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Some Initial Value Problems in Wave Propagation

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§ 1. Introduction

The present paper is a continuation to the previous paper[1], which has made use of an integral representation of $\sin pt/p$ for solving wave equations. An initial value problem is to find a solution to a given wave equation that fits to a given initial condition. For example, we take the following wave equation

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x_1^2}-\frac{\partial^2 u}{\partial x_2^2}-\cdots\cdots-\frac{\partial^2 u}{\partial x_n^2}=0 \tag{1}$$

with the initial condition

$$(u)_{t=0}=f(\boldsymbol{x}), \qquad (u_t)_{t=0}=g(\boldsymbol{x}).$$

If we multiply (1) by e^{-qt} and integrate it with respect to t from 0 to ∞, we get an equation to its Laplace transform

$$q^2\bar{u}-\frac{\partial^2 \bar{u}}{\partial x_1^2}-\cdots\cdots-\frac{\partial^2 \bar{u}}{\partial x_n^2}=g(\boldsymbol{x})+qf(\boldsymbol{x})$$

where we set

$$\bar{u}=\int_0^\infty ue^{-qt}dt$$

the real part of a complex variable q being positive enough to ensure the convergence of the integral. Thus, the differential equation and initial values are combined into one equation. On the other hand, the solution G to the nonhomogeous equation

$$\frac{\partial^2 G}{\partial t^2}-\frac{\partial^2 G}{\partial x_1^2}-\cdots\cdots-\frac{\partial^2 G}{\partial x_n^2}=\delta(t-t')\delta(\boldsymbol{x}-\boldsymbol{x}') \tag{2}$$

subject to the condition

$$G=0 \quad \text{for} \quad t<t'$$

gives its Laplace transform $\overline{G}$ satisfying

$$q^2\overline{G}-\frac{\partial^2 \overline{G}}{\partial x_1^2}-\cdots\cdots-\frac{\partial^2 \overline{G}}{\partial x_n^2}=e^{-qt'}\delta(\boldsymbol{x}-\boldsymbol{x}') \tag{3}$$

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We call the function G Green function of the wave equation. So, we can construct the solution u by means of the Green function $G(t, \boldsymbol{x}; t', \boldsymbol{x}')$ as follows

$$u(t, \boldsymbol{x}) = \int G(t, \boldsymbol{x}; 0, \boldsymbol{x}') g(\boldsymbol{x}') d\boldsymbol{x}' + \frac{\partial}{\partial t} \int G(t, \boldsymbol{x}; 0, \boldsymbol{x}') f(\boldsymbol{x}') d\boldsymbol{x}'.$$

We multiply both sides of the equation (3) by $e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}$ and integrate with respect to $\boldsymbol{x}$ over the entire space. Then we get an equation for Laplace-Fourier transform $\widetilde{G}$

$$(q^2 + p^2)\widetilde{G} = e^{-qt' - i\boldsymbol{p}\cdot\boldsymbol{x}'}, \qquad p^2 = p_1^2 + \cdots\cdots + p_n^2.$$

By an inverse Laplace-Fourier transformation we get

$$\begin{aligned} G(t, \boldsymbol{x}; t', \boldsymbol{x}') &= \frac{1}{(2\pi)^n 2\pi i} \int d\boldsymbol{p} \int_L dq \frac{e^{q(t-t') + i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')}}{q^2 + p^2} \\ &= \frac{1}{(2\pi)^n} \int d\boldsymbol{p} \frac{\sin p(t-t')}{p} e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \\ &= \frac{\Gamma(3/2)}{(2\pi)^n 2\pi i} \int_L \frac{ds}{s^{3/2}} \int d\boldsymbol{p} \exp\left[s(t-t')^2 - \frac{\boldsymbol{p}^2}{4s} + i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')\right] \\ &= \frac{\Gamma(3/2)}{\pi^{n/2}} \frac{1}{2\pi i} \int_L \exp s[(t-t')^2 - (\boldsymbol{x}-\boldsymbol{x}')^2] \cdot ds\, s^{(n-3)/2} \\ &= \frac{\Gamma(3/2)}{\pi^{n/2}} \delta^{(n-3)/2}[(t-t')^2 - (\boldsymbol{x}-\boldsymbol{x}')^2] \end{aligned}$$

where we set

$$\delta^\nu(x) = \frac{1}{2\pi i} \int_L e^{sx} s^\nu ds$$

remembering the notations in the previous paper. The sign L shows the path of integration along a straight line extending from $c - i\infty$ to $c + i\infty$, where c, which is the real part of s, is assumed usually positive. In passing we note here that

$$\delta^0(x) = \delta(x) \qquad \text{Dirac delta function}$$

$$\delta^{-1}(x) = \varepsilon(x) \qquad \text{Heaviside unit function.}$$

We encounter frequently with cases where we need to evaluate integrals of the type

$$\int_{-\infty}^{\infty} \int d\boldsymbol{p} \int_L dq \frac{e^{qt + i\boldsymbol{p}\cdot\boldsymbol{x}}}{(q^2 + A)(q^2 + B)\cdots\cdots(q^2 + C)}$$

$A, B, \cdots\cdots C$ being m positive quadratic forms in $\boldsymbol{p}$. The integration with respect to q runs along L, while integration with respect to $\boldsymbol{p}$ extends over the entire space. An integration with respect to q gives

$$\frac{1}{2\pi i} \int_L \frac{e^{qt}}{(q^2 + A)(q^2 + B)\cdots\cdots(q^2 + C)} dq$$

$$=\sum_{\text{cycl.}}\frac{\sin t\sqrt{A}}{\sqrt{A}}\frac{1}{(B-A)\cdots\cdots(C-A)}$$

$$=\frac{\Gamma(3/2)}{2\pi i}\int_L e^{st^2}\frac{ds}{s^{3/2}}\sum_{\text{cycl.}}\frac{e^{-A/4s}}{(B-A)\cdots\cdots(C-A)},\quad \mathscr{R}s>0.$$

The last term in the integrand may be transformed as follows

$$\sum_{\text{cycl.}}\frac{e^{-A/4s}}{(B-A)\cdots\cdots(C-A)}$$

$$=\frac{1}{2\pi i}\int_L\frac{e^{\tau}}{(\tau+A/4s)(\tau+B/4s)\cdots\cdots(\tau+C/4s)}\frac{d\tau}{(4s)^{m-1}}$$

$$\mathscr{R}\tau>0,\mathscr{R}(A/s)>0,\cdots\cdots,\mathscr{R}(C/s)>0$$

and further

$$=\frac{1}{2\pi i}\int_L\frac{e^{\tau}}{(4s)^{m-1}}d\tau\int_0^{\infty}\!\!\int\exp\Big[-u\Big(\tau+\frac{A}{4s}\Big)-v\Big(\tau+\frac{B}{4s}\Big)$$

$$-\cdots\cdots-w\Big(\tau+\frac{C}{4s}\Big)\Big]du\,dv\cdots\cdots dw$$

$$=\frac{1}{(4s)^{m-1}}\int_0^{\infty}\!\!\int\delta(1-u-v-\cdots\cdots-w)$$

$$\times\exp[-(uA+vB+\cdots\cdots+wC)/4s]du\,dv\cdots\cdots dw.$$

Therefore we get

$$\frac{1}{2\pi i}\int_L\frac{e^{qt}}{(q^2+A)(q^2+B)\cdots\cdots(q^2+C)}dq$$

$$=\frac{\Gamma(3/2)}{4^{m-1}}\int_0^{\infty}\!\!\int\delta(1-\sum u)\Pi du\frac{1}{2\pi i}\int_L\exp[st^2-\sum uA/4s]\frac{ds}{s^{m+1/2}}.\qquad(4)$$

Fourier transform of this expression is easy to calculate since m quadratic forms in $\boldsymbol{p}$ are on the shoulder of e. Likewise we get another formula

$$\frac{1}{2\pi i}\int_L\frac{e^{qt}}{q(q^2+A)(q^2+B)\cdots\cdots(q^2+C)}dq$$

$$=\frac{t\sqrt{\pi}}{4^m}\frac{1}{2\pi i}\int_0^{\infty}\!\!\int\varepsilon(1-\sum u)\Pi du\int_L\exp[st^2-\sum uA/4s]\frac{ds}{s^{m+1/2}}.$$

§2. Wave propagation in a crystal

Wave propagation in a crystal is governed by Maxwell equations

$$\begin{aligned}&\nabla\times\boldsymbol{E}+\frac{\partial\boldsymbol{B}}{\partial t}=0,\qquad \nabla\cdot\boldsymbol{B}=0\\ &\nabla\times\boldsymbol{H}-\frac{\partial\boldsymbol{D}}{\partial t}=0,\qquad \nabla\cdot\boldsymbol{D}=0\end{aligned}\qquad(5)$$

and $$\boldsymbol{B}=\mu\boldsymbol{H}, \qquad \boldsymbol{D}=\varepsilon\boldsymbol{E}.$$

Where $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{D}, \boldsymbol{B}$ stand for electric field strength, magnetic field strength, electric displacement, magnetic induction respectively, and the permeability μ is assumed to be a constant while the dielectric constant ε is assumed to be a diagonal tensor

$$\varepsilon=\begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}$$

in the crystal with suitably chosen coordinate axes.

We denote the Laplace-Fourier transform of a quantity $A(x, y, z, t)$ by $\tilde{A}(p_1, p_2, p_3, q)$, or

$$\tilde{A}(p_1, p_2, p_3, q)=\int_{-\infty}^{\infty}\int dx\int_0^{\infty} dt A(x, y, z, t)e^{-qt-ip\cdot x}$$

$$(\tilde{A})_0 \qquad =\int_{-\infty}^{\infty}\int dx A(x, y, z, 0)e^{-ip\cdot x}.$$

Then we have Maxwell equations

$$\begin{aligned} P\tilde{\boldsymbol{E}}+\mu q\tilde{\boldsymbol{H}}&=\mu(\tilde{\boldsymbol{H}})_0 \\ P\tilde{\boldsymbol{H}}-\varepsilon q\tilde{\boldsymbol{E}}&=-\varepsilon(\tilde{\boldsymbol{E}})_0, \end{aligned} \qquad P=\begin{pmatrix} 0 & -ip_3 & ip_2 \\ ip_3 & 0 & -ip_1 \\ -ip_2 & ip_1 & 0 \end{pmatrix}, \qquad (6)$$

in terms of Laplace-Fourier transforms while equations $\nabla\cdot\boldsymbol{B}=0$ and $\nabla\cdot\boldsymbol{D}=0$ are satisfied by virtue of these equations if they are satisfied at the initial epoch. We denote the matrix $\mu\varepsilon$ by σ. From (6) we get

$$\tilde{\boldsymbol{E}}=(q^2+\sigma^{-1}P^2)^{-1}\{q(\tilde{\boldsymbol{E}})_0+\mu\sigma^{-1}P(\tilde{\boldsymbol{H}})_0\}$$

$$\tilde{\boldsymbol{H}}=(q^2+P\sigma^{-1}P)^{-1}\left\{q(\tilde{\boldsymbol{H}})_0-\frac{1}{\mu}P(\tilde{\boldsymbol{E}})_0\right\}.$$

Here we note that

$$\det(q^2+\sigma^{-1}P^2)=\det(q^2+P\sigma^{-1}P)=q^2\{q^4+2Uq^2+p^2V\}$$

$$U=\frac{1}{2}\left(\frac{1}{\sigma_2}+\frac{1}{\sigma_3}\right)p_1^2+\frac{1}{2}\left(\frac{1}{\sigma_3}+\frac{1}{\sigma_1}\right)p_2^2+\frac{1}{2}\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}\right)p_3^2,$$

$$V=\frac{p_1^2}{\sigma_2\sigma_3}+\frac{p_2^2}{\sigma_3\sigma_1}+\frac{p_3^2}{\sigma_1\sigma_2}$$

$$p^2=p_1^2+p_2^2+p_3^2$$

and that two inverse matrices $(q^2+\sigma^{-1}P^2)^{-1}$ and $(q^2+P\sigma^{-1}P)^{-1}$ multiplied by det $(q^2+\sigma^{-1}P^2)$ have elements that are all polynomials in q and $\boldsymbol{p}$. We have $\boldsymbol{E}$ and $\boldsymbol{H}$ by an inverse Laplace-Fourier transform of $\tilde{\boldsymbol{E}}$ and $\tilde{\boldsymbol{H}}$ respectively,

$$\boldsymbol{E}=\frac{1}{(2\pi)^4 i}\int_{-\infty}^{\infty}\!\!\int d\boldsymbol{p}\int_L dq\,\tilde{\boldsymbol{E}}e^{qt+i\boldsymbol{p}\cdot\boldsymbol{x}}$$

$$\boldsymbol{H}=\frac{1}{(2\pi)^4 i}\int_{-\infty}^{\infty}\!\!\int d\boldsymbol{p}\int_L dq\,\tilde{\boldsymbol{H}}e^{qt+i\boldsymbol{p}\cdot\boldsymbol{x}}.$$

If we obtain an inverse Laplace-Fourier transform of $1/q^2(q^4+2Uq^2+Vp^2)$, or

$$\frac{1}{(2\pi)^4 i}\int_{-\infty}^{\infty}\!\!\int d\boldsymbol{p}\int_L dq\frac{e^{qt+i\boldsymbol{p}\cdot\boldsymbol{x}}}{q^2\{q^4+2Uq^2+Vp^2\}}\equiv G(\boldsymbol{x},t)$$

then $\boldsymbol{E}$ and $\boldsymbol{H}$ may be got by differentiating $G(\boldsymbol{x},t)$ with respect to $\boldsymbol{x}$ and t.

When two of dielectric constants are equal we can factorize $q^4+2Uq^2+Vp^2$ into two factors each quadratic in $\boldsymbol{p}$. We assume $\varepsilon_1=\varepsilon_2$, then we see that

$$q^4+2Uq^2+Vp^2=(q^2+A)(q^2+B),$$
$$A=(p_1^2+p_2^2+p_3^2)/\sigma_1,\qquad B=(p_1^2+p_2^2)/\sigma_3+p_3^2/\sigma_1$$

and

$$\frac{1}{2\pi i}\int_L\frac{e^{qt}}{q^2(q^4+2Uq^2+Vp^2)}dq$$
$$=\frac{\Gamma(3/2)}{4^2}\frac{1}{2\pi i}\int_0^{\infty}\!\!\int\varepsilon(1-u-v)\,du\,dv\int_L\exp\left[st^2-\frac{uA+vB}{4s}\right]ds\,s^{-7/2}.$$

Therefore we have, integrating with respect to $\boldsymbol{p}$ and s,

$$G(\boldsymbol{x},t)=\frac{1}{32\pi}\int_0^{\infty}\!\!\int\varepsilon(T)T\frac{1}{u/\sigma_1+v/\sigma_3}\sqrt{\frac{\sigma_1}{u+v}}\varepsilon(1-u-v)\,du\,dv,$$

$$T=t^2-\frac{x^2+y^2}{u/\sigma_1+v/\sigma_3}-\frac{\sigma_1 z^2}{u+v}.$$

A change of variables

$$u+v=w$$

$$u/\sigma_1+v/\sigma_3=w/\sigma$$

and the convention $\sigma_>=\mathrm{Max}(\sigma_1,\sigma_3)$, $\sigma_<=\mathrm{Min}(\sigma_1,\sigma_3)$ leads to the range of integration

$$0<w<1,\qquad \sigma_<<\sigma<\sigma_>$$

and

$$G(\boldsymbol{x},t)=\frac{1}{16\pi}\frac{\sqrt{\sigma_1}}{1/\sigma_<-1/\sigma_>}\int_{\sigma_<}^{\sigma_>}\frac{d\sigma}{\sigma}\varepsilon(t-X)(t-X)^2,$$

$$X=\sqrt{\{(x^2+y^2)\sigma+z^2\sigma_1\}}$$

$$=\frac{1}{16\pi}\frac{\sqrt{\sigma_1}}{1/\sigma_< - 1/\sigma_>}\Big[X^2-4tX+(t-\sqrt{\sigma_1 z^2})^2$$

$$\times\log(X-\sqrt{\sigma_1 z^2})+(t+\sqrt{\sigma_1 z^2})^2\log(X+\sqrt{\sigma_1 z^2})\Big]_\alpha^\beta.$$

The bracket $[\ \ ]_\alpha^\beta$ means the difference between the values of the expression within the bracket at $X=\beta$ and $X=\alpha$, where $\beta=\mathrm{Min}(X_>, t)$ and $\alpha=\mathrm{Min}(X_<, t)$.

By the way we add another formula

$$\frac{\partial^2 G(\boldsymbol{x}, t)}{\partial t^2}=\frac{1}{8\pi}\frac{\sqrt{\sigma_1}}{1/\sigma_< - 1/\sigma_>}\int_{\sigma_<}^{\sigma_>}\varepsilon(t-X)\frac{d\sigma}{\sigma}.$$

These integrals may be evaluated by means of elementary functions.

When all three dielectric constants are different from each other, evaluation of the functions $G(x, t)$, $\partial^2 G/\partial t^2$ needs numerical integration as has been shown in Courant-Hilbert's Methods of mathematical physics[2]. Our method differs from that by Courant-Hilbert a little. In this case we assume $\sigma_3>\sigma_2>\sigma_1$ and put

$$\frac{1}{\sigma_2}-\frac{1}{\sigma_3}=2\beta_1, \qquad \frac{1}{\sigma_3}-\frac{1}{\sigma_1}=2\beta_2, \qquad \frac{1}{\sigma_1}-\frac{1}{\sigma_2}=2\beta_3.$$

Then we see $\beta_1>0, \beta_2<0, \beta_3>0$ and

$$U^2-Vp^2=\beta_1^2 p_1^4+\beta_2^2 p_2^4+\beta_3^2 p_3^4-2\beta_2\beta_3 p_2^2 p_3^2-2\beta_3\beta_1 p_3^2 p_1^2-2\beta_1\beta_2 p_1^2 p_2^2$$

$$=(-\beta_1 p_1^2+\beta_2 p_2^2+\beta_3 p_3^2)^2-4\beta_2\beta_3 p_2^2 p_3^2. \qquad (7)$$

Since $\beta_2\beta_3$ is negative, U^2-Vp^2 is positive. Therefore

$$A=U+\surd(U^2-Vp^2) \quad \text{and} \quad B=U-\surd(U^2-Vp^2)$$

are real and positive for real p.

Remembering the formula (4) we may write

$$\frac{1}{2\pi i}\int_L \frac{e^{qt}}{q^2(q^2+A)(q^2+B)}dq$$

$$=\frac{\Gamma(3/2)}{16}\frac{1}{2\pi i}\int_0^\infty\!\!\int\varepsilon(1-u-v)\,du\,dv\int_L \exp\Big[st^2-\frac{uA+vB}{4s}\Big]ds\,s^{-7/2}.$$

A change of variables

$$u+v=w, \qquad u-v=w\rho$$

leads to

$$\int_0^\infty\!\!\int\exp\Big(-\frac{uA+vB}{4s}\Big)\varepsilon(1-u-v)\,du\,dv$$

$$=\frac{1}{2}\int_0^1 w\,dw\int_{-1}^1 d\rho\,\exp[-(U+\rho\surd(U^2-Vp^2))w/4s]. \qquad (\alpha)$$

Here we employ a device for any Z

$$\int_{-1}^{1} e^{-\rho Z} d\rho = \int_{-\infty}^{\infty} \varepsilon(1-\rho^2) e^{-\rho Z} d\rho$$

$$= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_L \exp[\xi(1-\rho^2) - \rho Z] d\rho d\xi / \xi$$

$$= \frac{1}{2\pi i} \int_L \exp[\xi + Z^2/4\xi] \sqrt{\pi} d\xi / \xi^{3/2}$$

and get

$$(\alpha) = \frac{\sqrt{\pi}}{2} \int_0^1 w dw \frac{1}{2\pi i} \int_L$$

$$\times \exp\left[\xi - \frac{wU}{4s} + (U^2 - Vp^2)\left(\frac{w}{4s}\right)^2 \frac{1}{4\xi}\right] d\xi / \xi^{3/2}. \qquad (\beta)$$

We need another device. Remembering the equality (7), we may write

$$\exp\left[(U^2 - Vp^2)\left(\frac{w}{4s}\right)^2 \frac{1}{4\xi}\right]$$

$$= \int_{-\infty}^{\infty} \int du dv \exp\left[-\xi u^2 + (-\beta_1 p_1^2 + \beta_2 p_2^2 + \beta_3 p_3^2) \frac{w}{4s} u - \xi v^2 + 2\gamma p_2 p_3 \frac{w}{4s} v\right] \cdot \xi / \pi$$

$$\gamma = \sqrt{(-\beta_2 \beta_3)}.$$

Inserting this equality in (β) and integrating with respect to ξ we get

$$(\beta) = \frac{1}{2\pi} \int_0^1 w dw \int_{-\infty}^{\infty} \int \frac{\varepsilon(1-u^2-v^2)}{\sqrt{1-u^2-v^2}} du dv$$

$$\times \exp\left[-\frac{w}{4s} \{(\alpha_1 p_1^2 + \alpha_2 p_2^2 + \alpha_3 p_3^2) + (-\beta_1 p_1^2 + \beta_2 p_2^2 + \beta_3 p_3^2) u + 2\gamma p_2 p_3 v\}\right],$$

$$\alpha_j = \frac{1}{2}\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} + \frac{1}{\sigma_3} - \frac{1}{\sigma_j}\right), \quad j = 1, 2, 3.$$

Hence we have, integrating with respect to $\boldsymbol{p}, s, w$ successively,

$$G(\boldsymbol{x}, t) = \frac{1}{(2\pi)^4 i} \int_{-\infty}^{\infty} \int d\boldsymbol{p} \int dq \frac{e^{qt + i\boldsymbol{p}\cdot\boldsymbol{x}}}{q^2(q^4 + 2Uq^2 + Vp^2)}$$

$$= \frac{1}{32\pi^2} \int_{-\infty}^{\infty} \int \frac{\varepsilon(1-u^2-v^2)}{\sqrt{1-u^2-v^2}} \varepsilon(t-X)(t-X)^2$$

$$\times \sqrt{\frac{1}{(\alpha_1 - \beta_1 u)\{(\alpha_2 + \beta_2 u)(\alpha_3 + \beta_3 u) - \gamma^2 v^2\}}} du dv$$

$$X^2 = \frac{x^2}{\alpha_1 - \beta_1 u} + \frac{(\alpha_3 + \beta_3 u) y^2 - 2\gamma v y z + (\alpha_2 + \beta_2 u) z^2}{(\alpha_2 + \beta_2 u)(\alpha_3 + \beta_3 u) - \gamma^2 v^2}.$$

The range of integration here is limited by the condition $t - X > 0$ and $1 - u^2 - v^2 > 0$. The expression of $\partial^2 G / \partial t^2$ may be got by replacing the term $(t-X)^2$ of the integrand by 2.

§ 3. Wave propagation in the presence of a plane boundary.

We take a scalar wave equation

$$\sum_{k=1}^{n} \frac{\partial}{\partial x_k} a(x) \frac{\partial u}{\partial x_k} - c(x) \frac{\partial^2 u}{\partial t^2} = -\delta(t-t') \prod_{k=1}^{n} \delta(x_k - x_k'). \qquad (8)$$

The quantities $a(x)$ and $c(x)$ are assumed to depend on only one coordinate x_1, which will be written x hereafter, dropping the suffix 1.

We assume

$$a(x)=a \quad \text{and} \quad c(x)=c \quad \text{for} \quad x>0$$

and further $$a(x)=a' \quad \text{and} \quad c(x)=c' \quad \text{for} \quad x<0.$$

The boundary condition at $x=0$ is assumed to be the continuity of $k(x)u$ and $h(x)\partial u/\partial x$ where $k(x)$ and $h(x)$ are given as follows

$$k(x)=k, \qquad h(x)=h \qquad \text{for} \quad x>0$$

$$k(x)=k', \qquad h(x)=h' \qquad \text{for} \quad x<0.$$

Another assumption is that

$$x_1' \equiv x' > 0 \quad \text{and} \quad u = \frac{\partial u}{\partial t} = 0 \quad \text{for} \quad t<t'.$$

We denote a Laplace-Fourier transform of u by $\bar{u}$, or

$$\bar{u} = \int_0^\infty dt \int_{-\infty}^{\infty} \int dx_2 \cdots dx_n u \exp[-qt - ip_2 x_2 - \cdots - ip_n x_n].$$

Initial values of u and $\partial u/\partial t$ are zero from the preceding assumption. The equation to be satisfied by $\bar{u}$ becomes then

$$\frac{d}{dx} a(x) \frac{d\bar{u}}{dx} - [c(x)q^2 + a(x)(p_2^2 + \cdots + p_n^2)]\bar{u} = -\delta(x-x')E \qquad (9)$$

$$E = \exp[-qt' - ip_2 x_2' - \cdots - ip_n x_n'].$$

We seek a solution to this equation which satisfies said boundary condition and the condition that the quantity $\bar{u}$ vanishes as $x\to\infty$ and $x\to-\infty$. We set

$$\int_0^\infty \bar{u} e^{-px} dx = \tilde{u}, \qquad \mathcal{R}p>0$$

and

$$\int_{-\infty}^0 \bar{u} e^{-px} dx = \tilde{u}', \qquad \mathcal{R}p<0$$

and assume

$$k(x)\bar{u} = B, \qquad h(x)\partial\bar{u}/\partial x = A$$

at $x=0$, both B and A being unknown quantities. Since $k(x)u(x)$

and $h(x)\partial u/\partial x$ are assumed to be continuous at $x=0, u(x)$ and $\partial u/\partial x$, and also $\bar{u}$ and $d\bar{u}/dx$ may not be continuous. We define two quantities λ and λ' by

$$\sqrt{(p_2^2+\cdots\cdot+p_n^2+cq^2/a)}=\lambda, \qquad \mathcal{R}\lambda>0$$
$$\sqrt{(p_2^2+\cdots\cdot+p_n^2+c'q^2/a')}=\lambda', \qquad \mathcal{R}\lambda'>0$$

both quantities being understood to have positive real parts.

Multiplying both sides of the equation (9) by e^{-px} and integrating either from 0 to ∞ or from $-\infty$ to 0, we get

$$(p^2-\lambda^2)\tilde{u} = -\frac{E}{a}e^{-px'}+\frac{A}{h}+p\frac{B}{k}$$
$$(p^2-\lambda'^2)\tilde{u}' = -\frac{A}{h'}-p\frac{B}{k'}.$$

The condition that $\bar{u}$ vanish as $x\to\infty$ and $\bar{u}$ vanish as $x\to-\infty$ determines unknown quantities A and B. Vanishing of $\bar{u}$ requires that $\tilde{u}$ should have no pole at $p=\lambda$, so that

$$\frac{A}{h}+\lambda\frac{B}{k}-\frac{E}{a}e^{-\lambda x'}=0.$$

Likewise, vanishing of $\bar{u}$ as $x\to-\infty$ leads to the condition that $\tilde{u}$ should have no pole at $p=-\lambda'$, so that

$$\frac{A}{h'}-\lambda'\frac{B}{k'} =0.$$

Hence we have

$$A=\frac{\lambda'\alpha' e^{-\lambda x'}}{\lambda\alpha+\lambda'\alpha'}\frac{Eh}{a}, \quad B=\frac{e^{-\lambda x'}}{\lambda\alpha+\lambda'\alpha'}\frac{Eh}{a}, \quad \alpha=h/k, \quad \alpha'=h'/k'$$

and

$$\tilde{u}=\frac{1}{p^2-\lambda^2}\left\{-e^{-px'}+\frac{\lambda'\alpha'+p\alpha}{\lambda\alpha+\lambda'\alpha'}e^{-\lambda x'}\right\}\frac{E}{a}$$
$$\tilde{u}'=-\frac{1}{p^2-\lambda'^2}\frac{\lambda'+p}{\lambda\alpha+\lambda'\alpha'}e^{-\lambda x'}\frac{Eh}{ak'}$$

and consequently

$$\bar{u}=\left(\frac{e^{-\lambda|x-x'|}-e^{-\lambda x-\lambda x'}}{2\lambda}+\frac{\alpha e^{-\lambda x-\lambda x'}}{\lambda\alpha+\lambda'\alpha'}\right)\frac{E}{a} \qquad x>0$$

$$\bar{u}=\frac{e^{-\lambda x'+\lambda' x}}{\lambda\alpha+\lambda'\alpha'}\frac{Eh}{ak'} \qquad x<0.$$

If we define two functions G and H

$$G(t,y)=\frac{1}{(2\pi)^n i}\int\int dp_2\cdots dp_n\int_L dq\frac{e^{-\lambda y}}{2\lambda}$$

$$\times\exp[qt+ip_2(x_2-x_2')+\cdots\cdot+ip_n(x_n-x_n')]$$

$$H(t,y,y')=\frac{1}{(2\pi)^n i}\iint dp_2\cdots\cdot dp_n\int_L dq\frac{e^{-\lambda y-\lambda' y'}}{\lambda\alpha+\lambda'\alpha'}$$

$$\times\exp[qt+ip_2(x_2-x_2')+\cdots\cdot+ip_n(x_n-x_n')],$$

the solution u may be represented as follows

$$u=\frac{1}{a}\{G(t-t',|x-x'|)-G(t-t',x+x')+\alpha H(t-t',x+x',0)\},\quad x>0$$

$$u=\frac{h}{ak'}H(t-t',x',-x)\qquad x<0.$$

To evaluate G and H, we employ an integral representation of $e^{-\lambda y}/2\lambda$ for $y>0$,

$$\frac{e^{-\lambda y}}{2\lambda}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ipy}}{p^2+\lambda^2}dp,\qquad y>0,\qquad \mathcal{R}\lambda>0,$$

and a formula based on the assumption $\mathcal{R}\lambda>0$ and $\mathcal{R}\lambda'>0$

$$\frac{1}{\lambda\alpha+\lambda'\alpha'}=\int_0^{\infty}e^{-r(\lambda\alpha+\lambda'\alpha')}dr.$$

These formulas lead to the following representation

$$\frac{e^{-\lambda y-\lambda' y'}}{\lambda\alpha+\lambda'\alpha'}=\int_0^{\infty}e^{-\lambda(y+r\alpha)-\lambda'(y'+r\alpha')}dr$$

$$=\frac{\partial^2}{\partial y\partial y'}\frac{1}{\pi^2}\int_0^{\infty}dr\int_{-\infty}^{\infty}\int dp\,dp'\frac{e^{ip(y+r\alpha)+ip'(y'+r\alpha')}}{(p^2+\lambda^2)(p'^2+\lambda'^2)}.$$

We have then

$$G(t,y)=\frac{\Gamma(3/2)}{\pi^{n/2}}\left(\frac{c}{a}\right)^{\frac{n}{2}-1}\delta^{(n-3)/2}\left[t^2-\frac{c}{a}\{y^2+(x_2-x_2')^2+\cdots\cdot+(x_n-x_n')^2\}\right]$$

and

$$H(t,y,y')=\frac{\Gamma(3/2)}{\pi^{(n+1)/2}}\sqrt{\frac{aa'}{cc'}}\frac{\partial^2}{\partial y\partial y'}\int_0^{\infty}dr$$

$$\times\int_0^1\frac{du}{\sqrt{u(1-u)}}\left(\frac{1}{au/c+a'(1-u)/c'}\right)^{\frac{n-1}{2}}\delta^{(n-4)/2}(\mathrm{T}),$$

$$T=t^2-\frac{c(y+r\alpha)^2}{au}-\frac{c'(y'+r\alpha')^2}{a'(1-u)}-\frac{(x_2-x_2')^2+\cdots\cdot+(x_n-x_n')^2}{au/c+a'(1-u)/c'}.$$

The range of integration with respect to r is formally from 0 to ∞, but it is actually limited by the condition $T>0$, since the integral H vanishes when $T<0$.

§ 4. Electromagnetic waves in the presence of a plane boundary

The plane $z=0$ is supposed to separate two media. The medium for $z>0$ has dielectric constant ε, permeability μ and conductivity σ while the medium for $z<0$ has dielectric constant ε', permeability μ' and conductivity σ'. Maxwell equations for $z>0$ are

$$\begin{aligned} &\nabla\times\boldsymbol{E}+\mu\frac{\partial\boldsymbol{H}}{\partial t}=0, \qquad \nabla\cdot(\mu\boldsymbol{H})=0 \\ &\nabla\times\boldsymbol{H}-\varepsilon\frac{\partial\boldsymbol{E}}{\partial t}=\sigma\boldsymbol{E}, \qquad \nabla\cdot(\varepsilon\boldsymbol{E})=0 \end{aligned} \tag{10}$$

Physical quantities for $z<0$ are marked by a prime. Boundary conditions at $z=0$ are known to be[3)]

$$\begin{aligned} \varepsilon E_z&=\varepsilon' E_z', & \mu H_z&=\mu' H_z' \\ E_x&=E_x', & H_x&=H_x' \\ E_y&=E_y', & H_y&=H_y'. \end{aligned}$$

We define here a Laplace-Fourier transform $\tilde{A}$ of a quantity A as

$$\int_0^\infty dt\int_{-\infty}^\infty\int dx\,dy\int_0^\infty dz\,A\exp(-qt-ip_1x-ip_2y-ip_3z)=\tilde{A},$$

$$\mathscr{R}q>0, \qquad \mathscr{I}p_3<0,$$

and likewise

$$\int_0^\infty dt\int_{-\infty}^\infty\int dx\,dy\int_{-\infty}^0 dz\,A'\exp(-qt-ip_1x-ip_2y-ip_3z)=\tilde{A}',$$

$$\mathscr{R}q>0, \qquad \mathscr{I}p_3>0.$$

Further we define $\mathscr{E}$, $\mathscr{H}$ from field quantties $\boldsymbol{E}$ and $\boldsymbol{H}$ at $t=0$ as

$$\int_{-\infty}^\infty\int dx\,dy\int_0^\infty dz(\boldsymbol{E})_{t=0}\exp(-i\boldsymbol{p}\cdot\boldsymbol{x})=\mathscr{E}$$

$$\int_{-\infty}^\infty\int dx\,dy\int_0^\infty dz(\boldsymbol{H})_{t=0}\exp(-i\boldsymbol{p}\cdot\boldsymbol{x})=\mathscr{H}$$

and e, h from $\boldsymbol{E}, \boldsymbol{H}$ at $z=0$ as

$$\int_0^\infty dt\int_{-\infty}^\infty\int dx\,dy(\boldsymbol{E})_{z=0}\exp[-qt-ip_1x-ip_2y]=\boldsymbol{e}$$

$$\int_0^\infty dt\int_{-\infty}^\infty\int dx\,dy(\boldsymbol{H})_{z=0}\exp[-qt-ip_1x-ip_2y]=\boldsymbol{h}.$$

Here we note that $\mathscr{E}$ and $\mathscr{H}$ are known from given initial values of $\boldsymbol{E}$ and $\boldsymbol{H}$ while $\boldsymbol{e}$ and $\boldsymbol{h}$ are unknown since the values of E and H at the boundary can not be given. Similar quantities with a prime are defined for the region $z<0$. We derive then from Maxwell equations (10) the following relations

$$P\tilde{\boldsymbol{E}}+\mu q\tilde{\boldsymbol{H}}=\mu\mathscr{H}+\boldsymbol{a}$$
$$P\tilde{\boldsymbol{H}}-(\varepsilon q+\sigma)\tilde{\boldsymbol{E}}=-\varepsilon\mathscr{E}+\boldsymbol{b}$$

for the region $z>0$, where we use the following symbols

$$P=\begin{pmatrix} 0 & -ip_3 & ip_2 \\ ip_3 & 0 & -ip_1 \\ -ip_2 & ip_1 & 0 \end{pmatrix},\quad \boldsymbol{a}=\begin{pmatrix} -e_y \\ e_x \\ 0 \end{pmatrix},\quad \boldsymbol{b}=\begin{pmatrix} -h_y \\ h_x \\ 0 \end{pmatrix}$$

and

$$P\tilde{\boldsymbol{E}}'+\mu' q\tilde{\boldsymbol{H}}'=\mu'\mathscr{H}'-\boldsymbol{a}$$
$$P\tilde{\boldsymbol{H}}'-(\varepsilon' q+\sigma')\tilde{\boldsymbol{E}}'=-\varepsilon'\mathscr{E}'-\boldsymbol{b}$$

for the region $z<0$. The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are common to said two regions because of the continuity of E_x, E_y, H_x, H_y at the boundary. The difference in the sign stems from the difference in the range of integration with respect to z in $\tilde{A}$ and $\tilde{A}'$.

These relations give

$$\tilde{\boldsymbol{E}}=\frac{1}{\tau+P^2}\{\mu q(\varepsilon\mathscr{E}-\boldsymbol{b})+P(\mu\mathscr{H}+\boldsymbol{a})\}$$

$$\tilde{\boldsymbol{H}}=\frac{1}{\tau+P^2}\{-P(\varepsilon\mathscr{E}-\boldsymbol{b})+(\varepsilon q+\sigma)(\mu\mathscr{H}+\boldsymbol{a})\}$$

$$\tau=\mu q(\varepsilon q+\sigma)$$

and

$$\tilde{\boldsymbol{E}}'=\frac{1}{\tau'+P^2}\{\mu' q(\varepsilon'\mathscr{E}'+\boldsymbol{b})+P(\mu'\mathscr{H}'-\boldsymbol{a})\}$$

$$\tilde{\boldsymbol{H}}'=\frac{1}{\tau'+P^2}\{-P(\varepsilon'\mathscr{E}'+\boldsymbol{b})+(\varepsilon' q+\sigma')(\mu'\mathscr{H}'-\boldsymbol{a})\}$$

$$\tau'=\mu' q(\varepsilon' q+\sigma').$$

To determine two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ we use the condition that the field quantities $\boldsymbol{E}$ and $\boldsymbol{H}$ vanish as $z\to\infty$ and $\boldsymbol{E}'$ and $\boldsymbol{H}'$ vanish as $z\to-\infty$. Then the inverse Fourier transforms

$$\int\tilde{\boldsymbol{E}}\exp(ip_3 z)dp_3,\qquad \int\tilde{\boldsymbol{H}}\exp(ip_3 z)dp_3,\qquad \mathscr{I}p_3<0 \tag{11}$$

and

$$\int\tilde{\boldsymbol{E}}'\exp(ip_3 z)dp_3,\qquad \int\tilde{\boldsymbol{H}}'\exp(ip_3 z)dp_3,\qquad \mathscr{I}p_3>0 \tag{12}$$

should vanish as $z\to\infty$ and $z\to-\infty$ respectively. The inverse of the matrix $\tau+P^2$ turns out to be

$$\frac{1}{\tau+P^2}=\frac{1}{\tau(\tau+p^2)}R$$

$$R=\begin{pmatrix} \tau+p_1^2 & p_1p_2 & p_1p_3 \\ p_2p_1 & \tau+p_2^2 & p_2p_3 \\ p_3p_1 & p_3p_2 & \tau+p_3^2 \end{pmatrix}.$$

The factor $\tau+p^2=\tau+p_1^2+p_2^2+p_3^2$ vanishes at $p_3=\lambda$ and $p_3=-\lambda$,

$$\lambda=\sqrt{(-p_1^2-p_2^2-\tau)}, \qquad \mathscr{I}\lambda>0.$$

For the inverse Fourier transforms (11) to vanish as $z\to\infty$, the integrands $\tilde{\boldsymbol{E}}$ and $\tilde{\boldsymbol{H}}$ should not have a pole at $p_3=-\lambda$, because the term $e^{-i\lambda z}\to\infty$ as $z\to\infty$ due to the convention $\mathscr{I}\lambda>0$. When $\tau+p^2$ vanishes, we see

$$R=\begin{pmatrix} -p_2^2-p_3^2 & p_1p_2 & p_1p_3 \\ p_2p_1 & -p_3^2-p_1^2 & p_2p_3 \\ p_3p_1 & p_3p_2 & -p_1^2-p_2^2 \end{pmatrix}=-P^2$$

so that vanishing of $\boldsymbol{E}$ and $\boldsymbol{H}$ as $z\to\infty$ leads to the conditions

$$P^2\{\mu q(\varepsilon\mathcal{E}-\boldsymbol{b})+P(\mu\mathcal{H}+\boldsymbol{a})\}=0 \qquad \text{for} \quad p_3=-\lambda$$

and

$$P^2\{-P(\varepsilon\mathcal{E}-\boldsymbol{b})+(\varepsilon q+\sigma)(\mu\mathcal{H}+\boldsymbol{a})\}=0 \qquad \text{for} \quad p_3=-\lambda.$$

The matrix P satisfies the equation $P^3=p^2P$ and p^2 is equal to $-\tau$ when $p_3=-\lambda$, so these two conditions turn out to be equivalent each to the other. We choose the second condition which may be written

$$P^2\boldsymbol{a}-\mu qP\boldsymbol{b}+\mu\varepsilon qP\mathcal{E}+\mu P^2\mathcal{H}=0 \qquad \text{for} \quad p_3=-\lambda. \tag{13}$$

For the region $z<0$ we define

$$\lambda'=\sqrt{(-p_1^2-p_2^2-\tau')}, \qquad \mathscr{I}\lambda'>0.$$

The condition that $\boldsymbol{E}'$ and $\boldsymbol{H}'$ vanish as $z\to-\infty$ entails that the numerators of $\tilde{\boldsymbol{E}}'$ and $\tilde{\boldsymbol{H}}'$ should vanish for $p_3=\lambda'$, consequently

$$P^2\boldsymbol{a}-\mu'qP\boldsymbol{b}-\mu'\varepsilon'qP\mathcal{E}'-\mu'P^2\mathcal{H}'=0, \qquad \text{for} \quad p_3=\lambda'. \tag{14}$$

Each of these two vector equations (13), (14) has three components, but the third components of $\boldsymbol{a}$ and $\boldsymbol{b}$ vanish. So we discard the third components of these vector equations and get

$$\begin{pmatrix} p_2^2+\lambda^2 & -p_1p_2 & 0 & -i\mu q\lambda \\ -p_2p_1 & p_1^2+\lambda^2 & i\mu q\lambda & 0 \\ p_2^2+\lambda'^2 & -p_1p_2 & 0 & i\mu'q\lambda' \\ -p_2p_1 & p_1^2+\lambda'^2 & -i\mu'q\lambda' & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix}=\begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{B} \end{pmatrix} \tag{15}$$

where two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are deprived of their third components and

$$\boldsymbol{A}=[-\mu\varepsilon qP\mathcal{E}-\mu P^2\mathcal{H}]_{p_3=-\lambda}$$
$$\boldsymbol{B}=[\mu'\varepsilon'qP\mathcal{E}'+\mu'P^2\mathcal{H}']_{p_3=\lambda'}$$

are similarly deprived of their third components. We denote the matrix on the left side of (15) by Q. The determinant of the matrix Q is computed to be

$$\det Q=q^2\lambda\lambda'(\mu\lambda'+\mu'\lambda)(\tau\mu'\lambda'+\tau'\mu\lambda)$$

and two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are given by

$$\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{pmatrix}=\frac{1}{Q}\begin{pmatrix}\boldsymbol{A}\\ \boldsymbol{B}\end{pmatrix}.$$

Then $\tilde{\boldsymbol{E}},\tilde{\boldsymbol{H}},\tilde{\boldsymbol{E}}',\tilde{\boldsymbol{H}}'$ are fully determined, so that we can establish formal solutions $\boldsymbol{E},\boldsymbol{H}$, and $\boldsymbol{E}',\boldsymbol{H}'$ for given initial values. But the expressions of $\tilde{\boldsymbol{E}},\tilde{\boldsymbol{H}}$ and $\tilde{\boldsymbol{E}}'$, $\tilde{\boldsymbol{H}}'$ depend on λ and λ', which are radicals having their positive imaginary parts. So the computation is very difficult.

We try to bring the formal solutions into a form more easy to compute. The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ depend linearly on $\boldsymbol{A}$ and $\boldsymbol{B}$, which depend, in turn, linearly on initial values of $\boldsymbol{E},\boldsymbol{H},\boldsymbol{E}',\boldsymbol{H}'$. So if we establish the solution to a special initial value where one of field quantities $=\delta(\boldsymbol{x}-\boldsymbol{x}')$ and others vanish, desired solutions may be obtained by a linear superposition of similar solutions. Hence the main task is to obtain an inverse Laplace-Fourier transform of a typical term

$$\frac{1}{\tau(\tau+p^2)}\frac{1}{\det Q}\exp[-ip_1x'-ip_2y'+i\lambda\zeta+i\lambda'\zeta']$$

ζ,ζ' being to be replaced by z' or 0 after integration. Even now, computation is formidable. When conductivities of the media vanish $\sigma=\sigma'=0$, the task may be easier. We are faced then with the evaluation of

$$K=\frac{1}{(2\pi)^4 i}\int d\boldsymbol{p}\int_L dq\frac{e^{qt+i\boldsymbol{p}\cdot\boldsymbol{x}+i\lambda\zeta+i\lambda'\zeta'}}{q^6(q^2+p^2/\mu\varepsilon)(\mu'\lambda+\mu\lambda')(\varepsilon'\lambda+\varepsilon\lambda')\lambda\lambda'}.$$

Since the imaginary parts of λ and λ' are positive, both $\mu'\lambda+\mu\lambda'$ and $\varepsilon'\lambda+\varepsilon\lambda'$ have positive imaginary parts. So we may have the following integral representations

$$\frac{1}{\mu'\lambda+\mu\lambda'}=\frac{1}{i}\int_0^\infty e^{i(\mu'\lambda+\mu\lambda')u}du$$

$$\frac{1}{\varepsilon'\lambda+\varepsilon\lambda'}=\frac{1}{i}\int_0^\infty e^{i(\varepsilon'\lambda+\varepsilon\lambda')v}dv$$

and

$$\frac{e^{i\lambda\zeta+i\lambda'\zeta'}}{(\mu'\lambda+\mu\lambda')(\varepsilon'\lambda+\varepsilon\lambda')\lambda\lambda'}=\frac{1}{\pi^2}\int_0^\infty\int dudv\int_{-\infty}^\infty\int drdr'\frac{e^{ir(\zeta+\mu'u+\varepsilon'v)}}{r^2-\lambda^2}\frac{e^{ir'(\zeta'+\mu u+\varepsilon v)}}{r'^2-\lambda'^2}.$$

With the use of these expressions we have

$$K=\frac{1}{(2\pi)^4 i\pi^2}\frac{1}{\mu\varepsilon\mu'\varepsilon'}\int_0^\infty\int dudv\int_{-\infty}^\infty\int d\boldsymbol{p}drdr'$$

$$\times\int_L dq\frac{\exp[qt+i\boldsymbol{p}\cdot\boldsymbol{x}+ir(\zeta+\mu'u+\varepsilon'v)+ir'(\zeta'+\mu u+\varepsilon v)]}{q^6(q^2+p^2/\mu\varepsilon)(q^2+(r^2+p_1^2+p_2^2)/\mu\varepsilon)(q^2+(r'^2+p_1^2+p_2^2)/\mu'\varepsilon')}.$$

The denominator of the integrand is of the sixth degree in q^2, so we introduce six variables ρ_i, $i=1,\cdots,6$ corresponding to each factor in q^2, with the help of the formula (4), and get

$$K=\frac{\Gamma(3/2)}{4^5(2\pi)^4\pi^2 i}\frac{1}{\mu\varepsilon\mu'\varepsilon'}\int_0^\infty\int dudv\int_{-\infty}^\infty\int d\boldsymbol{p}drdr'\int ds s^{-13/2}\int_0^\infty\int\delta\Big(1-\sum_1^6\rho_k\Big)\prod_1^6 d\rho_k$$

$$\times\exp\Big[st^2-\Big(\frac{p^2}{\mu\varepsilon}\rho_1+\frac{r^2+p_1^2+p_2^2}{\mu\varepsilon}\rho_2+\frac{r'^2+p_1^2+p_2^2}{\mu'\varepsilon'}\rho_3\Big)\Big/4s$$

$$+i\boldsymbol{p}\cdot\boldsymbol{x}+ir(\zeta+\mu'u+\varepsilon'v)+ir'(\zeta'+\mu u+\varepsilon v)\Big].$$

Integration with respect to variables ρ_4,ρ_5,ρ_6 gives

$$\int_0^\infty\int\delta\Big(1-\sum_1^6\rho_k\Big)d\rho_4d\rho_5d\rho_6=\varepsilon(1-\rho_1-\rho_2-\rho_3)(1-\rho_1-\rho_2-\rho_3)^2/2!$$

and integration with respect to $\boldsymbol{p},r,r'$ is immediate. We have then, after integration with respect to s,

$$K=\frac{1}{2^9\pi^2 2!\,3!}\sqrt{\frac{1}{\mu'\varepsilon'}}\int_0^\infty\int dudv\int_0^\infty\int\varepsilon(1-\rho_1-\rho_2-\rho_3)(1-\rho_1-\rho_2-\rho_3)^2$$

$$\times\varepsilon(T)T^3\frac{1}{(\rho_1+\rho_2)/\mu\varepsilon+\rho_3/\mu'\varepsilon'}\sqrt{\frac{1}{\rho_1\rho_2\rho_3}}d\rho_1d\rho_2d\rho_3,$$

$$T=t^2-\frac{x^2+y^2}{(\rho_1+\rho_2)/\mu\varepsilon+\rho_3/\mu'\varepsilon'}-\frac{\mu\varepsilon z^2}{\rho_1}-\frac{\mu\varepsilon(\zeta+\mu'u+\varepsilon'v)^2}{\rho_2}$$

$$-\frac{\mu'\varepsilon'(\zeta'+\mu u+\varepsilon v)^2}{\rho_3}.$$

Then, remaining integration variables are u,v,ρ_1,ρ_2,ρ_3. The presence of $\varepsilon(T)$ and $\varepsilon(1-\rho_1-\rho_2-\rho_3)$ reduces the range of integration to a finite region. Now we change variables from ρ_1,ρ_2,ρ_3 to w,α,β by

$$\rho_1+\rho_2+\rho_3=w,\qquad \rho_1+\rho_2=w\alpha,\qquad \rho_2=w\alpha\beta$$

or

$$\rho_1 = w\alpha(1-\beta), \qquad \rho_2 = w\alpha\beta, \qquad \rho_3 = w(1-\alpha)$$

and get

$$K = \frac{1}{3\cdot 2^{11}\pi^2}\sqrt{\frac{1}{\mu'\varepsilon'}}\int_0^\infty\int du\,dv\int_0^1\int_0^1 \frac{d\alpha}{\sqrt{1-\alpha}}\frac{d\beta}{\sqrt{\beta(1-\beta)}}\frac{1}{\alpha/\mu\varepsilon+(1-\alpha)/\mu'\varepsilon'}$$
$$\times\int_0^1 (1-w)^2\varepsilon(T)T^3\frac{dw}{\sqrt{w}}$$

$$T = t^2 - \frac{1}{w}X^2, \qquad X^2 = \frac{x^2+y^2}{\alpha/\mu\varepsilon+(1-\alpha)/\mu'\varepsilon'} + \frac{\mu\varepsilon z^2}{\alpha(1-\beta)}$$
$$+\frac{\mu\varepsilon(\zeta+\mu' u+\varepsilon' v)^2}{\alpha\beta} + \frac{\mu'\varepsilon'(\zeta'+\mu u+\varepsilon v)^2}{1-\alpha}.$$

Integration with respect to w yields

$$\int_0^1 (1-w)^2\varepsilon(T)T^3 dw/\sqrt{w} = 16\varepsilon(t-X)(t-X)^6/15$$

consequently

$$K = \frac{1}{2^7\cdot 45\pi^2}\sqrt{\frac{1}{\mu'\varepsilon'}}\int_0^\infty\int du\,dv\int_0^1\int \frac{d\alpha}{\sqrt{1-\alpha}}\frac{d\beta}{\sqrt{\beta(1-\beta)}}\frac{1}{\alpha/\mu\varepsilon+(1-\alpha)/\mu'\varepsilon'}$$
$$\times\varepsilon(t-X)(t-X)^6$$

the range of integration with respect to u and v being limited by the condition $t-X>0$ in the first quadrant of the u-v plane.

§ 5. Elastic waves in an isotropic medium with a free plane boundary

We suppose the region $z>0$ is filled with an isotropic medium and no stress is applied to the boundary $z=0$. We denote the displacement of a point by $(u,v,w)=\boldsymbol{v}$ and assume the equations of motion to be

$$\rho\frac{\partial^2 \boldsymbol{v}}{\partial t^2} - \mu\Delta\boldsymbol{v} - (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{v}) = \boldsymbol{f}$$

where ρ, λ, μ denote the density and the Lamé constants respectively, $\boldsymbol{f}$ standing for the body force. No stress at $z=0$ entails

$$\mu\Big(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\Big)=0,\ \ \mu\Big(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\Big)=0,\ \ \lambda\Big(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\Big)+2\mu\frac{\partial w}{\partial z}=0 \text{ at } z=0.$$

In the following we abbreviate $\mu/\rho=\alpha$, $(\lambda+\mu)/\rho=\beta$.

Solutions to the initial value problem of the equations with the body force may be constructed with the aid of the matrix Green function satisfying the following equation

$$\Big[\frac{\partial^2}{\partial t^2} - \alpha\Delta - \beta D\Big]G = 1\cdot\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t') \tag{16}$$

$$D=\begin{pmatrix} \dfrac{\partial^2}{\partial x^2} & \dfrac{\partial^2}{\partial x\partial y} & \dfrac{\partial^2}{\partial x\partial z} \\ \dfrac{\partial^2}{\partial y\partial x} & \dfrac{\partial^2}{\partial y^2} & \dfrac{\partial^2}{\partial y\partial z} \\ \dfrac{\partial^2}{\partial z\partial x} & \dfrac{\partial^2}{\partial z\partial y} & \dfrac{\partial^2}{\partial z^2} \end{pmatrix}$$

and the condition

$$G=0 \qquad \text{for} \qquad t<t'$$

and G vanish as $z\to\infty$.

We denote the first column of the matrix G by (U, V, W), which should satisfy

$$\left.\begin{aligned} &\frac{\partial^2 U}{\partial t^2}-\alpha\Delta U-\beta\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}\right)=\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t') \\ &\frac{\partial^2 V}{\partial t^2}-\alpha\Delta V-\beta\frac{\partial}{\partial y}\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}\right)=0 \\ &\frac{\partial^2 W}{\partial t^2}-\alpha\Delta W-\beta\frac{\partial}{\partial z}\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}\right)=0 \end{aligned}\right\} \tag{17}$$

and the boundary conditions

$$\frac{\partial U}{\partial z}+\frac{\partial W}{\partial x}=0,\ \frac{\partial V}{\partial z}+\frac{\partial W}{\partial y}=0,\ (\beta-\alpha)\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}\right)+(\beta+\alpha)\frac{\partial W}{\partial z}=0$$

at $z=0$.

We introduce a Laplace-Fourier transform $\bar{A}$ of a quantity A by

$$\bar{A}=\int_0^\infty dt\int_{-\infty}^\infty\int dx\,dy\,A\exp(-qt-ip_1x-ip_2y)$$

and rewrite (17) as

$$(q^2+\alpha p_1^2+\alpha p_2^2)\bar{U}-\alpha\frac{\partial^2\bar{U}}{\partial z^2}+\beta p_1(p_1\bar{U}+p_2\bar{V})-i\beta p_1\frac{\partial\bar{W}}{\partial z}$$

$$=\delta(z-z')\exp(-ip_1x'-ip_2y'-qt')$$

$$(q^2+\alpha p_1^2+\alpha p_2^2)\bar{V}-\alpha\frac{\partial^2\bar{V}}{\partial z^2}+\beta p_2(p_1\bar{U}+p_2\bar{V})-i\beta p_2\frac{\partial\bar{W}}{\partial z}=0$$

$$(q^2+\alpha p_1^2+\alpha p_2^2)\bar{W}\qquad -i\beta\frac{\partial}{\partial z}(p_1\bar{U}+p_2\bar{V})-(\alpha+\beta)\frac{\partial^2\bar{W}}{\partial z^2}=0$$

remembering $U=V=W=0$ for $t<t'$. The boundary conditions become

$$\frac{\partial\bar{U}}{\partial z}+ip_1\bar{W}=0,\ \frac{\partial\bar{V}}{\partial z}+ip_2\bar{W}=0,\ i(\beta-\alpha)(p_1\bar{U}+p_2\bar{V})+(\beta+\alpha)\frac{\partial\bar{W}}{\partial z}=0$$

at $z=0$. We set

$$(\bar{U})_{z=0}=ia, \qquad (\bar{V})_{z=0}=ib, \qquad (\bar{W})_{z=0}=ic$$

then we see

$$\left(\frac{\partial \bar{U}}{\partial z}\right)_{z=0}=p_1c,\ \left(\frac{\partial \bar{V}}{\partial z}\right)_{z=0}=p_2c,\ \left(\frac{\partial \bar{W}}{\partial z}\right)_{z=0}=\frac{\beta-\alpha}{\beta+\alpha}(p_1a+p_2b).$$

Further we introduce a Fourier transform $\tilde{A}$ of $\bar{A}$ by

$$\tilde{A}=\int_0^\infty \bar{A}\exp[-ip_3z]\,dz, \qquad \mathcal{I}p_3<0$$

then we see

$$\int_0^\infty \frac{\partial \bar{U}}{\partial z}\exp(-ip_3z)\,dz=-ia+ip_3\tilde{U}$$

$$\int_0^\infty \frac{\partial^2 \bar{U}}{\partial z^2}\exp(-ip_3z)\,dz=-p_1c+p_3a-p_3^2\tilde{U}$$

$$\int_0^\infty \frac{\partial \bar{V}}{\partial z}\exp(-ip_3z)\,dz=-ib+ip_3\tilde{V}$$

$$\int_0^\infty \frac{\partial^2 \bar{V}}{\partial z^2}\exp(-ip_3z)\,dz=-p_2c+p_3b-p_3^2\tilde{V}$$

$$\int_0^\infty \frac{\partial \bar{W}}{\partial z}\exp(-ip_3z)\,dz=-ic+ip_3\tilde{W}$$

$$\int_0^\infty \frac{\partial^2 \bar{W}}{\partial z^2}\exp(-ip_3z)\,dz=-\frac{\beta-\alpha}{\beta+\alpha}(p_1a+p_2b)+p_3c-p_3^2\tilde{W}$$

consequently

$$\begin{pmatrix} q^2+\alpha p^2+\beta p_1^2 & \beta p_1p_2 & \beta p_1p_3 \\ \beta p_2p_1 & q^2+\alpha p^2+\beta p_2^2 & \beta p_2p_3 \\ \beta p_3p_1 & \beta p_3p_2 & q^2+\alpha p^2+\beta p_3^2 \end{pmatrix}\begin{pmatrix}\tilde{U}\\ \tilde{V}\\ \tilde{W}\end{pmatrix}$$

$$=\begin{pmatrix} \alpha p_3 & 0 & (\beta-\alpha)p_1 \\ 0 & \alpha p_3 & (\beta-\alpha)p_2 \\ \alpha p_1 & \alpha p_2 & (\alpha+\beta)p_3 \end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}+\begin{pmatrix}E\\ 0\\ 0\end{pmatrix} \tag{18}$$

$$p^2=p_1^2+p_2^2+p_3^2, \qquad E=\exp(-qt'-ip_1x'-ip_2y'-ip_3z').$$

We denote the matrix on the left side by S and the matrix on the right side by Y, and represent (18) as follows

$$S\begin{pmatrix}\tilde{U}\\ \tilde{V}\\ \tilde{W}\end{pmatrix}=Y\begin{pmatrix}a\\ b\\ c\end{pmatrix}+\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}E.$$

The vector (a, b, c) is unknown. To determine the vector, we use the condition that U, V, W vanish as $z\to\infty$. If we write $q^2+\alpha p^2 = s$, then we have $\det S=s^2(s+\beta p^2)$ and

$$\frac{1}{S}=\frac{1}{s(s+\beta p^2)}R, \qquad R=\begin{pmatrix} s+\beta p^2-\beta p_1^2 & -\beta p_1 p_2 & -\beta p_1 p_3 \\ -\beta p_2 p_1 & s+\beta p^2-\beta p_2^2 & -\beta p_2 p_3 \\ -\beta p_3 p_1 & -\beta p_3 p_2 & s+\beta p^2-\beta p_3^2 \end{pmatrix},$$

$$\begin{pmatrix} \widetilde{U} \\ \widetilde{V} \\ \widetilde{W} \end{pmatrix}=\frac{1}{s(s+\beta p^2)}R\left\{Y\begin{pmatrix} a \\ b \\ c \end{pmatrix}+\begin{pmatrix} E \\ 0 \\ 0 \end{pmatrix}\right\}.$$

The factor $s=q^2+\alpha(p_1^2+p_2^2+p_3^2)$ vanishes at

$$p_3=\kappa \text{ and } p_3=-\kappa, \qquad \kappa=\surd(-p_1^2-p_2^2-q^2/\alpha), \qquad \mathcal{I}\kappa>0$$

and the factor $s+\beta p^2=q^2+(\alpha+\beta)(p_1^2+p_2^2+p_3^2)$ vanishes at

$$p_3=\kappa' \text{ and } p_3=-\kappa', \quad \kappa'=\surd(-p_1^2-p_2^2-q^2/(\alpha+\beta)), \quad \mathcal{I}\kappa'>0$$

where both κ and κ' are chosen so as to have positive imaginary parts. So, for $\bar{U}=\int \widetilde{U}\exp(ip_3 z)dp_3/2\pi$ to vanish as $z\to\infty$, $\widetilde{U}$ should not have a pole at $p_3=-\kappa$ and at $p_3=-\kappa'$, since the poles at these points give rise to terms diverging as $z\to\infty$. Therefore we have the following conditions

$$R(-\kappa)\left\{Y(-\kappa)\begin{pmatrix} a \\ b \\ c \end{pmatrix}+\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}E(-\kappa)\right\}=0 \tag{19}$$

$$R(-\kappa')\left\{Y(-\kappa')\begin{pmatrix} a \\ b \\ c \end{pmatrix}+\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}E(-\kappa')\right\}=0 \tag{20}$$

where $R(-\kappa)$ means the value of R when $p_3=-\kappa$. When $s=0$, the matrix R is of rank 2, so we use the first and second rows of R in (19). When $s+\beta p^2=0$, the matrix R is of rank 1, so we use only the third row of R in (20). Combining these three rows, we have the relation to determine the vector (a, b, c)

$$Q\begin{pmatrix} a \\ b \\ c \end{pmatrix}-FZ\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}=0, \quad Q=\begin{pmatrix} q^2+2\alpha p_1^2 & 2\alpha p_1 p_2 & -2\alpha p_1\kappa \\ 2\alpha p_1 p_2 & q^2+2\alpha p_2^2 & -2\alpha p_2\kappa \\ 2\alpha p_1\kappa' & 2\alpha p_2\kappa' & q^2+2\alpha(p_1^2+p_2^2) \end{pmatrix},$$

$$F=\begin{pmatrix} E(-\kappa)/\kappa & 0 & 0 \\ 0 & E(-\kappa)/\kappa & 0 \\ 0 & 0 & E(-\kappa')/\kappa' \end{pmatrix},$$

$$Z=\begin{pmatrix} q^2/\alpha+p_1^2 & p_1p_2 & -p_1\kappa \\ p_2p_1 & q^2/\alpha+p_2^2 & -p_2\kappa \\ p_1\kappa' & p_2\kappa' & q^2/(\alpha+\beta)+p_1^2+p_2^2 \end{pmatrix}.$$

So we have

$$\begin{pmatrix} \widetilde{U} \\ \widetilde{V} \\ \widetilde{W} \end{pmatrix}=\frac{1}{s(s+\beta p^2)}R\left\{Y\frac{1}{Q}FZ+1\cdot E\right\}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

1 : unit matrix.

In a similar manner we have

$$\widetilde{G}=\frac{1}{s(s+\beta p^2)}R\left\{Y\frac{1}{Q}FZ+1\cdot E\right\}.$$

The solution G will be given by an inverse Laplace-Fourier transform

$$G=\frac{1}{(2\pi)^3 2\pi i}\int_{-\infty}^{\infty}\int d\boldsymbol{p}\int_L dq\widetilde{G}\exp\left[qt+i\boldsymbol{p}\cdot\boldsymbol{x}\right].$$

Integration, however, is very difficult. At first we compute the determinant of the matrix Q to be

$$\det Q=q^2\{q^4+2fq^2+f^2+2\alpha f\kappa\kappa'\},\quad f=2\alpha(p_1{}^2+p_2{}^2).$$

So the determinant of Q involves κ and κ', each of which is a radical of a quadratic form in q, p_1 and p_2. The presence of radicals in the denominator of $\widetilde{G}$ is awkward. To get rid of radicals from the denominator we multiply $\det Q$ by its second factor with the sign of $2\alpha f\kappa\kappa'$ changed and have

$$\begin{aligned}&\det Q\cdot\{(q^2+f)^2-2\alpha f\kappa\kappa'\}\\ &=q^4\{q^6+4fq^4+(2\alpha+6\beta)f^2q^2/(\alpha+\beta)+2\beta f^3/(\alpha+\beta)\}\\ &=q^4\{q^6+4fq^4+(2+2/(1-\sigma))f^2q^2+f^3/(1-\sigma)\}\end{aligned}$$

remembering the relation $2\beta/(\alpha+\beta)=1/(1-\sigma)$, σ here denoting the Poisson ratio. The cubic equation in $x=q^2/f$

$$x^3+4x^2+(2+2/(1-\sigma))x+1/(1-\sigma)=0$$

has three negative roots for $0<\sigma<\sigma_c=0.26308$[4] and one negative and two complex conjugate roots for $\sigma>\sigma_c$. We assume here $\sigma<\sigma_c$ and denote three negative roots by $-\nu_1$, $-\nu_2$, $-\nu_3$, and rationalize the denominator of $1/\det Q$

$$\frac{1}{\det Q}=\frac{q^4+2fq^2+f^2-2\alpha f\kappa\kappa'}{q^4(q^2+\nu_1 f)(q^2+\nu_2 f)(q^2+\nu_3 f)}$$

so that evaluation of G needs evaluation of the integral K

$$K(t,x,y,z,\zeta,\zeta')=\frac{1}{(2\pi)^4 i}\iint d\boldsymbol{p}\int_L dq\cdot J\cdot N\cdot M$$

$$J=\frac{\exp[qt+i\boldsymbol{p}\cdot\boldsymbol{x}]}{(q^2+\alpha p^2)(q^2+(\alpha+\beta)p^2)}$$

$$N=1/q^4(q^2+\nu_1 f)(q^2+\nu_2 f)(q^2+\nu_3 f)$$

$$M=\frac{e^{i\kappa\zeta}}{\kappa}\cdot\frac{e^{i\kappa'\zeta'}}{\kappa'}$$

and its differenciation with respect to six variables t, x, y, z, ζ, ζ'. When ζ and ζ' are positive we use the following representations

$$\frac{e^{i\kappa\zeta}}{\kappa}=\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{e^{ir\zeta}}{r^2-\kappa^2}dr=\frac{\alpha}{\pi i}\int_{-\infty}^{\infty}\frac{e^{ir\zeta}}{q^2+\alpha(p_1^2+p_2^2+r^2)}dr$$

$$\frac{e^{i\kappa'\zeta'}}{\kappa'}=\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{e^{ir'\zeta'}}{r'^2-\kappa'^2}dr'=\frac{\alpha+\beta}{\pi i}\int_{-\infty}^{\infty}\frac{e^{ir'\zeta'}}{q^2+(\alpha+\beta)(p_1^2+p_2^2+r'^2)}dr'.$$

Then the denominator of the integrand of K has nine factors, each quadratic in q. So allotting nine variables $\lambda_i, i=1,2,\cdots,9$ to nine factors and employing the formula (4), we have

$$K=\frac{\Gamma(3/2)}{(2\pi)^4 i4^8}\frac{\alpha(\alpha+\beta)}{(\pi i)^2}\int_0^{\infty}\int\delta\Big(1-\sum_1^9\lambda_k\Big)\Pi d\lambda_k\int_L ds/s^{19/2}\cdot H$$

$$H=\iint_{-\infty}^{\infty}\int dr\,dr'\,d\boldsymbol{p}\exp\Big[st^2-\frac{L}{4s}+i\boldsymbol{p}\cdot\boldsymbol{x}+ir\zeta+ir'\zeta'\Big]$$

where we put

$$L=\alpha p^2\lambda_1+(\alpha+\beta)p^2\lambda_2+\alpha(p_1^2+p_2^2+r^2)\lambda_3+(\alpha+\beta)(p_1^2+p_2^2+r'^2)\lambda_4$$
$$+2\alpha(p_1^2+p_2^2)(\nu_1\lambda_5+\nu_2\lambda_6+\nu_3\lambda_7)$$
$$\equiv(p_1^2+p_2^2)\rho_1+p_3^2\rho_2+r^2\rho_3+r'^2\rho_4$$
$$\rho_1=\alpha\lambda_1+(\alpha+\beta)\lambda_2+\alpha\lambda_3+(\alpha+\beta)\lambda_4+2\alpha(\nu_1\lambda_5+\nu_2\lambda_6+\nu_3\lambda_7)$$
$$\rho_2=\alpha\lambda_1+(\alpha+\beta)\lambda_2$$
$$\rho_3=\alpha\lambda_3$$
$$\rho_4=(\alpha+\beta)\lambda_4.$$

Integration with respect to variables $\boldsymbol{p}, r$ and r' gives

$$H=(4\pi s)^{5/2}\exp(sT)\cdot\frac{1}{\rho_1}\sqrt{\frac{1}{\rho_2\rho_3\rho_4}},$$

$$T=t^2-\frac{x^2+y^2}{\rho_1}-\frac{z^2}{\rho_2}-\frac{\zeta^2}{\rho_3}-\frac{\zeta'^2}{\rho_4}.$$

Integration with respect to s gives

$$\frac{1}{2\pi i}\int_L \exp sT\cdot ds\,s^{-7}=\varepsilon(T)T^6/6!.$$

Remaining nine variables $\lambda_i, i=1, \cdots, 9$ may be reduced to ρ_1, ρ_2, ρ_3, ρ_4 as follows. For any function $A(\lambda)$ we see

$$\int_0^\infty \int \delta\Big(1-\sum_1^9 \lambda_k\Big)\Pi d\lambda_k A(\lambda)=\int_0^\infty \int \delta(1-\sum\lambda_k)\Pi d\lambda_k A\int_0^\infty \cdots \int_0^\infty$$

$$\times \delta[\rho_1-\alpha\lambda_1-(\alpha+\beta)\lambda_2-\alpha\lambda_3-(\alpha+\beta)\lambda_4-2\alpha(\nu_1\lambda_5+\nu_2\lambda_6+\nu_3\lambda_7)]d\rho_1$$

$$\times \delta(\rho_2-\alpha\lambda_1-(\alpha+\beta)\lambda_2)d\rho_2\delta(\rho_3-\alpha\lambda_3)d\rho_3\delta(\rho_4-(\alpha+\beta)\lambda_4)d\rho_4$$

$$=\int_0^\infty \int d\rho_1 d\rho_2 d\rho_3 d\rho_4 AW(\rho_1, \rho_2, \rho_3, \rho_4)\varepsilon(\rho),$$

$$\rho=\rho_1-\rho_2-\rho_3-\rho_4,$$

$$W(\rho_1, \rho_2, \rho_3, \rho_4)=\frac{1}{2\alpha^2\beta(\alpha+\beta)}\frac{1}{4!}\sum_{\text{cycl.}}\frac{\nu_k}{(\nu_k-\nu_i)(\nu_k-\nu_j)}$$

$$\times\left\{\left[\left[1-\frac{\rho_3}{\alpha}-\frac{\rho_4}{\alpha+\beta}-\frac{\rho_2}{\alpha+\beta}-\frac{\rho}{2\alpha\nu_k}\right]\right]^4-\left[\left[1-\frac{\rho_3}{\alpha}-\frac{\rho_4}{\alpha+\beta}-\frac{\rho_2}{\alpha}-\frac{\rho}{2\alpha\nu_k}\right]\right]^4\right\},$$

where the symbol $[[x]]$ is meant to denote $\varepsilon(x)x$ and $\sum$ means the sum over three cyclic permutations of $(k, i, j)=(1, 2, 3)$. Finally we have

$$K=-\frac{1}{2^{15}\pi^2}\frac{\alpha(\alpha+\beta)}{6!}\int_0^\infty \int d\rho_1 d\rho_2 d\rho_3 d\rho_4 T^6\varepsilon(T)\varepsilon(\rho)W/\rho_1(\rho_2\rho_3\rho_4)^{1/2}.$$

The range of integration with respect to $\rho_1, \rho_2, \rho_3, \rho_4$ is formally 0 to ∞, but it is actually confined to a finite region by virtue of the factor $\delta(1-\sum\lambda_i)$, further limited by $T>0$ and $\rho=\rho_1-\rho_2-\rho_3-\rho_4>0$.

When the Poisson ratio σ exceeds the critical value σ_c, two of ν_1, ν_2, ν_3 become complex conjugates, and evaluation of K becomes difficult.

§ 6. Remarks

Techniques used for solving some wave equations in physics in the present paper are

1) Use of integral transforms. This technique enables to combine wave equations with initial values and boundary values into united equations, as is well known.

2) Use of integral representations of $\sin pt/p$, $e^{-\lambda y}/\lambda$ etc. This technique makes some integrations easy. In particular, use of the integral representation of $\sin pt/p$ leads to the appearance of discontinuous factors, which is inevitable in the process of wave propagation.

3) Use of the formula (4). The formula allows straightfoward calculations of some Fourier transforms. However, when the deno-

minator cannot be factorized rationally, more powerful techniques need to be invented.

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