\setminus N_\infty$, we can put

$r(x):=\min\{r\in \boldsymbol{N};k^r(x)=k^s(x) \text{ for some } s\in \boldsymbol{N} \text{ with } r>s\geqq 0\}$

$s(x):=\min\{s\in \boldsymbol{N}\cup\{0\};k^r(x)=k^s(x) \text{ for some } r\in \boldsymbol{N} \text{ with } r>s\}$

$n(x):=r(x)-s(x)$ and

$p(x):=\min\{p\cdot n(x);p\cdot n(x)\geqq s(x), p\in \boldsymbol{N}\}$

as finite numbers. Define

$$P:\overline{\partial_e X}\setminus N_\infty\to\overline{\partial_e X}\setminus N_\infty$$

by $Px=k^{p(x)}(x)$. Put

$$S_n:=\{x\in\overline{\partial_e X}\setminus\tilde{\theta}(\overline{N_0});n(x)=n\}$$

and

$$P_n:=\{x\in S_n;Px=x\}.$$

By [8, Lemma 4], there exists $M_0>0$ such that

$$\bigcup_{n=1}^{M_0}S_n=\overline{\partial_e X}\setminus\tilde{\theta}(\overline{N_0}).$$

Let M be the least common multiple of the set $\{n(x);x\in\overline{\partial_e X}\setminus N_0\}$ [8, Lemma 4].

Let E_1 be the smallest Banach sublattice of E'' containing E and let F be the space $\{f\in C(\overline{\partial_e X});f(x_\alpha)=c_\alpha\cdot f(x'_\alpha)$ for all $\alpha\in\Delta\}$, where $\{(x_\alpha,x'_\alpha,c_\alpha)\}_{\alpha\in\Delta}$ is a subset of $\overline{\partial_e X}\times\overline{\partial_e X}\times(0,1]$ consisting of all the triple $(x_\alpha,x'_\alpha,c_\alpha)$ such that $f(x_\alpha)=c_\alpha\cdot f(x'_\alpha)$ holds for any $f\in E$. Then there is a lattice isomorphism ϕ of E_1 onto F [6, Theorem 1] and $\tilde{T}:=\phi T''\phi^{-1}$ is a lattice homomorphism of F [8, Lemma 1]. Furthermore $\int g\,d\mu_x=\phi^{-1}g(x)$ holds for any $g\in F$ and any $x\in\overline{\partial_e X}$.

Let F_0 be the space $\{f\in F;f(x)=0$ for all $x\in N_0\}$. Define $\tilde{P}\in\mathfrak{L}(F_0)$ by

$$\tilde{P}g(x)=\begin{cases}\gamma(p(x),x)\cdot g(Px) & x\in\overline{\partial_e X}\setminus N_0\\ 0 & x\in N_0\end{cases}$$

for $g\in F_0$. Then $\tilde{P}g\in F_0$ [8, Lemma 6] and $\tilde{P}\tilde{T}g=\tilde{T}\tilde{P}g$ holds for any $g\in F_0$.

For $p,q\in N$, the notation (p,q) means the greatest common divisor of p and q.

§ 3. Some properties of a simplex space and of a simplex homomorphism.

Since $\tilde{T}$ is a lattice homomorphism, we have the following

PROPOSITION 1. *Let E be a simplex space and T be a simplex homomorphism. Put $\alpha=\exp\left(\frac{q}{p}2\pi i\right)$ for some $p,q\in N$ with $(p,q)=1$. Suppose $Tf=\alpha f$ for some $f\in E$. Put*

$$g_j(x)=\begin{cases}\dfrac{f(x)^j}{|f(x)|^{j-1}} & \text{if } f(x)\neq 0\\ 0 & \text{if } f(x)=0\end{cases}$$

for $x\in\overline{\partial_e X}$ and $1\leqq j\leqq p$.

Then $g_j \in F$ *and* $\tilde{T}g_j = \alpha^j g_j$.

PROOF. g_j is continuous on $\{x \in \overline{\partial_e X}; f(x) \neq 0\}$. If $f(x)=0$, then $\lim_{y\to x} |g_j(y)-0| = \lim_{y\to x} |f(y)| = 0$ holds. So g_j is continuous on $\overline{\partial_e X}$. If $x_\alpha = c_\alpha x'_\alpha$, then $g_j(x_\alpha) = c_\alpha \cdot g_j(x'_\alpha)$ holds by definition. So $g_j \in F$.

Next, we have

$$\tilde{T}g_j(x) = \gamma(x)\cdot g_j \circ k(x) = \gamma(x)\frac{(f\circ k(x))^j}{|f\circ k(x)|^{j-1}} = \frac{(Tf(x))^j}{|Tf(x)|^{j-1}} = \alpha^j g_j(x). \quad //$$

In general, g_j does not belong to E as shown in a counter-example in [8].

Hereafter we shall investigate under what condition g_j belongs to E, or whether there is any eigenvector in E pertaining to α^j when g_j does not belong to E.

The next theorem shows the condition that g_j belongs to E.

THEORM 1. *Let* E *be a simplex space satisfying the condition* (C1) *and let* T *be a simplex homomorphism of* E *satisfying the conditions* (C2) *and* (C3). *Put* $\alpha = \exp\left(\frac{q}{p} 2\pi i\right)$ *for some* $p, q \in N$ *with* $(p, q)=1$. *Let* $Tf = \alpha f$ *for some* $f \in E$. *For* $j \in N$, *put* $s_j = \frac{p}{(j, p)}$. *If* $\{\overline{\partial_e X} \setminus \tilde{\theta}(\partial_e X)\} \cap \bigcup_{t\geq 1} S_{s_j t} = \phi$, *then* g_j *defined at Proposition* 1 *belongs to* E *and* $\alpha^j \in P_\sigma(T)$.

Before proving the above theorem, we shall investigate some properties of a simplex space.

From now on let E be a simplex space satisfying the condition (C1) and let T be a simplex homomorphism of E satisfying the conditions (C2) and (C3).

Now we have

LEMMA 1. $\bigcup_{t\geq 1} S_{pt}$ *is* θ*-open for any* $p \in N$.

PROOF. If $p=1$, then $\bigcup_{t\geq 1} S_{pt} = \overline{\partial_e X} \setminus (\tilde{\theta}(\overline{N_0})$ is θ-open. For $x \in S_{pt}$ $(p \geq 2)$, we have $k^j(x) \neq k^{j'}(x)$ for $0 \leq j < j' \leq r(x)-1$ $(=s(x)+pt-1)$. It may happen that $k^j(x)$ belongs to $\tilde{\theta}(\overline{N_0})$ for some $j \in N$. Put $K_\gamma = \{x \in \overline{\partial_e X}; \gamma(x) \neq 0\}$. Then K_γ is θ-open and $k^j(x) \in K_\gamma$ holds for any $n \in N$. So there are θ-open subsets $\{U_j\}_{0\leq j\leq r(x)-1}$ of K_γ, satisfying

$$U_0 \subset \overline{\partial_e X} \setminus \tilde{\theta}(\overline{N_0}),\ k^j(x) \in U_j \text{ and } U_j \cap U_{j'} = \phi \ (j \neq j').$$

Since $x \in S_{pt} \subset \overline{\partial_e X} \setminus \tilde{\theta}(\overline{N_0})$, there exists $n \in N$ such that $\overline{\lim_{m\to\infty}} \|T'^m x\| > \frac{1}{n}$.

By [8, Corollary to Lemma 3], there exists $m_0 \in N$ such that $\sup\{r(y); y \in B_{n,m_0}\} < \infty$, where $B_{n,m_0} = \left\{y \in \overline{\partial_e X}; \sup_{m \geqq m_0} \|T'^m y\| > \frac{1}{n}\right\}$. Put $c_n = \sup\{r(y); y \in B_{n,m_0}\}$ and $s_0 = \left[\frac{c_n}{pt}\right] + 1$. Put $V_0 := \bigcap_{s=0}^{s_0} \bigcap_{j=0}^{pt-1} k^{-pts-j-s(x)}(U_{j+s(x)}) \cap U_0$. Then V_0 is a non-empty θ-open set, since k is θ-continuous on K_r. For $y \in V_0 \cap B_{n,m_0}$ we have $k^{pts+j+s(x)}(y) \in U_{j+s(x)}$ for $0 \leqq j \leqq pt-1$, $s \in N$ such that $pts+j+s(x) \leqq c_n$, which implies $y \in \bigcup_{s \geqq 1} S_{pts} \subset \bigcup_{t \geqq 1} S_{pt}$ and $V_0 \cap B_{n,m_0} \subset \bigcup_{t \geqq 1} S_{pt}$. Since B_{n,m_0} is an open set containing x, $\bigcup_{t \geqq 1} S_{pt}$ is open. Since $x = cx'$ and $x \in S_{pt}$ implies $x' \in S_{pt}$, $\bigcup_{t \geqq 1} S_{pt}$ is θ-open. //

LEMMA 2. *P_n is a Borel subset of $\overline{\partial_e X}$ for any $n \in N$.*

PROOF. It holds that

$$P_n = \left\{\ker(T'^n - I) \setminus \bigcup_{j=1}^{n-1} \ker(T'^j - I)\right\} \cap \{\overline{\partial_e X} \setminus \tilde{\theta}(\overline{N_0})\}.$$

So P_n is a Borel set. //

LEMMA 3. *$\bigcup_{n \geqq 1} P_n$ is a relatively closed subset of $\overline{\partial_e X} \setminus \tilde{\theta}(\overline{N_0})$.*

PROOF. It holds that $\bigcup_{n \geqq 1} P_n = \bigcup_{n=1}^{M} \ker(T'^n - I) \cap (\overline{\partial_e X} \setminus \tilde{\theta}(\overline{N_0}))$. So $\bigcup_{n \geqq 1} P_n$ is a relatively closed subset of $\overline{\partial_e X} \setminus \tilde{\theta}(\overline{N_0})$. //

LEMMA 4. *Let K be a closed subset of $\{x \in \overline{\partial_e X}; \gamma(x) \neq 0\}$. Then $\tilde{\theta}(k(K))$ is θ-closed.*

PROOF. By the property of k, $k(K) = k(\tilde{\theta}(K))$ holds. Since $\tilde{\theta}(K) = [a, \infty)K \cap \overline{\partial_e X}$ is θ-compact and k is θ-continuos on $\{x \in \overline{\partial_e X}; \gamma(x) \neq 0\}$, $k(\tilde{\theta}(K))$ is θ-compact. It is enough to show that for any $p \notin \tilde{\theta}(k(\tilde{\theta}(K)))$, there exists a θ-open set $V(p)$ such that $V(p) \cap \tilde{\theta}(k(\tilde{\theta}(K))) = \phi$. For any $q \in \tilde{\theta}(k(\tilde{\theta}(K)))$, there exist θ-open sets U_q and V_q such that $\tilde{\theta}(q) \subset U_q$, $\tilde{\theta}(p) \subset V_q$ and $U_q \cap V_q = \phi$, since $\tilde{\theta}(p) \cap \tilde{\theta}(q) = \phi$. Since $k(\tilde{\theta}(K))$ is θ-compact, there exist finite θ-open sets $U_{q_1}, \cdots, U_{q_n}$ such that $\bigcup_{j=1}^{n} U_{q_j} \supset k(\tilde{\theta}(K))$. Since U_{q_j} is θ-open, we have $\tilde{\theta}(k(\tilde{\theta}(K))) \subset \bigcup_{j=1}^{n} U_{q_j}$. Put $V(p) = \bigcap_{j=1}^{n} V_{q_j}$. Then $V(p)$ is a θ-open set containing $\tilde{\theta}(p)$ and $V(p) \cap \tilde{\theta}(k(\tilde{\theta}(K))) = \phi$. //

By [7, Proposition], $\overline{\partial_e X}$ equipped with θ-topology is T_4 space. So we have the following

PROPOSITION 2. *Let K be a closed subset of $\overline{\partial_e X}$ and $x_0 \in \partial_e X \setminus \tilde{\theta}(K)$. Then there exists $f \in F$ such that* $0 \leqq f(x) \leqq 1$ for any $x \in \overline{\partial_e X}$, $f(x_0)=1$ and $f \mid \tilde{\theta}(K)=0$, where $f \mid \tilde{\theta}(K)$ is the restriction of f to $\tilde{\theta}(K)$.

PROOF. Since $\tilde{\theta}(x_0)$ and $\tilde{\theta}(K)$ are disjoint and θ-topology is T_4, there exists $g \in C(\overline{\partial_e X})$ satisfying $0 \leqq g \leqq 1$, $g \mid \tilde{\theta}(x_0)=1$ and $g \mid \tilde{\theta}(K)=0$. Since $\{0, x_0\}$ is a closed subset of $\partial_e X$, there exists $h \in E$ satisfying $h(x_0)=1$ and $0 \leqq h(x) \leqq 1$ for $x \in \overline{\partial_e X}$ by [1, 28.6 (viii)]. Put $f(x)= h(x) \cdot g(x)$ for any $x \in \overline{\partial_e X}$. Then $f \in F$ is a desired one. //

COROLLARY 1. *Let K be a closed subset of $\overline{\partial_e X}$. For any $y_1, \cdots, y_n \in \partial_e X \setminus \tilde{\theta}(K)$ and any $c_1, \cdots, c_n \in \boldsymbol{R}$, there exists $f \in F$ satisfying*

$$f(y_j)=c_j \ (1 \leqq j \leqq n) \text{ and } f \mid \tilde{\theta}(K)=0.$$

PROOF. Put $K_j=K \cup \{y_1, \cdots, y_n\} \setminus \{y_j\}$. Then by Proposition 2, we have $f_j \in F$ satisfying $f_j(y_j)=c_j$ and $f_j \mid \tilde{\theta}(K_j)=0$. Put $f=\sum_{j=1}^{n} f_j$. Then f is a desired one. //

COROLLARY 2. *Let K_1 and K_2 be disjoint θ-closed subsets of $\overline{\partial_e X}$. Then for any $g \in F$, there exists $f \in F$ such that $f \mid K_1=g \mid K_1$, $f \mid K_2=0$ and $\|f\| \leqq \|g\|$.*

As for the relation between μ_x and $\mu_{k^j(x)}$, we have

PROPOSITION 3. *Let B be a θ-Borel subset of $\overline{\partial_e X} \setminus \overline{N_0}$. Then we have*

$$\gamma(j, x) \int_B g \, d\mu_{k^j(x)} = \int_{k^{-j}B} \tilde{T}^j g \, d\mu_x$$

for any $g \in F$, any $x \in \overline{\partial_e X}$ and any $j \in \boldsymbol{N}$.

PROOF. At first we consider the case of $j=1$. Since k is θ-continuous, $k^{-1}B$ is a θ-Borel set. Since μ_x and $\mu_{k(x)}$ are regular measures on $\overline{\partial_e X}$, there exist weak*-closed subsets K_1, K_2 and open subsets U_1, U_2 of $\overline{\partial_e X}$ satisfying

$$\overline{\partial_e X} \setminus \overline{N_0} \supset U_1 \supset B \supset K_1, \qquad \mu_{k(x)}(U_1 \setminus K_1) < \varepsilon,$$

$$\overline{\partial_e X} \setminus \overline{N_0} \supset U_2 \supset k^{-1}B \supset K_2 \text{ and } \mu_x(U_2 \setminus K_2) < \varepsilon.$$

Then $U_1 \supset \theta(U_1) \supset B \supset \tilde{\theta}(K_1) \supset K_1$ and $\theta(U_2) \supset k^{-1}B \supset \tilde{\theta}(K_2)$.
Therefore $\mu_{k(x)}(\theta(U_1) \setminus \tilde{\theta}(K_1)) < \varepsilon$ and $\mu_x(\theta(U_2) \setminus \tilde{\theta}(K_2)) < \varepsilon$.
Put $K=\tilde{\theta}(K_1) \cup \tilde{\theta}(k(\tilde{\theta}(K_2)))$ and $U=\theta(U_1) \cap (\overline{\partial_e X} \setminus \tilde{\theta}(k(\overline{\partial_e X} \setminus \theta(U_2))))$.
Then K is θ-closed by Lemma 4, U is θ-open, $\tilde{\theta}(K_1) \subset K \subset B \subset U \subset \theta(U_1)$

and $\tilde{\theta}(K_2)\subset k^{-1}K\subset k^{-1}B\subset k^{-1}U\subset\theta(U_2)$. Therefore we have $\mu_{k(x)}(U\setminus K)<\varepsilon$ and $\mu_x(k^{-1}U\setminus k^{-1}K)<\varepsilon$. By Corollary 2 to Proposition 2, we have $f\in F$ satisfying $f|K=g|K$, $f|(\overline{\partial_e X}\setminus U)=0$ and $\|f\|\leqq\|g\|$. So we have $\tilde{T}f|k^{-1}K=\tilde{T}g|k^{-1}K$, $\tilde{T}f|(\overline{\partial_e X}\setminus k^{-1}U)=0$ and $\|Tf\|\leqq\|T\|\,\|g\|$. By using the relation

$$\int_{k^{-1}U}\tilde{T}f\,d\mu_x=\int\tilde{T}f\,d\mu_x=T''\phi^{-1}f(x)=\phi^{-1}f(T'x)$$

$$=\gamma(x)\cdot\phi^{-1}f\circ k(x)=\gamma(x)\int f\,d\mu_{k(x)},$$

we have

$$\left|\int_{k^{-1}B}\tilde{T}g\,d\mu_x-\gamma(x)\int_B g\,d\mu_{k(x)}\right|\leqq\left|\int_{k^{-1}U}\tilde{T}f\,d\mu_x-\gamma(x)\int_U f\,d\mu_{k(x)}\right|+\left|\int_{k^{-1}U\setminus k^{-1}B}\tilde{T}f\,d\mu_x\right|$$

$$+\left|\int_{k^{-1}B\setminus k^{-1}K}\tilde{T}(f-g)\,d\mu_x\right|+\gamma(x)\left|\int_{U\setminus B}f\,d\mu_{k(x)}\right|+\gamma(x)\left|\int_{B\setminus K}(f-g)\,d\mu_{k(x)}\right|$$

$$\leqq 6\|T\|\,\|g\|\varepsilon,$$

where $\int$ implies $\int_{\overline{\partial_e X}}$.

Since $\varepsilon>0$ is arbitrary, we have

$$\gamma(x)\int_B g\,d\mu_{k(x)}=\int_{k^{-1}B}\tilde{T}g\,d\mu_x.$$

For $j\geqq 2$, we get the desired relation by iterating the result of the case $j=1$. //

For $f\in F$, it is a problem under what condition f belongs to E. The following proposition shows one condition.

PROPOSION 4. *Let $f\in F$ satisfy*

$$f|N_0=0 \quad \textit{and} \quad \int f\,d\mu_x=f(x) \textit{ for } x\in P(\overline{\partial_e X}\setminus(N_0\cup\tilde{\theta}(\partial_e X))).$$

Then there exists $\lim_{j\to\infty}\tilde{T}^{jM}f$ (norm convergence) as an element of E, where M is the number defined at § 2.

PROOF. In the same way as the proof of [8, Lemma 6], we can show that $\{\tilde{T}^{jM}f\}_{j=1}^{\infty}$ is a norm convergent sequence and $f_0(x):=\lim_{j\to\infty}\tilde{T}^{jM}f(x)=\gamma(p(x),x)\cdot f(Px)$ exists.

It is clear that $f\in F$. So all that remains is to show that

$$f_0(x)=\int f_0\,d\mu_x \text{ holds for all } x\in\overline{\partial_e X}\setminus\tilde{\theta}(\partial_e X).$$

For $x\in\overline{\partial_e X}\setminus(N_0\cup\tilde{\theta}(\partial_e X))$, there exists $j_x\in N$ such that $j_xM\geqq p(x)$. We have for $j\geqq j_x$,

$$\int\tilde{T}^{jM}f\,d\mu_x=\gamma(jM,x)\int f\,d\mu_{k^{jM}(x)}=\gamma(p(x),x)\int f\,d\mu_{Px}$$

by Proposition 3 and [8, Lemma 5]. By assumption, we have

$$f_0(x)=\gamma(p(x),x)\cdot f(Px)=\gamma(p(x),x)\int f\,d\mu_{Px}.$$

Since f_0 is the uniform limit of $\tilde{T}^{jM}f$, we have $f_0(x)=\int f_0\,d\mu_x$.

For $x\in N_0$, we have $f_0(x)=0$. For any $\varepsilon>0$, there exists n_0 such that $\|T'^n x\|<\varepsilon$ for $n\geqq n_0$. If $k^n(x)\neq 0$, $\gamma(n,x)=\dfrac{\|T'^n x\|}{\|k^n(x)\|}<\dfrac{\varepsilon}{a}$.
So by using Proposition 3, we have

$$\left|\int\tilde{T}^{jM}f\,d\mu_x\right|=\left|\gamma(jM,x)\int f\,d\mu_{k^{jM}(x)}\right|\leqq\frac{\varepsilon}{a}\|f\|\quad\text{for}\quad jM\geqq n_0.$$

Since f_0 is the uniform limit of $\tilde{T}^{jM}f$, we have $\int f_0\,d\mu_x=0$.

Therefore $f_0(x)=\int f_0\,d\mu_x$ holds for any $x\in\overline{\partial_e X}\setminus\tilde{\theta}(\partial_e X)$. //

For $p,q\in N$ such that $(p,q)=1$, put $\alpha=\exp\left(\frac{q}{p}2\pi i\right)$. Then we have

LEMMA 5. *Snppose $Tf=\alpha f$ holds for some $f\in E$. Then*
(i) *if $(n(x),p)<p$ holds for $x\in\overline{\partial_e X}$, then $f(x)=0$.*
(ii) *For $x\in N_0$, we have $f(x)=0$.*

PROOF. (i) Since $(n(x),p)<p$ implies that $\alpha^{n(x)}$ is not real, we have $f(Px)=0$ by the relation:

$$\alpha^{n(x)}f(Px)=T^{n(x)}f(Px)=\gamma(n(x),Px)\cdot f(Px).$$

Furthermore we have $f(x)=T^{p\cdot p(x)}f(x)=\gamma(p\cdot p(x),x)\cdot f(Px)=0$.
(ii) For any $n\in N$, we have $|f(x)|=|T^n f(x)|\leqq\|f\|\,\|T'^n x\|$, which implies $f(x)=0$ for $x\in N_0$. //

Put $I_p=\{f\in E; f(x)=0 \text{ for } x\in(\overline{\partial_e X}\setminus\bigcup_{j\geqq 1}S_{pj})\}$. Then I_p is a T-invariant subspace (but I_p is not necessarily an ideal). So we can consider the restriction $T|I_p$ of T to I_p. Then as for the point spectra of T and $T|I_p$, Lemma 5 implies the following

PROPOSITION 5. *The following are equivalent.*
(i) $\alpha\in P_\sigma(T)$.
(ii) $\alpha\in P_\sigma(T|I_p)$.

In order to check whether $f\in F$ belongs to E, the following is useful.

LEMMA 6. *Let $f\in F$ satisfy $f|N_0=0$ and $\tilde{T}f=\alpha f$. If $(n(x),p)<p$, then we have $\int f d\mu_{Px}=0$.*

PROOF. Put $n=n(x)$. By using Proposition 3, we have

$$\begin{aligned} np\int f\,d\mu_{Px} &= \sum_{s=0}^{p-1}\sum_{j=0}^{n-1}\int \alpha^{-(sn+j)}\tilde{T}^{sn+j} f\,d\mu_{Px} \\ &= \sum_{s=0}^{p-1}\sum_{j=0}^{n-1}\alpha^{-(sn+j)}\gamma(sn,Px)\int \tilde{T}^{j} f\,d\mu_{k^{sn}(Px)} \\ &= \sum_{s=0}^{p-1}\alpha^{-sn}\sum_{j=0}^{n-1}\alpha^{-j}\int \tilde{T}^{j} f\,d\mu_{Px}. \end{aligned}$$

Hence $\int f d\mu_{Px}=0$ holds, since $\sum_{s=0}^{p-1}\alpha^{-sn}=0$. //

Now we can prove Theorem 1.

PROOF OF THEOREM 1. For $j\in N$, put $q'=\frac{qj}{(p,j)}$ and $\beta=\alpha^j$. Then $\beta=\exp\left(\frac{q'}{s_j}2\pi i\right)$, $(q',s_j)=1$ and $\tilde{T}g_j=\beta g_j$. For $x\in\overline{\partial_e X}\setminus\{N_0\cup\tilde{\theta}(\partial_e X)\}$, $(n(x),s_j)<1$ holds by assumption. By Lemma 5, we have $g_j(x)=0$ and $g_j|N_0=0$. By lemma 6, we have $\int g_j\,d\mu_{Px}=0$. Therefore $g_j(Px)$ $=\frac{\beta^{P(x)}}{\gamma(p(x),x)}g_j(x)=0=\int g_j\,d\mu_{Px}$. Since M is a multiple of s_j, $\tilde{T}^M g_j=g_j$. By Proposition 4, we have $g_j\in E$, which implies $\beta=\alpha^j\in P_\sigma(T)$. //

Next we shall investigate whether there is an eigenvector of T pertaining to α^j when g_j does not belong to E. In that case, the property of an operator in a finite dimensional space is essential. At next section we shall examine about it.

§ 4. The spectrum of an operator in a finite-dimensional space.

Let $W=(w_{jt})$ be an m×n-matrix with $w_{jt}\geqq 0$ $(1\leqq j\leqq m, 1\leqq t\leqq n)$ and $N(W)$ be the set $\{v=(v(1),\cdots,v(m))\in C^m; vW=0\}$. Let p be a divisor of m and τ be a permutation of the set $\{1,\cdots,m\}$, satisfying that for any j $(1\leqq j\leqq m)$, $n(j)$ is a multiple of p, where $n(j)$ is the least number such that $\tau^{n(j)}(j)=j$.

Let $\beta(j)$ $(1\leqq j\leqq m)$ be a positive number satisfying $\prod_{t=1}^{n(j)}\beta(\tau^t(j))=1$

($1 \leqq j \leqq m$). We shall say that j and j' are *τ-disjoint integers* if $\tau^t(j) \neq j'$ for any t ($0 \leqq t \leqq m$). Let $o(\tau)$ be a number of mutually τ-disjoint integers of the set $\{1, 2, \cdots\cdots, m\}$.

Let $\hat{T} \in \mathfrak{L}(N(W))$ be defined by

$$(\hat{T}v)(j) = \beta(j) \cdot v(\tau(j)) \text{ for } 1 \leqq j \leqq m \text{ and } v \in N(W).$$

Let q be a number such that $(p, q) = 1$. Put $\alpha = \exp\left(\frac{q}{p} 2\pi i\right)$, $\beta(t, j) = \prod_{s=0}^{t-1} \beta(\tau^s(j))$ and $a_{jt}(\alpha) = \sum_{s=1}^{n(j)} \frac{\alpha^s}{\beta(s, j)} w_{\tau^s(j)t}$ for $1 \leqq j \leqq m$, $1 \leqq t \leqq n$.

Let $A = (a_{jt}(\alpha))$ be a $m \times n$-matrix.

As for the spectrum $\sigma(\hat{T})$ of $\hat{T}$, we have

PROPOSITION 6. *The following are equivalent.*

(i) rank $A < o(\tau)$.

(ii) $\alpha \in \sigma(\hat{T})$.

PROOF. Let $\{s_r\}_{1 \leqq r \leqq o(\tau)}$ be a set of τ-disjoint integers. Put $b_{rt} = a_{s_r t}(\alpha)$ and let $B = (b_{rt})$ be a $o(\tau) \times n$-matrix. Since $n(s_r)$ is a multiple of p, we have

$$\text{rank } A = \text{rank } B. \qquad (*)$$

(i)$\Rightarrow$(ii) Put

$$v_r(j) = \begin{cases} \dfrac{\alpha^t}{\beta(t, s_r)} & j = \tau^t(s_r) \text{ for some } t \ (1 \leqq t \leqq n(s_r)) \\ 0 & \text{otherwise} \end{cases}$$

$$\text{for } 1 \leqq r \leqq o(\tau) \text{ and } 1 \leqq j \leqq m.$$

Then $\boldsymbol{v}_r \in \boldsymbol{C}^m$ and $b_{rt} = \sum_{j=1}^{m} v_r(j) w_{jt}$. Since rank B is less than $o(\tau)$, there exists $\boldsymbol{c} = (c_1, \cdots\cdots, c_{o(\tau)}) \in \boldsymbol{C}^{o(\tau)}$ such that $\boldsymbol{c} \neq \boldsymbol{0}$ and $\boldsymbol{c}B = 0$. Put $\boldsymbol{v}_0 = \sum_{r=1}^{o(\tau)} c_r \boldsymbol{v}_r \in \boldsymbol{C}^m$. Since $\boldsymbol{v}_r (1 \leqq r \leqq o(\tau))$ are linearly independent, $\boldsymbol{v}_0 \neq \boldsymbol{0}$ and we also have $\boldsymbol{v}_0 W = \boldsymbol{c}B = 0$, which implies $\boldsymbol{v}_0 \in N(W)$. We have $(\hat{T}\boldsymbol{v}_0)(\tau^t(s_r)) = \alpha \boldsymbol{v}_0(\tau^t(s_r))$ for $1 \leqq r \leqq o(\tau)$ and $1 \leqq t \leqq n(s_r)$. Therefore we have $\hat{T}\boldsymbol{v}_0 = \alpha \boldsymbol{v}_0$.

(ii)$\Rightarrow$(i) Suppose rank $A \geqq o(\tau)$ (**) and $\hat{T}\boldsymbol{v} = \alpha \boldsymbol{v}$ for some $\boldsymbol{v} \in N(W)$. Then by using the relation

$$v(\tau^j(s_r)) = \frac{(\hat{T}^j \boldsymbol{v})(s_r)}{\beta(j, s_r)} = \frac{\alpha^j}{\beta(j, s_r)} v(s_r),$$

we have

$$\sum_{r=1}^{o(\tau)} a_{s_r t}(\alpha) v(s_r) = \sum_{r=1}^{o(\tau)} \sum_{j=1}^{n(s_r)} \frac{\alpha^j}{\beta(j, s_r)} w_{\tau^j(s_r)t} v(s_r) = \sum_{j=1}^{m} v(j) w_{jt} = 0 \qquad (***)$$

for $1\leqq t\leqq n$. The relation $(*)$, $(**)$ and $(***)$ imply $v(s_r)=0$ for $1\leqq r\leqq o(\tau)$. Therefore $v=0$, which implies $\alpha\notin P_\sigma(\hat{T})$. //

§ 5. Peripheral point spectrum.

Now we assume the condition

(C4) $\overline{\partial_e X}\setminus\tilde{\theta}(\partial_e X)$ is an intersection of $\overline{\partial_e X}$ with finite rays.

When E is a separable simplex space with order unit, it is known [4, § 3, Theorem] that $\int_{\overline{\partial_e X}} f d\mu_x = \int_{\partial_e X} f d\mu_x$ holds for any $f\in E$.

When E is not separable, under the condition (C4) we have the following

LEMMA. 7. *$\int f d\mu_x = \int_{\tilde{\theta}(\partial_e X)} f d\mu_x$ holds for any $f\in F$.*

PROOF. Let μ_1 be the restriction of μ_x to $\tilde{\theta}(\partial_e X)$ and put $\nu_x=\frac{1}{\mu_1(\tilde{\theta}(\partial_e X))}\mu_1$. Then we see $\nu_x\succ\mu_x$ [1, § 26.6]. Since μ_x is maximal, we have $\nu_x=\mu_x$. //

For $p\in N$, put

$B_p:=\bigcup_{t\geqq 1} P_{pt}\cap\partial_e X$ and $C_p:=\{x\in(\overline{\partial_e X}\setminus\tilde{\theta}(\partial_e X))\cap P(\tilde{\theta}(\overline{N_0})\setminus N_0)\,;\, n(x)=jp$ for some $j\in N\}$. By the condition (C4), there exists $\{x_1,\cdots,x_n\}$ such that $C_p=\{x_1,\cdots,x_n\}$. We will show that μ_x has no support in $\bigcup_{t\geqq 1} S_{pt}\setminus\tilde{\theta}(B_p)$ for $x\in P(\overline{\partial_e X}\setminus N_0)$, if B_p is closed.

LEMMA 8. *For $p\in N$, let $S=\bigcup_{t\geqq 1} S_{pt}$ and suppose B_p is closed. Then*

$$\int_S f d\mu_x = \int_{\tilde{\theta}(B_p)} f d\mu_x$$

holds for any $f\in F$ and any $x\in P(\overline{\partial_e X}\setminus N_0)$.

PROOF. It is clear that $S\supset\tilde{\theta}(B_p)$, Suppose there exist $f\in F$ and $x\in P(\overline{\partial_e X}\setminus N_0)$ such that

$$\left|\int_S f d\mu_x - \int_{\tilde{\theta}(B_p)} f d\mu_x\right| = c > 0.$$

Since $S\setminus\tilde{\theta}(B_p)$ is θ-open by Lemma 1 and assumption, there exists a θ-closed set K satisfying

$$\mu_x((S\setminus\tilde{\theta}(B_p))\setminus K) < \frac{c}{3\|f\|} \text{ and } K\subset S\setminus\tilde{\theta}(B_p).$$

By Corollary 2 to Proposition 2, there exists $g\in F$ satisfying $g\mid(\overline{\partial_e X})\setminus(S\setminus\tilde{\theta}(B_p)))=0$, $g\mid K=f\mid K$ and $\|g\|\leqq\|f\|$. Since we have

$$\int g\,d\mu_x=\phi^{-1}g(Px)=\int_{\tilde{\theta}(\partial_e X)}\tilde{P}\phi^{-1}g\,d\mu_x$$

by Lemma 7 and $\tilde{P}\phi^{-1}g(y)=\phi^{-1}g(Py)=0$ holds for $y\in\tilde{\theta}(\partial_e X)$, it follows that $\int g\,d\mu_x=0$. On the other hand,

$$\left|\int g\,d\mu_x\right|\geqq\left|\int_K g\,d\mu_x\right|-\left|\int_{(S\setminus\tilde{\theta}(B_p))\setminus K} g\,d\mu_x\right|>\frac{2c}{3}-\frac{c}{3}=\frac{c}{3},$$

which is a contradiction. //

a) In this section, we consider the case that the potency $\overline{\overline{B_p}}$ of B_p is finite, that is, $B_p=\{y_1,\cdots,y_m\}$.

Let $\alpha=\exp\left(\frac{q}{p}2\pi i\right)$ for $p,q\in N$ with $(p,q)=1$. By condition (C4), there exists $\{z_1,\cdots,z_s\}$ such that

$$\bigcup_{j=1}^{s}\tilde{\theta}(z_j)=\bigcup_{t\geqq1}S_{pt}\cap(\overline{\partial_e X}\setminus\tilde{\theta}(\partial_e X)).$$

Then as for the relation between elements of E and F, we have the following

PROPOSITION 7. *Let U be a θ-open subset of $\bigcup_{t\geqq1}S_{pt}\setminus\tilde{\theta}(B_p)$ containing $\bigcup_{j=1}^{s}\tilde{\theta}(z_j)$. If $f\in F$ satisfies $\int f\,d\mu_x=0$ for $x\in C_p$, $f\mid(\overline{\partial_e X}\setminus\bigcup_{t\geqq1}S_{pt})=0$ and $\tilde{T}f(x)=\alpha f(x)$ for $x\in B_p$, then there exists $h\in E$ such that $h\mid(\overline{\partial_e X}\setminus U)=f\mid(\overline{\partial_e X}\setminus U)$.*

PROOF. For j $(1\leqq j\leqq s)$, there exists a θ-open set V_j satisfying $z_j\in V_j\subset U$ and $V_j\cap V_{j'}=\phi$ if $j\neq j'$. Put $c_j=\int f\,d\mu_{z_j}-f(z_j)$ for $1\leqq j\leqq s$ and

$$v_j=\begin{cases}1 & \text{if } c_j=0\\ c_j/|c_j| & \text{if } c_j\neq0.\end{cases}$$

By Proposition 2, there exists $g_j\in F$ satisfying

$g_j(z_j)=|c_j|$, $0\leqq g_j(x)\leqq|c_j|$ for $x\in V_j$ and $g_j(x)=0$ for $x\in\overline{\partial_e X}\setminus V_j$.

Put $g_0=f+\sum_{j=1}^{s}v_j g_j$. Then we have $g_0\in F$, $g_0(z_j)=\int f\,d\mu_{z_j}$ $(1\leqq j\leqq s)$ and $g_0\mid(\overline{\partial_e X}\setminus U)=f\mid(\overline{\partial_e X}\setminus U)$, which implies $g_0\mid(\overline{\partial_e X}\setminus\bigcup_{t\geqq1}S_{pt})=0$, $g_0\mid N_0=0$ and $g_0\mid\tilde{\theta}(B_p)=f\mid\tilde{\theta}(B_p)$.

We shall show that $g_0(x)=\int g_0\,d\mu_x$ holds for

$x \in P(\overline{\partial_e X} \setminus (N_0 \cup \tilde{\theta}(\partial_e X)))$. By using Lemma 8, we have

$$\int g_0 d\mu_x = \int_{\tilde{\theta}(B_p)} g_0 d\mu_x = \int_{\tilde{\theta}(B_p)} f d\mu_x = \int f d\mu_x.$$

In case of $(n(x), p) = p$ and $x \in \tilde{\theta}(\overline{N_0})$, that is, $x \in C_p$, we have $g_0(x) = 0$ and $\int f_0 d\mu_x = 0$ by assumption. So $g_0(x) = \int g_0 d\mu_x$.

In case of $(n(x), p) = p$ and $x \in \overline{\partial_e X} \setminus \tilde{\theta}(\overline{N_0})$, there is some j such that $x = z_j$ and $\int g_0 d\mu_{z_j} = \int f d\mu_{z_j} = g_0(z_j)$.

In case of $(n(x), p) < p$, there is a finite set B such that $k^j B \cap k^{j'} B = \phi$ if $1 \leqq j \neq j' \leqq p$ and $\bigcup_{j=0}^{p-1} k^j B = B_p$, since B_p is a finite set. Then we have, by using Proposition 3, Lemma 8 and [8, Lemma 5],

$$\int g_0 d\mu_x = \sum_{j=1}^{p=1} \int_{\tilde{\theta}(k^j B)} f d\mu_x = \frac{1}{n(x)} \sum_{s=0}^{n(x)-1} \sum_{j=0}^{p-1} \int_{\tilde{\theta}(k^{s+n(x)j} B)} f d\mu_x$$

$$= \frac{1}{n(x)} \sum_{s=0}^{n(x)-1} \sum_{j=0}^{p-1} \frac{1}{\gamma(n(x)j, x)} \int_{\tilde{\theta}(k^s B)} \tilde{T}^{n(x)j} f d\mu_x$$

$$= \frac{1}{n(x)} \sum_{s=0}^{n(x)-1} \sum_{j=0}^{p-1} \alpha^{n(x)j} \int_{\tilde{\theta}(k^s B)} f d\mu_x.$$

Since $(n(x), p) < p$ implies $x \notin \bigcup_{t \geqq 0} S_{pt}$ and $\sum_{j=0}^{p-1} \alpha^{n(x)j} = 0$, we have $\int g_0 d\mu_x = 0 = g_0(x)$. By using Proposition 4, we have $h := \lim_{j \to \infty} T^{jM} g_0 \in E$ and h is a desired one. //

Put $w_{jt} := \mu_{x_t}(y_j)$ for $1 \leqq j \leqq m$ and $1 \leqq t \leqq n$, where y_j and x_t are elements of B_p and C_p, respectively. Let $W = (w_{jt})$ be a $m \times n$-matrix and put $N_p(W) := \{v \in C^m; vW = 0\}$. Define $\tau^* : \{1, \cdots, m\} \to B_p$ by $\tau^*(j) = y_j$. Put $\tau(j) = \tau^{*-1} k \tau^*(j)$, $\beta(j) = \gamma(\tau^*(j))$ and $n(j) = n(y_j)$. Then $y_j \in \partial_e X \setminus \tilde{\theta}(\overline{N_0})$ implies $\gamma(\tau^*(j)) > 0$ for $1 \leqq j \leqq m$ and $y_j \in \bigcup_{s \geqq 1} P_{ps}$ implies $\beta(n(j), j) := \prod_{t=0}^{n(j)-1} \beta(\tau^t(j)) = \gamma(n(y_j), y_j) = 1$. So we can consider an operator $\hat{T}_p$ on $N_p(W)$ by $(\hat{T}_p v)(j) = \beta(j) \cdot v(\tau(j))$ $(1 \leqq j \leqq m)$ for $v \in N_p(W)$. We call $\hat{T}_p$ an operator induced by T on $N_p(W)$. Now we have

PROPOSITION 8. *The following are equivalent.*

(i) $\alpha \in P_\sigma(T | I_p)$

(ii) $\alpha \in P_\sigma(\hat{T}_p)$.

PROOF. (i)⇒(ii) Suppose $\alpha \in P_\sigma(T | I_p)$. Then there exists nonzero $f\in I_p$ such that $Tf=\alpha f$. Put $v(j):=f(y_j)$ for $1\leqq j\leqq m$ and $\boldsymbol{v}=(v(1),\cdots,v(m))$. Then we have

$$(\boldsymbol{v}W)(t)=\sum_{j=1}^{m}v(j)w_{jt}=\sum_{j=1}^{m}f(y_j)\mu_{x_t}(y_j)=\int_{B_p}f\,d\mu_{x_t}=f(x_t)=0 \text{ for } 1\leqq t\leqq n$$

by using Lemma 8. So $\boldsymbol{v}\in N_p(W)$ and $(\hat{T}_p\boldsymbol{v})(j)=\alpha v(j)$ is easily obtained for $1\leqq j\leqq m$, which implies $\alpha\in P_\sigma(\hat{T}_p)$.

(ii)⇒(i) Suppose $\alpha\in P_\sigma(\hat{T}_p)$. Then there exists $\boldsymbol{v}\in N_p(W)$ satisfying $\hat{T}_p\boldsymbol{v}=\alpha\boldsymbol{v}$. Since $\bigcup_{j\geqq 1}S_{pj}$ is θ-open by Lemma 1, $\overline{\partial_e X}\setminus\bigcup_{j\geqq 1}S_{pj}$ is a closed set not containing B_p. By Corollary 1 to Proposition 2, there exists $f\in F$ satisfying $f(y_j)=v(j)$ $(1\leqq j\leqq m)$ and $f\,|\,(\overline{\partial_e X}\setminus\bigcup_{j\geqq 1}S_{pt})=0$. For $y_j\in B_p$ $(1\leqq j\leqq m)$. $\tilde{T}f(y_j)=\beta(j)\cdot v(\tau(j))=\alpha f(y_j)$.
For $x_t\in C_p$ $(1\leqq t\leqq n)$, we have by using Lemma 8,

$$\int f\,d\mu_{x_t}=\int_{B_p}f\,d\mu_{x_t}=\sum_{j=1}^{m}v(j)w_{jt}=0,$$

since $\boldsymbol{v}\in N(W)$. Since $\bigcup_{j\geqq 1}S_{pj}\setminus\tilde{\theta}(B_p)$ is a θ-open set containing $\sum_{j=1}^{s}\tilde{\theta}(z_j)$, there exists a θ-open subset U of $\bigcup_{j\geqq 1}S_{pj}\setminus\tilde{\theta}(B_p)$, containing $\sum_{j=1}^{s}\tilde{\theta}(z_j)$. By Proposition 7, there exists $h\in E$ satisfying $h\,|\,(\overline{\partial_e X}\setminus U)=f\,|\,(\overline{\partial_e X}\setminus U)$. Then for $x\in\overline{\partial_e X}\setminus\bigcup_{j\geqq 1}S_{pj}$, $h(x)=0$. For $x\in\bigcup_{j\geqq 1}S_{pj}$, we have $Th(x)=\gamma(p(x),x)\cdot Tf(Px)=\gamma(p(x),x)\cdot\alpha f(Px)=\alpha h(x)$, by using the relations $Px\in B_p$ and $h(x)=\gamma(p(x),x)\cdot f(Px)$. So $Th=\alpha h$. //

b) In this section, we consider the case that B_p contains infinite points, i. e. $\overline{\overline{B_p}}=\infty$. The following proposition is useful when we investigate eigenfunctions of T pertaining to α, where $\alpha=\exp\left(\frac{q}{p}2\pi i\right)$ for $p,q\in N$ such that $(p,q)=1$. By the condition (C4), there exists a set $\{z_1,\cdots,z_s\}$ of k-disjoint points (i. e. $k^j(z_t)\neq z_{t'}$ if $1\leqq t\neq t'\leqq s$ and $j\in N$) such that $\bigcup_{t=1}^{s}\bigcup_{j=1}^{M}k^j(z_t)=(\overline{\partial_e X}\setminus\tilde{\theta}(\partial_e X))\cap\bigcup_{m\geqq 1}P_{pm}$, where M is the number defined at §2. Then we have

PROPOSITION 9. *Let $y_0\in B_p\cap\partial_e X$ and let U be a θ-open subset of $\overline{\partial_e X}\setminus\tilde{\theta}(\overline{N_0})$ such that $y_0\in U$, $\bigcup_{t=1}^{s}z_t\in U$ and $k^j(y_0)\notin U$ for $1\leqq j\leqq n(y_0)-1$, $k^j(z_t)\notin U$ for $1\leqq j\leqq n(z_t)-1$ and $1\leqq t\leqq s$. Then there exists nonzero $h\in F$ satisfying*

$$\tilde{T}h=\alpha h,\qquad \int h\,d\mu_x=h(x)\quad for\quad x\in PU$$

and
$$h\mid(\overline{\partial_e X}\setminus\bigcup_{j=0}^{\infty}k^{-j}(U))=0.$$

PROOF. By Proposition 2, there exists $g\in F$ satisfying $0\leqq g\leqq 1$, $g(y_0)=1$ and $g\mid(\overline{\partial_e X}\setminus U)=0$. Put $\tilde{g}:=\sum_{j=0}^{M-1}\alpha^{-j}\tilde{T}^j\tilde{P}g$. For $n\in N$, there exists a θ-open set V_n satisfying $\{z_1,\cdots,z_s\}\subset V_n\subset U\setminus\{y_0\}$ and $\sum_{j=0}^{M-1}\mu_{k^j(z_t)}(V_n)<\frac{1}{(n+1)C}$ for $1\leqq t\leqq s$, where $C=\sup_m\|T^m\|$. Let $g_1\in F$ satisfy $g_1(z_1)=\frac{n(z_t)}{M}\int\tilde{g}\,d\mu_{z_t}$ for $1\leqq t\leqq s$ and $g_1\mid(\overline{\partial_e X}\setminus V_1)=g\mid(\overline{\partial_e X}\setminus V_1)$. Put $\tilde{g}_1=\sum_{j=0}^{M-1}\alpha^{-j}\tilde{T}^j\tilde{P}g_1$. For $n\geqq 2$, inductively define $f_n\in F$ and $g_n\in F$ satisfying $f_n(z_t)=\frac{n(z_t)}{M}\int\tilde{g}_{n-1}\,d\mu_{z_t}-g_{n-1}(z_t)$ for $1\leqq t\leqq s$, $f_n\mid(\overline{\partial_e X}\setminus V_n)=0$, $\|f_n\|\leqq\max\{|f_n(z_t)|;t=1,\cdots,s\}$, $g_n=g_{n-1}+f_n\in F$ and $\tilde{g}_n=\sum_{j=0}^{M-1}\alpha^{-j}\tilde{T}^j\tilde{P}g_n$. Put $B=\bigcup_{n\geqq 1}P_n$. Then in a similar way to the proof of Lemma 8, we have $\int f\,d\mu_{z_t}=\int_{\tilde{\theta}(B)}f\,d\mu_{z_t}$ for $f\in F$, by using the relation $z_t=Pz_t$. For $y\in B$, $\tilde{T}^j\tilde{P}f(y)=\tilde{T}^jf(y)$ holds for $f\in F$ and $j\in N$. So we have $|f_n(z_t)|\leqq\frac{1}{n!}(\|g_1\|+1)$ for $n\in N$ and $t=1,\cdots,s$, by using the relation

$$\begin{aligned}f_n(z_t)&=\frac{n(z_t)}{M}\int_{\tilde{\theta}(B)}(\tilde{g}_{n-1}-\tilde{g}_{n-2})\,d\mu_{z_t}\\&=\frac{n(z_t)}{M}\sum_{j=0}^{M-1}\alpha^{-j}\gamma(j,z_t)\int_{V_{n-1}}f_{n-1}\,d\mu_{k^j(z_t)}.\end{aligned}$$

Then $\|g_n-g_{n-1}\|=\|f_n\|\leqq\frac{1}{n!}(\|g_1\|+1)\to 0$ as $n\to\infty$. So there is $g_0\in F$ such that $g_n\to g_0$ as $n\to\infty$. Put $h=\sum_{j=0}^{M-1}\alpha^{-j}\tilde{T}^j\tilde{P}g_0$. Then we can easily check that $h(y_0)=\frac{M}{n(y_0)}\neq 0$, $h\in F$ and $\tilde{T}f=\alpha f$. For $x\in PU\cap(\overline{\partial_e X}\setminus\tilde{\theta}(\partial_e X))$ with $(n(x),p)=p$, there is t $(1\leqq t\leqq s)$ such that $x=z_t$ and so $h(x)=\int h\,d\mu_x$. For $x\in PU\cap(\overline{\partial_e X}\setminus\tilde{\theta}(\partial_e X))$ with $(n(x),p)<p$, we have $h(x)=0=\int h\,d\mu_x$ by Lemma 6. So h is a desired one. //

As for the point spectrum of T, we have

PROPOSITION 10. *For $p,q\in N$ such that $(p,q)=1$, put*

$\alpha=\exp\left(\frac{q}{p}2\pi i\right)$. *Suppose* $\overline{\overline{B_p}}=\infty$. *Then we have* $\alpha\in P_\sigma(T)$.

PROOF. Let $\{z_1,\cdots,z_s\}$ be k-disjoint points such that

$$\bigcup_{t=1}^{s}\bigcup_{j=1}^{M}k^j(z_t)=(\overline{\partial_e X}\setminus\tilde{\theta}(\partial_e X)\cap\bigcup_{m\geqq 1}P_{pm}.$$

Put $C_p=\{x_1,\cdots,x_n\}$. By assumption, there are $n+1$ k-disjoint points $y_1,\cdots,y_{n+1}$ of B_p. Since $\tilde{\theta}(\partial_e X)\setminus\tilde{\theta}(\overline{N_0})$ and $(\tilde{\theta}(\partial_e X)\setminus\tilde{\theta}(\overline{N_0}))\cup\tilde{\theta}(z_t)$ are θ-open by (C4), there are θ-open sets V_j $(1\leqq j\leqq n+1)$ and W_t $(1\leqq t\leqq s)$ satisfying

$$y_j\in V_j\subset\tilde{\theta}(\partial_e X)\setminus\tilde{\theta}(\overline{N_0})$$

$$z_t\in W_t\subset(\tilde{\theta}(\partial_e X)\setminus\tilde{\theta}(\overline{N_0}))\cup\tilde{\theta}(z_t)$$

$k^m(y_j)\notin V_{j'}$ for $1\leqq j\neq j'\leqq n+1$, $m\in N$, and $k^m(y_j)\notin W_t$ for $1\leqq t\leqq s$, $1\leqq j\leqq n+1$, $m\in N$. By Proposition 9, there exists nonzero $f_j\in F$ satisfying $\tilde{T}f_j=\alpha f_j$, $f_j\Big|\Big(\overline{\partial_e X}\setminus\bigcup_{m=0}^{\infty}k^{-m}\Big(V_j\cup\bigcup_{t=1}^{s}W_t\Big)\Big)=0$ and $\int f_j\,d\mu_{z_t}=f_j(z_t)$ for $1\leqq t\leqq s$. Put $a_{jt}=\int f_j\,d\mu_{z_t}$ for $1\leqq j\leqq n+1$, $1\leqq t\leqq n$. Then there exists $\mathbf{c}=(c_1,\cdots,c_{n+1})\in\boldsymbol{R}^{n+1}$ satisfying $\sum_{j=1}^{n+1}c_j a_{jt}=0$ for $1\leqq t\leqq s$ and $\mathbf{c}\neq\mathbf{0}$. Put $f_0=\sum_{j=1}^{n+1}c_j f_j$. Then $f_0\in F$ and $f_0\neq 0$, since $f_j(y_{j'})=0$ if $j\neq j'$ and $f_j(y_j)=1$. The relation $\tilde{T}f_0=\alpha f_0$ is easily obtained. All that remains is to show that $f_0\in E$. Since $f_0\,|\,N_0=0$ holds, we will show that $\int f_0\,d\mu_x=f_0(x)$ holds for any $x\in P(\overline{\partial_e X}\setminus(N_0\cup\tilde{\theta}(\partial_e X)))$.

If $(n(x),p)=p$, then $x\in C_p$ or $x=k^r(z_t)$ for some t $(1\leqq t\leqq s)$ and some $r\in\mathrm{N}$. If $x=x_t\in C_p$, it holds that

$$\int f_0\,d\mu_x=\sum_{j=1}^{n+1}c_j\int f_j\,d\mu_{x_t}=\sum_{j=1}^{n+1}c_j a_{jt}=0,$$

which implies $f_0(x_t)=\int f_0\,d\mu_{x_t}$. If $x=k^r(z_t)$ $(1\leqq t\leqq s, r\in\mathrm{N})$, it holds that

$$f_0(k^r(z_t))=\frac{\tilde{T}^r f_0(z_t)}{\gamma(r,z_t)}=\frac{\alpha^r}{\gamma(r,z_t)}\sum_{j=1}^{n+1}c_j f_j(z_t)$$

$$=\frac{1}{\gamma(r,z_t)}\int\tilde{T}^r f_0\,d\mu_{z_t}=\int f_0\,d\mu_{k^r(z_t)},$$

by using Proposition 3.

If $(n(x),p)<p$, then we have $\int f_0\,d\mu_{Px}=0=f_0(Px)$ by using Lemma 6. By the definition of f_0, $f_0=\tilde{T}^{jM}f_0$ holds for any $j\in N$. So by Proposition 4, we have $f_0\in E$. //

From Propositions 5, 8 and 10, we have

THEOREM 2. *Let E be a simplex space such that* $\inf\{\|x\|; x\in\overline{\partial_e X}\setminus\{0\}\}>0$ *aud* $\overline{\partial_e X}\setminus\tilde{\theta}(\partial_e X)$ *is the intersection of finite rays with* $\overline{\partial_e X}$. *Let T be a simplex homomorphism of E satisfying* $\sup_n\|T^n\|<\infty$ *and* $\sigma(T)\cap\Gamma\neq\Gamma$. *Let* $\alpha=\exp\left(\frac{q}{p}2\pi i\right)$ *for some* $p,q\in N$ *with* $(p,q)=1$.

Then the following are equivalent.

(i) $\alpha\in P_\sigma(T)$.

(ii) *Either of the following is satisfied.*

a) $\overline{\overline{B_p}}=\infty$, *where* $B_p=\bigcup_{m\geqq 1}P_{pm}\cap(\partial_e X\setminus\tilde{\theta}(N_0))$.

b) *If* $\overline{\overline{B_p}}<\infty$, *then* $\alpha\in\sigma(\hat{T}_p)$, *where $\hat{T}_p$ is an operator induced by T in a finite dimensional space $N_p(W)$ as defined before Proposition* 8.

DEFINITION. $\sigma(T)$ is said to have the *property* (QC) if $\alpha=\exp\left(\frac{q}{p}2\pi i\right)\in\sigma(T)$, $(p,q)=1$ and $(p,s)=1$ imply $\alpha^s\in\sigma(T)$.

Then we have

THEOREM 3. *Let E and T satisfy the assumption of Theorem* 2. *Then the following are equivalent.*

(i) $P_\sigma(T)$ *has the property* (QC).

(ii) *For any* $p\in N$ *such that* $\overline{\overline{B_p}}<\infty$ *and* $\exp\left(\frac{q}{p}2\pi i\right)\in\sigma(\hat{T}_p)$ *with* $(p,q)=1$, $\sigma(\hat{T}_p)$ *has the property* (QC).

PROOF. (i)$\Rightarrow$(ii): Suppose $\alpha=\exp\left(\frac{q}{p}2\pi i\right)\in\sigma(\hat{T}_p)$, $(p,q)=1$, $(p,s)=1$ and $\overline{\overline{B_p}}<\infty$. By Propositions 5 and 8, we have $\alpha\in P_\sigma(T)$. Since $P_\sigma(T)$ has the *property* (QC), we have $\alpha^s=\exp\left(\frac{qs}{p}2\pi i\right)\in P_\sigma(T)$, which implies $\alpha^s\in\sigma(\hat{T}_p)$.

(ii)$\Rightarrow$(i): Suppose $\alpha=\exp\left(\frac{q}{p}2\pi i\right)\in P_\sigma(T)$, $(p,q)=1$ and $(p,s)=1$. Then $(p,qs)=1$. If $\overline{\overline{B_p}}=\infty$, we have $\alpha^s=\exp\left(\frac{qs}{p}2\pi i\right)\in P_\sigma(T)$ by Proposition 10. If $\overline{\overline{B_p}}<\infty$, we have $\alpha\in\sigma(\hat{T}_p)$ by Propositions 5 and 8. Since $\sigma(\hat{T}_p)$ has the *property* (QC), we have $\alpha^s\in\sigma(\hat{T}_p)$, which implies $\alpha^s\in P_\sigma(T)$ by Propositions 5 and 8. //

THEOREM 4. *Let E and T satisfy the assumption of Theorem* 2. *Then the following are equivalent.*

(i) $P_\sigma(T)\cap\Gamma$ *is cyclic.*

(ii) *If p satisfies* $\overline{\overline{B_p}}<\infty$ *and* $\exp\left(\frac{q}{p}2\pi i\right)\in\sigma(\hat{T}_p)$ *with some* $q\in N$ *such that* $(p,q)=1$, *then for any divisor* s *of* p *with* $\overline{\overline{B_s}}<\infty$, $\exp\left(\frac{1}{s}2\pi i\right)$ *belongs to* $\sigma(\hat{T}_s)$ *and* $\sigma(\hat{T}_s)$ *has the property (QC).*

PROOF. (i)⇒(ii): Suppose $\overline{\overline{B_p}}<\infty$ and $\alpha:=\exp\left(\frac{q}{p}2\pi i\right)\in\sigma(\hat{T}_p)$ with $(p,q)=1$. Then by Propositions 5 and 8, we have $\alpha\in P_\sigma(T)$. For any divisor s of p, $\beta:=\exp\left(\frac{1}{s}2\pi i\right)$ belongs to $P_\sigma(T)$, since $P_\sigma(T)\cap\Gamma$ is cyclic. If $\overline{\overline{B_s}}<\infty$, we have $\beta\in\sigma(\hat{T}_s)$ by Propositions 5 and 8. Since $P_\sigma(T)$ has *the property* (*QC*), $\sigma(\hat{T}_s)$ has the *property* (*QC*) by Theorem 3.

(ii)⇒(i): Suppose $\alpha\in P_\sigma(T)\cap\Gamma$. Then there are $p\in N$ and $q\in N$ such that $\alpha=\exp\left(\frac{q}{p}2\pi i\right)$ and $(p,q)=1$ since $\sigma(T)\cap\Gamma\neq\Gamma$ holds. If $\overline{\overline{B_p}}=\infty$, then α^n belongs to $P_\sigma(T)$ by Proposition 10, since $\overline{\overline{B_s}}>\overline{\overline{B_p}}$ holds with $s=\frac{p}{(p,n)}$. If $\overline{\overline{B_p}}<\infty$, put $s_n=\frac{p}{(p,n)}$ for any $n\in N$. Then $(q,s_n)\leqq(q,p)$ implies $(q,s_n)=1$. When $\overline{\overline{B_{s_n}}}=\infty$, we have $\alpha^n=\exp\left(\frac{p}{s_n}2\pi i\right)\in P_\sigma(T)$ by Proposition 10. When $\overline{\overline{B_{s_n}}}<\infty$, α^n belongs to $P_\sigma(T)$ by Propositions 5 and 8, since $\exp\left(\frac{1}{s_n}2\pi i\right)\in\sigma(\hat{T}_{s_n})$ and $\sigma(\hat{T}_{s_n})$ has the property (QC). //

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