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On an Integration Called Dirichlet Totalization

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Introduction. This is a continuation of our recent papers [1] to [3]. We shall be concerned with a new integration which resembles the powerwise integration in the wide sense and which will be called the Dirichlet totalization. The reason for introducing this integration is the fact that we can establish an integration by parts theorem for the new integral, whereas it seems difficult to obtain an analogous result for the powerwise integral in the wide sense. The validity of such a theorem is plainly a desirable property of any integration theory. On the other hand, the connection between the two integrations is not elucidated yet.

§ 1. Dirichlet totalization.

By a *function*, by itself, we shall mean any mapping of the real line $\boldsymbol{R}$ into $\boldsymbol{R}$, unless explicitly stated to the contrary. The increment of a function $\varphi(x)$ over the closure of an open interval H will be denoted by $\varphi(H)$. The letter p will always signify a generic real number >1. We shall write $\xi\square^{p}$ for $|\xi|^{p}\operatorname{sgn}\xi$, where ξ is any real number. All sets considered in this paper will be linear, i.e. contained in the real line. A set $(\mathfrak{F}_\sigma)$ will be called *sigma-closed set* in this paper.

Let $\varphi(x)$ be a function and Q a closed set. We begin by defining two quantities $\Lambda(\varphi;p;Q)$ and $\Upsilon(\varphi;p;Q)$ as follows. If Q is connected, then these quantities are to mean zero. If Q is not connected, we denote by H a generic open interval contiguous to Q and we write by definition

$$\Lambda(\varphi;p;Q)=\sum_{H}|\varphi(H)|^{p} \quad \text{and} \quad \Upsilon(\varphi;p;Q)=\sum_{H}\varphi(H)\square^{p},$$

where we understand $\Upsilon(\varphi;p;Q)$ to have a meaning when and only when the series $\sum\varphi(H)\square^{p}$ converges absolutely. This last condition comes to the same thing as $\Lambda(\varphi;p;Q)<+\infty$.

Given a function $\varphi(x)$ and a set Q, we shall call Q to be a *Dirichlet set* for the function $\varphi(x)$, if Q is closed and if the following four conditions are fulfilled:

(i) $\Lambda(\varphi;p;Q)$ is finite for every p (subject to the condition $p>1$);

(ii) we have $\Lambda(\varphi;p;Q)=o\left(\dfrac{1}{p-1}\right)$ as $p\to 1$;

(iii) if the end points of a closed interval A belong to Q, then

$$\Upsilon(\varphi;p;Q\frown A)\longrightarrow\varphi(A)\quad\text{as}\quad p\to 1;$$

(iv) there exists a constant $\delta>0$ such that we have

$$|\Upsilon(\varphi;p;Q\frown A)|<\delta^{-1}\qquad\text{for every}\quad p<1+\delta$$

and for every closed interval A.

A few remarks are appropriate to these conditions. The interval A is kept fixed in condition (iii), while it varies arbitrarily in (iv). The quantity $\Upsilon(\varphi;p;Q\frown A)$ appearing above has a meaning; in fact, every open interval (if existent) contiguous to the closed set $Q\frown A$ is at the same time contiguous to the set Q, so that $\Lambda(\varphi;p;Q\frown A)<+\infty$ by condition (i). Again, condition (iii) shows that $\varphi(x)$ is a constant over each closed interval (if existent) contained in Q. Further, every closed set on which a function $\varphi(x)$ is a constant, is a Dirichlet set for $\varphi(x)$.

THEOREM 1. *If Q is a Dirichlet set for each of two given functions, then Q is so also for any linear combination, with constant coefficients, of these functions.*

PROOF. This follows immediately from Lemma 1 of [1], according to which we have the inequalities $|\alpha+\beta|^p\leqq 2^{p-1}(|\alpha|^p+|\beta|^p)$ and

$$|(\alpha+\beta)\square^{p}-\alpha\square^{p}-\beta\square^{p}|\leqq 2^{p-1}(p-1)(|\alpha|^p+|\beta|^p)$$

for any two real numbers α and β.

A function $\varphi(x)$ will be termed *Dirichlet admissible* on a set E, if the function is continuous on E and if there corresponds to each closed set $Q\subset E$ a function $\psi(x;Q)$ approximately derivable to zero at almost all points of Q, such that Q is a Dirichlet set for $\psi(x;Q)$ and that the difference $\varphi(x)-\psi(x;Q)$ is absolutely continuous on Q.

Plainly this property of a function is hereditary, i. e. *a function which is Dirichlet admissible on a set is necessarily so also on every subset of this set.* On the other hand, *any function which is* AC *on a set, is Dirichlet admissible on this set.*

THEOREM 2. *Every function $\varphi(x)$ which is Dirichlet admissible on a measurable set E is* AD *at almost all points of E.*

PROOF. By a well-known theorem, the set E contains a null set S

such that $E\setminus S$ is expressible as the union of an infinite sequence of closed sets, say $\langle Q_n; n\in \boldsymbol{N}\rangle$. With the same notation as above, the function $\psi(x;Q_n)$ is, for each n, AD to 0 at almost all points of Q_n. On the other hand, the function $\varphi(x)-\psi(x;Q_n)$, which is AC on Q_n, must be AD at almost all points of Q_n, on account of the Denjoy-Khintchine Theorem (see Saks [5], p. 222). Consequently $\varphi(x)$ is AD at almost all points of Q_n, and hence at almost all points of the union $Q_1\cup Q_2\cup\cdots=E\setminus S$. This completes the proof, since S is a null set.

We shall say that a function is *generalized Dirichlet admissible,* or briefly GD *admissible,* on a set Z, if the function is continuous on Z and if Z is expressible as the union of a sequence of closed sets on each of which the function is Dirichlet admissible.

When this is the case, the set Z must be sigma-closed and the function is GD *admissible on every sigma-closed set contained in Z.* Again, *a function which is* GD *admissible on a set, is necessarily* AD *at almost all points of this set,* as we find at once by means of Theorem 2. On the other hand, *any function which is* GAC *on a sigma-closed set, is* GD *admissible on this set.*

THEOREM 3. *Every linear combination of two functions Dirichlet* [*or generalized Dirichlet*] *admissible on a set Z, is itself so on this set.*

THEOREM 4. *In order that a function which is continuous on a nonvoid closed set Z be generalized Dirichlet admissible on Z, it is necessary and sufficient that every nonvoid closed subset of Z contain a portion on which the function is Dirichlet admissible.*

Of the above two theorems, the former is immediate from Theorem 1 and the latter admits a proof quite similar to that for Theorem (9.1) on p. 233 of Saks [5]. As to the latter theorem we remark that *a portion of a closed set is necessarily sigma-closed.*

Inspecting the proof of Theorem 3 of [1], we get easily the following

THEOREM 5. *If a function is* BV *on a closed set which is a Dirichlet set for the function, then the function is* AC *on this set.*

THEOREM 6. *A function $\varphi(x)$ which is generalized Dirichlet admissible and* GBV *on a set Z, is necessarily* GAC *on Z. If in addition the set Z is compact and the fnnction is* BV *on Z, then it is* AC *on Z.*

PROOF. The second part of the assertion follows in a routine way from the first, to which we may therefore confine ourselves.

Since $\varphi(x)$ is GD admissible on Z, this set is the union of a sequence of closed sets on each of which $\varphi(x)$ is Dirichlet admissible. It thus suffices to ascertain that $\varphi(x)$ is GAC on each constituent E of this sequence. We shall keep E fixed in what follows.

The function $\varphi(x)$, which is GBV on Z, is so also on E. Hence E is the union of a sequence of bounded sets on each of which $\varphi(x)$ is BV. The proof is thus reduced to showing that $\varphi(x)$ is AC on each set M appearing in this sequence. We shall keep M fixed in the sequel.

The function $\varphi(x)$ is continuous on Z and hence on E, which is a closed set. We may therefore assume M closed. Then there is a function $\psi(x)$ for which M is a Dirichlet set and whose difference from $\varphi(x)$ is AC on M. Since $\varphi(x)$ is BV on M, so is also the function $\psi(x)$. Consequently $\psi(x)$, and hence $\varphi(x)$ also, is AC on M, on account of the preceding theorem. This completes the proof.

THEOREM 7. *Given a function $F(x)$ which is* AC *on a compact set Q and given a function $G(x)$ for which Q is a Dirichlet set, suppose that $F(x)$ is approximately derivable to $F'_{\text{ap}}(x)\geqq 0$ at almost every point of Q and that the sum $\xi(x)=F(x)+G(x)$ has a nonnegative increment over each closed interval (if existent) contiguous to Q.*

Then $\xi(x)$ is monotone nondecreasing over Q.

This may be established as in Theorem 6 of [3]. But the proof of the following theorem is somewhat different from that of the corresponding Theorem 8 of [3].

THEOREM 8. *Given two functions $\varphi(x)$ and $\psi(x)$ which are generalized Dirichlet admissible on a closed interval I, if we have $\varphi'_{\text{ap}}(x)\leqq\psi'_{\text{ap}}(x)$ at almost every point x of I at which both the functions are* AD, *then the difference $\psi(x)-\varphi(x)$ is* AC *and nondecreasing, on the interval I.*

Consequently, if two functions are generalized Dirichlet admissible on a closed interval I and if they are approximately equiderivable almost everywhere on I, then the functions differ over I only by an additive constant.

PROOF. The function $\xi(x)=\psi(x)-\varphi(x)$ is GD admissible on I by Theorem 3 and possesses a finite approximate derivative $\xi'_{\text{ap}}(x)\geqq 0$ almost everywhere on I.

Let us define a subset S of I as follows: a point of I belongs to S if and only if there is no open interval V containing this point and such that the function $\xi(x)$ is GAC on the interval $I\cap V$. We find, by Theorem 7 of [3], that S is a perfect set and that $\xi(x)$ is both GAC and nondecreasing on each finite interval contained in I and disjoint with S. It thus suffices to show that S is void; in fact, the function $\xi(x)$ will then be GAC and nondecreasing on the whole interval I, and hence, by a routine inference, also AC on I.

Suppose therefore, if possible, that S is nonvoid. There exists by Theorem 4 an open interval H such that $S\cap H$ is a portion of S and that $\xi(x)$ is Dirichlet admissible on $S\cap H$. We shall show that $\xi(x)$ is AC on the intersection $Q=S\cap J$ for each closed interval $J\subset H$.

The set Q is closed and contained in $S\cap H$. Hence there exists a function $G(x)$ approximately derivable to zero at almost all points of Q, such that Q is a Dirichlet set for $G(x)$ and that the difference $F(x)=\xi(x)-G(x)$ is AC on Q. Then $F(x)$ is AD at almost all points of Q, on account of the Denjoy-Khintchine Theorem (see Saks [5], p. 222). It follows further that $F'_{\mathrm{ap}}(x)$ fulfils the relation

$$F'_{\mathrm{ap}}(x)=F'_{\mathrm{ap}}(x)+G'_{\mathrm{ap}}(x)=\xi'_{\mathrm{ap}}(x)\in[0,+\infty)$$

at almost every point x of Q. On the other hand, each open interval D contiguous to Q is at the same time contiguous to S. Hence $\xi(x)$ is GAC and nondecreasing on such an interval D. This together with the continuity of $\xi(x)$ on I, shows that $\xi(x)$ is nondecreasing on the closure of D and in particular that $\xi(D)\geqq 0$. Combining the above results and applying the foregoing Theorem 7, we find that $\xi(x)$ is nondecreasing over Q.

This fact implies that the function $G(x)=\xi(x)-F(x)$ is BV on Q. But Q is a Dirichlet set for $G(x)$. It thus follows from Theorem 5 that $G(x)$, and hence $\xi(x)$ also, is AC on Q, as announced above.

Since S is a perfect set which has $S\cap H$ for a portion, the closed interval $J\subset H$ considered above can be so chosen that the set $Q=S\cap J$ is infinite. Let L be the minimal closed interval containing Q, so that $L\subset I$. As already proved, the function $\xi(x)$ is AC on Q and GAC on each open interval contiguous to Q. Hence $\xi(x)$ is GAC on the interval L. This contradicts the definition of the set S, since Q is an infinite subset of $S\cap L$. The proof is thus complete.

We now proceed to give the descriptive definition of the *Dirichlet totalization.* A function $f(x)$ will be termed *Dirichlet totalizable* on a closed interval I, if there exists a function $\varphi(x)$ which is generalized Dirichlet admissible on I and which has $f(x)$ for its approximate derivative almost everywhere on I. Any such function $\varphi(x)$ is then called *indefinite Dirichlet total* of $f(x)$ on I. By the *definite Dirichlet total* of $f(x)$ over I we shall mean the increment $\varphi(I)$ of its indefinite total. This number $\varphi(I)$, which is uniquely determined by the function $f(x)$ and the interval I on account of the foregoing Theorem 8, will be denoted by $(\mathfrak{T})\int_I f(x)\,dx$.

All the properties, except Theorem 19, of the powerwise integral that are stated on pp. 16–17 of [1] are shared also by the Dirichlet total, as may readily be verified. In particular, *the Dirichlet totalization includes the Denjoy integration.*

THEOREM 9. *The Dirichlet totalization is strictly wider than the Denjoy integration.*

PROOF. We observe first the following simple fact. Given a function $\varphi(x)$ and a compact nonconnected set E, suppose that $\varphi(x)$ is Dirichlet continuous on E and AD to zero at almost all points of E. If there exists a constant $\delta>0$ such that

$$|\Upsilon(\varphi;p;Q)|<\delta^{-1} \qquad \text{for} \quad 1<p<1+\delta$$

and for every closed set $Q\subset E$, then $\varphi(x)$ is Dirichlet admissible on E.

Consider the function $\Omega(x)=\Theta(x;3^{-1})$ and the set $\Gamma=U_0\cap U_1\cap\cdots$, where the notation is the same as in § 2 of [3]. This function being strongly Dirichlet continuous on Γ by Lemma 16 of [3], what was stated above will apply to $\Omega(x)$ and Γ, if only we ascertain the existence of the constant δ. But this is an immediate consequence of the appraisal

$$|\Upsilon(\Omega;p;Q)|<13^p\cdot\sqrt{p-1}+1,$$

where $p>1$ and Q is any closed subset of Γ.

To prove the validity of this inequality, we may assume that Q is nonconnected. Let $[u,v]$ be the minimal closed interval containing Q. The function $\Omega(x)$ is everywhere nonnegative and we have $\Omega(0)=\Omega(1)=0$. Hence, writing $R=Q\cup\{0,1\}$, we find at once that

$$\begin{aligned}\Upsilon(\Omega;p;R)&=\Upsilon(\Omega;p;Q)+\Omega([0,u])\square^p+\Omega([v,1])\square^p\\&=\Upsilon(\Omega;p;Q)+\Omega^p(u)-\Omega^p(v),\end{aligned}$$

where $p>1$. This, combined with Lemma 15 of [3], gives

$$|\Upsilon(\Omega;p;Q)|<13^p\cdot\sqrt{p-1}+|\Omega^p(v)-\Omega^p(u)|.$$

The assertion will therefore follow if we show that $\Omega(x)<1$ for $x\in\Gamma$.

By definition, the value of $\Omega(x)$ for $x\in\Gamma$ is expressed by

$$\Omega(x)=\sum_{n=1}^{\infty}\frac{a_n(x)}{n}\left(\frac{1}{3}\right)^n, \quad \text{where} \quad \langle a_n(x);n\in\boldsymbol{N}\rangle=\sigma(x).$$

Recalling that $\sigma(x)$ is a binary sequence, we obtain for $\Omega(x)$ the following appraisal, which completes the proof of the inequality in hand:

$$\Omega(x) \leqq \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{3}\right)^n < \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n = \frac{1}{2}.$$

The function $\Omega(x)$ is thus Dirichlet admissible on the set Γ. But $\Omega(x)$ is so also on every open interval J contiguous to Γ, since it is linear, and hence AC, on J. $\Omega(x)$ is therefore GD admissible on the interval $U_0=[0, 1]$, which is the minimal closed interval containing Γ. Consequently, if $f(x)$ is any function such that $\Omega'_{\mathrm{ap}}(x)=f(x)$ almost everywhere on U_0, then $f(x)$ is Dirichlet totalizable on U_0. On the other hand, $f(x)$ fails to be Denjoy integrable on U_0, since $\Omega(x)$ is not powerwise continuous, and hence not GAC, on U_0 (see Lemma 8 of [3]). The theorem is thus established.

§2. Integration by parts.

LEMMA 1. *Let $M(x)$ be a function of bounded variation and $C(x)$ a function continuous on a closed interval $I=[a, b]$.*

(i) *If $M(x)$ is nondecreasing, then there is a point $\xi \in I$ such that*

$$(\mathfrak{S})\int_I C(x)\, dM(x) = C(\xi)\cdot M(I).$$

(ii) *We have the relation $(\mathfrak{S})\int_I C(x)\, dM(x)=\int_I C(x)\, dM(x)$, where the integral on the right is a Riemann-Stieltjes one.*

PROOF. We observe first that a function "of bounded variation" means one which is BV on every closed interval.

re (i): This follows at once from the first half of Theorem (2.1) on p. 244 of Saks [5].

re (ii): In the case where $M(x)$ is nondecreasing, the assertion is immediate from part (i) and the additivity of the $\mathfrak{S}$-integral with respect to the interval.

In the general case, on the other hand, the function $M(x)$ is expressible in the form $M(x)=U(x)-L(x)$, where $U(x)$ is its upper variation and $-L(x)$ its lower variation. Since the $\mathfrak{S}$-integral of a function with respect to $M(x)$ over $I=[a, b]$ does not depend on the values taken by $M(x)$ outside the interval I, we may assume that

$$M(x)=M(a) \quad \text{for } x<a \quad \text{and} \quad M(x)=M(b) \quad \text{for } x>b.$$

Then a similar assumption is automatically fulfilled by each of the functions $U(x)$ and $L(x)$. Consequently, by the definition of the $\mathfrak{S}$-integral and by that of the Lebesgue-Stieltjes integral (Saks [5], p. 65), we find that

$$(\mathfrak{S})\int_I C(x)\, dM(x)=\int_I C(x)\, dM^*(x)=\int_I C(x)\, dU^*(x)-\int_I C(x)\, dL^*(x)$$

$$=(\mathfrak{S})\int_I C(x)\,dU(x)-(\mathfrak{S})\int_I C(x)\,dL(x).$$

Both $U(x)$ and $L(x)$ being nondecreasing, it follows finally that

$$(\mathfrak{S})\int_I C(x)\,dM(x)=\int_I C(x)\,dU(x)-\int_I C(x)\,dL(x)=\int_I C(x)\,dM(x).$$

LEMMA 2. *Given a function $M(x)$ of bounded variation and given a function $C(x)$ which is continuous on a closed interval $I=[a,b]$, let $K(x)$ be any function such that*

$$K(x)=M(x)C(x)-\int_a^x C(t)\,dM(t) \qquad \textit{for} \quad x\in I,$$

where the integral is a Riemann-Stieltjes one as above.

(i) *The function $K(x)$ is then continuous on the interval I.*

(ii) *If the function $C(x)$ is absolutely continuous on a set $E\subset I$, so is the function $K(x)$ also.*

(iii) *We have $K'_{\mathrm{ap}}(x)=M(x)C'_{\mathrm{ap}}(x)$ at almost every point $x\in I$ at which the function $C(x)$ is approximately derivable.*

PROOF. We may clearly assume $M(x)$ to be nondecreasing. Writing

$$D(x)=\int_a^x C(t)\,dM(t)=(\mathfrak{S})\int_a^x C(t)\,dM(t) \qquad \text{for} \quad x\in I,$$

where the second equality is ensured by part (ii) of Lemma 1, and defining $D(x)$ arbitrarily for x outside I, we have

$$K(x)=M(x)C(x)-D(x) \qquad \text{for} \quad x\in I.$$

re (i) and (ii): These results are stated respectively on p. 245 and on p. 246 of Saks [5], in the course of the proof for Lemma (2.2).

re (iii): The second half of Theorem (2.1) on p. 244 of Saks [5] shows that the function $D(x)$ is derivable to $D'(\xi)=C(\xi)\cdot M'(\xi)$ at every point ξ interior to I and at which $M(x)$ is derivable.

Suppose that ξ is such a point and that $C(x)$ is AD at this point. Approximately deriving at ξ both sides of the relation $K(x)=M(x)C(x)-D(x)$, we obtain at once $K'_{\mathrm{ap}}(\xi)=M(\xi)\cdot C'_{\mathrm{ap}}(\xi)$. This establishes the assertion, since $M(x)$ is derivable almost everywhere on $\boldsymbol{R}$.

LEMMA 3. *Suppose that a function $U(x)$ is Riemann integrable on a closed interval $I=[a,b]$ and that a function $V(x)$ is, on this interval, an indefinite integral of a function $v(x)$ summable over I. Then $U(x)$ is Riemann-Stieltjes integrable over I with respect to $V(x)$ and we have*

$$\int_I U(x)\,dV(x)=\int_I U(x)v(x)\,dx,$$

where the integral on the right is a Lebesgue one (see [4], p. 254).

LEMMA 4. *Given a function $M(x)$ of bounded variation and given a function $C(x)$ which is continuous on a closed interval $I=[a,b]$, let $K(x)$ be any function such that*

$$K(x)-K(a)=\int_a^x M(t)\,dC(t) \qquad \textit{for} \quad x\in I,$$

the integral being a Riemann-Stieltjes one. Further let Q be a closed subset of I.

If Q is a Dirichlet set for $C(x)$ and if $C(x)$ is linear on each closed interval contiguous to Q, then Q is a Dirichlet set for $K(x)$, too.

PROOF. Suppose first that the set Q is connected, so that the linearity hypothesis on the function $C(x)$ is vacuously true. Then $C(x)$ is a constant over Q and hence so must also be the function $K(x)$, for which therefore Q is certainly a Dirichlet set. Thus we may suppose Q to be nonconnected.

Plainly we need only treat the case where the function $M(x)$ is nondecreasing and where I is the minimal closed interval containing Q.

If σ and τ are constants and if $\sigma>0$, then the function $N(x)$ determined by $M(x)=\sigma N(x)+\tau$ is likewise nondecreasing and we have

$$K(x)-K(a)=\int_a^x M(t)\,dC(t)=\sigma\int_a^x N(t)\,dC(t)+\tau[C(x)-C(a)]$$

for $x\in I$. On the other hand, choosing σ and τ suitably, we can ensure that $0<N(x)<3^{-1}$ for $x\in I$. This, together with Theorem 1, allows us to suppose that $M(x)$ itself fulfils the condition $0<M(x)<3^{-1}$ for $x\in I$.

Furthermore, we may clearly add the assumption that $C(a)=0$.

Let J stand in the sequel for a generic open interval contiguous to Q.

We shall begin by appraising the series $\Lambda(K;p;Q)=\sum|K(J)|^p$. By hypothesis, the function $C(x)$ is linear on the closed interval $[u,v]$, if we write $J=(u,v)$. Consequently, denoting by ω a positive upper bound for $|M(x)|$ on the interval I, we find at once that

$$K(J)=\int_u^v M(x)\,dC(x)=\frac{C(J)}{|J|}\int_u^v M(x)\,dx, \quad \text{whence } |K(J)|\leqq\omega|C(J)|.$$

On the other hand, Q is a Dirichlet set for $C(x)$, by hypothesis. Hence it follows that, as $p\to1$,

$$\Lambda(K;p;Q)\leqq\omega^p\cdot\Lambda(C;p;Q)=o\left(\frac{\omega^p}{p-1}\right)=o\left(\frac{1}{p-1}\right).$$

We proceed to verify that for each closed interval A with end points belonging to Q, we have $\Upsilon(K;p;Q\cap A)\to K(A)$ as $p\to1$. For this purpose, we need some preliminaries.

For each interval J considered above, we write for brevity

$$\rho(J)=\rho(J;p)=\frac{1}{|J|}[C(J)]^{p}$$

and we define a function $\gamma(x)=\gamma(x;p)$ by the following conditions:

(a) $\gamma(x)=\rho(J)$, if x belongs to an interval J;
(b) $\gamma(x)=0$, if x belongs to no interval J.

The function $\gamma(x)$, thus constructed, is evidently Borel measurable on $\boldsymbol{R}$. Further, it is summable over $\boldsymbol{R}$, since

$$\int_{-\infty}^{+\infty}|\gamma(x)|dx=\sum_{J}|\rho(J)|\cdot|J|=\sum_{J}|C(J)|^{p}=\Lambda(C;p;Q)<+\infty.$$

This being so, let us consider the function

$$\Gamma(x)=\Gamma(x;p)=\int_{a}^{x}\gamma(t)\,dt, \qquad \text{where} \quad x\in\boldsymbol{R}.$$

This function vanishes for $x=a$ and is absolutely continuous (i.e. AC on every closed interval). Furthermore, we have the relation

$$\Gamma(A)=\Gamma(A;p)=\Upsilon(C;p;Q\cap A)$$

for every closed interval A whose end points belong to the set Q. In fact, $\Gamma(A)=0=\Upsilon(C;p;Q\cap A)$ when $A\subset Q$, while in the opposite case

$$\Gamma(A)=\sum_{J\subset A}\rho(J)\cdot|J|=\sum_{J\subset A}C(J)]^{p}=\Upsilon(C;p;Q\cap A).$$

We shall show that if we keep fixed any point ξ of I, then $\Gamma(\xi;p)$ tends to $C(\xi)$ as $p\to1$. On account of $\Gamma(a;p)=0=C(a)$, we may assume that $a<\xi\leqq b$; moreover, we need only consider the case in which $\xi\in Q$, both functions $C(x)$ and $\Gamma(x)$ being linear on the closure of each interval J. Writing $L=[a,\xi]$, we find that

$$\Gamma(\xi;p)=\Gamma(L;p)=\Upsilon(C;p;Q\cap L).$$

But this last quantity tends to $C(L)=C(\xi)-C(a)=C(\xi)$ as $p\to1$, since by hypothesis Q is a Dirichlet set for the function $C(x)$.

The same hypothesis on Q ensures further the existence of a number $\delta>0$ such that for every closed interval R we have

$$|\Upsilon(C;p;Q\cap R)|<\delta^{-1} \qquad \text{whenever} \quad 1<p<1+\delta.$$

Specializing R to the above interval $L=[a,\xi]$, we obtain $|\Gamma(\xi;p)|<\delta^{-1}$. We thus see that $|\Gamma(x;p)|<\delta^{-1}$ for every $x\in Q$, and hence for every $x\in I$, provided $1<p<1+\delta$.

This being so, now suppose given an infinite sequence of real numbers, $\langle p_n; n\in \boldsymbol{N}\rangle$, tending to 1 and such that $1<p_n<1+\delta$ for $n\in\boldsymbol{N}$. It follows from what was shown just now that

(1) $|\Gamma(x;p_n)|<\delta^{-1}$ on the interval I for $n\in\boldsymbol{N}$,
(2) $\lim\Gamma(x;p_n)=C(x)$ for each $x\in I$.

Given a closed interval $A=[r,s]$, with r and s belonging to the set Q, suppose that A itself is not contained in Q. Let us consider the function $H(x)$ coinciding on the whole A with $M(x)$ and such that $H(x)=M(r)$ for $x<r$, $H(x)=M(s)$ for $x>s$. The function $H(x)$, thus defined, is evidently nondecreasing. It is further continuous from the left at $x=r$ and from the right at $x=s$.

Now the well-known integration by parts formula for the Riemann-Stieltjes integral shows that

$$K(A)=\int_A M(x)\,dC(x)=\Big[M(x)C(x)\Big]_r^s-\int_A C(x)\,dM(x),$$

where the last integral is expressible as a Lebesgue-Stieltjes one:

$$\int_A C(x)\,dM(x)=\int_A C(x)\,dH(x)=(\mathfrak{S})\int_A C(x)\,dH(x)=\int_A C(x)\,dH^*(x).$$

Hence
$$K(A)=\Big[M(x)C(x)\Big]_r^s-\int_A C(x)\,dH^*(x).$$

If we replace the function $C(x)$ by $\Gamma(x;p_n)$, we get similarly

$$\int_A M(x)\,d\Gamma(x;p_n)=\Big[M(x)\Gamma(x;p_n)\Big]_r^s-\int_A\Gamma(x;p_n)dH^*(x)$$

for every $n\in\boldsymbol{N}$. Utilizing the above properties (1) and (2) of the sequence $\langle\Gamma(x;p_n);n\in\boldsymbol{N}\rangle$ and appealing to Lebesgue's Theorem on termwise integration, we find that

$$\lim_n\int_A M(x)\,d\Gamma(x;p_n)=K(A).$$

Since this is valid for every sequence $\langle p_1,p_2,\cdots\rangle$ of the above kind, we obtain the following relation, where we assume $p>1$ as hitherto:

$$\lim_{p\to1}\int_A M(x)\,d\Gamma(x;p)=K(A).$$

Let us transform this integral. The function $M(x)$ is Riemann integrable over A, since it is nondecreasing. On the other hand, the function $\Gamma(x;p)$ was defined to be an indefinite integral of the summable function $\gamma(x;p)$. Hence it follows from Lemma 3 that

$$\int_A M(x)\,d\Gamma(x;p)=\int_A M(x)\gamma(x;p)\,dx.$$

But the function $\gamma(x;p)$ vanishes over the set Q. Consequently

$$\int_A M(x)\,d\Gamma(x;p)=\sum_{J\subset A}\int_J M(x)\gamma(x;p)\,dx$$

$$=\sum_{J\subset A}\left[\frac{C(J)\square^p}{|J|}\int_J M(x)\,dx\right]=\sum_{J\subset A}\mu(J)\cdot C(J)\square^p,$$

where we write $\mu(J)=\dfrac{1}{|J|}\displaystyle\int_J M(x)\,dx$ for each J. Thus $\mu(J)$ is the average value of $M(x)$ over J. We observe that $K(J)=\mu(J)\cdot C(J)$ for each J.

By means of $\mu(J)$ we now define on $\boldsymbol{R}$ a function $W(x)$ as follows:

(a) $W(x)=\mu(J)$, if there exists a J to which x belongs;
(b) $W(x)=M(x)$, if x belongs to no J.

We find easily that the function $W(x)$, thus constructed, is nondecreasing over $\boldsymbol{R}$. On the other hand, we have $0<M(x)<3^{-1}$ for $x\in I$ by assumption, and we may clearly replace here $M(x)$ by $W(x)$.

This being so, let us consider for $p>1$ the following two integrals:

$$S(p)=\int_A W(x)\,d\Gamma(x;p)\quad\text{and}\quad T(p)=\int_A W^p(x)\,d\Gamma(x;p).$$

Using Lemma 3 we transform these integrals and we have

$$S(p)=\int_A W(x)\gamma(x;p)\,dx=\sum_{J\subset A}\mu(J)\cdot C(J)\square^p=\int_A M(x)\,d\Gamma(x;p),$$

$$T(p)=\int_A W^p(x)\gamma(x;p)\,dx=\sum_{J\subset A}\mu^p(J)\cdot C(J)\square^p$$

$$=\sum_{J\subset A}\{\mu(J)\cdot C(J)\}\square^p=\sum_{J\subset A}K(J)\square^p=\Upsilon(K;p;Q\cap A).$$

Integrating by parts, we find on the other hand that

$$S(p)-T(p)=\int_A\{W(x)-W^p(x)\}\,d\Gamma(x;p)$$

$$=\Big[\{W(x)-W^p(x)\}\,\Gamma(x;p)\Big]_r^s-\int_A\Gamma(x;p)\,d\{W(x)-W^p(x)\}.$$

But the monotonity of the function $W(x)$, together with the inequalities $0<W(x)<3^{-1}$, where $x\in I$, imply that the function $W(x)-W^p(x)$ is nondecreasing over I and that $0<W(x)-W^p(x)<3^{-1}-3^{-p}$ for $x\in I$. This is an immediate consequence of the fact that if $0<u<v<e^{-1}$, where e is the base of the natural logarithm, we have

$$u-u^p<v-v^p,\quad\text{i.e.}\quad v^p-u^p<v-u.$$

Indeed, by the mean value theorem, there is a number ξ such that

$$u<\xi<v \quad \text{and} \quad \frac{v^p-u^p}{v-u}=p\,\xi^{p-1}.$$

Then $\xi<e^{-1}$ and hence we find that

$$\xi^{p-1}<\left(\frac{1}{e}\right)^{p-1}=\frac{1}{e^{p-1}}<\frac{1}{p}, \quad \text{whence} \quad \frac{v^p-u^p}{v-u}<1.$$

We now combine the results obtained above. In view of the relation $|\Gamma(x;p)|<\delta^{-1}$, where $x\in I$ and $1<p<1+\delta$, we derive at once the following appraisal for the same values of p:

$$|S(p)-T(p)|<\frac{1}{\delta}(1-3^{1-p}).$$

We thus have $\lim_{p\to 1}[S(p)-T(p)]=0$. This, together with the above expressions for $S(p)$ and $T(p)$, leads to the relation

$$\lim_{p\to 1}\Upsilon(K;p;Q\cap A)=\lim_{p\to 1}T(p)=\lim_{p\to 1}S(p)=\lim_{p\to 1}\int_A M(x)\,d\Gamma(x;p),$$

provided that the rightmost limit exists. But it does exist and equals $K(A)$, as already established.

We assumed in the above that the set Q does not contain the interval A. If $A\subset Q$, however, the quantity $\Upsilon(K;p;Q\cap A)$ vanishes and the function $C(x)$ is a constant over A. Hence we still have

$$\lim_{p\to 1}\Upsilon(K;p;Q\cap A)=\int_A M(x)\,dC(x)=K(A).$$

We have thus proved that this relation is true for any closed interval A, if only its end points belong to Q.

It remains to show further the existence of a positive constant η such that $|\Upsilon(K;p;Q\cap R)|<\eta^{-1}$ for every closed interval R, whenever $1<p<1+\eta$. But $\Upsilon(K;p;Q\cap R)$ vanishes if $Q\cap R$ is connected. Hence we need only consider such R as makes $Q\cap R$ nonconnected.

This being so, let $A=[r,s]$ be the minimal closed interval containing the compact nonconnected set $Q\cap R$. Then $r\in Q$, $s\in Q$, and $Q\cap R=Q\cap A$. Consequently it follows from what was already proved that

$$\Upsilon(K;p;Q\cap R)=\Upsilon(K;p;Q\cap A)=\int_A W^p(x)\,d\Gamma(x;p).$$

Integrating by parts, we transform this last integral into

$$\Big[W^p(x)\Gamma(x;p)\Big]_r^s-\int_A \Gamma(x;p)\,dW^p(x).$$

But $W(x)$ is nondecreasing. Moreover $0<W(x)<3^{-1}$ and $|\Gamma(x;p)|<\delta^{-1}$ for $x\in I$, where $1<p<1+\delta$ in the latter inequality. We thus find finally that $|\Upsilon(K;p;Q\cap R)|<\delta^{-1}$ for every closed interval R, provided $1<p<1+\delta$. Thus it suffices to take simply $\eta=\delta$, and this completes the proof.

Concerning integration by parts for the Denjoy integral, we have the following well-known theorem (see Saks [5], p. 246):

THEOREM. *If $M(x)$ is a function of bounded variation and $f(x)$ a function which is integrable in the Denjoy sense (wide or restricted) on a closed interval $I=[a,b]$, then the function $M(x)f(x)$ is integrable on I in the same sense, and moreover denoting by $F(x)$ any indefinite Denjoy integral of $f(x)$ on I, we have*

$$(\mathfrak{D})\int_I M(x)f(x)\,dx=\Big[M(x)F(x)\Big]_a^b-(\mathfrak{S})\int_I F(x)\,dM(x).$$

The author endeavoured without effect to obtain a similar theorem for the powerwise integral in the wide sense. Subsequently it occurred to him that there is another way to deal with the problem. The present paper is the result of this change of way.

We are now in a position to prove the following main theorem.

THEOREM 10. *If $M(x)$ is a function of bounded variation and if $f(x)$ is a function Dirichlet totalizable on a closed interval $I=[a,b]$, then the function $M(x)f(x)$ shares this latter property with $f(x)$, and moreover denoting by $F(x)$ any indefinite Dirichlet total of $f(x)$ on I, we have*

$$(\mathfrak{T})\int_I M(x)f(x)\,dx=\Big[M(x)F(x)\Big]_a^b-\int_I F(x)\,dM(x).$$

REMARK. If we integrate by parts the Riemann-Stieltjes integral on the right, this equality is transformed into the following form which is the essential import of the above theorem:

$$(\mathfrak{T})\int_I M(x)f(x)\,dx=\int_I M(x)\,dF(x).$$

PROOF. The function $F(x)$ is continuous on the interval I, since it is generalized Dirichlet admissible on I. Let $G(x)$ be any function such that

$$G(x)=M(x)F(x)-\int_a^x F(t)\,dM(t)\qquad\text{for}\quad x\in I.$$

Then we have

$$G(I)=\Big[M(x)F(x)\Big]_a^b-\int_I F(x)\,dM(x),$$

and the proof comes to showing that $G(x)$ is an indefinite Dirichlet total of $M(x)f(x)$ over I, or in other words, that

(1) the function $G(x)$ is GD admissible on the interval I,
(2) we have $G'_{\text{ap}}(x)=M(x)f(x)$ almost everywhere on I.

Assertion (2) is immediate from part (iii) of Lemma 2, since we have $F'_{\text{ap}}(x)=f(x)$ almost everywhere on I. Accordingly we shall be concerned, in what follows, with proving assertion (1).

The function $F(x)$ being GD admissible on I, this interval is expressible as the union of a sequence of closed sets on each of which $F(x)$ is Dirichlet admissible. Let E be any one of these sets. Then to each closed set $Q\subset E$ there correspond two functions $\Phi(x)$ and $\Psi(x)$ such that

(a) we have $F(x)=\Phi(x)+\Psi(x)$ for $x\in\boldsymbol{R}$,
(b) $\Phi(x)$ is absolutely continuous on Q,
(c) $\Psi(x)$ is AD to zero at almost all points of Q,
(d) Q is a Dirichlet set for $\Psi(x)$.

The continuity of $F(x)$ on I, together with the above condition (b), shows that the function $\Psi(x)$ is continuous on Q. Hence there exists a function $\lambda(x)$ which is (i) continuous on I, (ii) coinciding with $\Psi(x)$ on Q, and (iii) linear on each closed interval (if existent) contiguous to Q. Such a function $\lambda(x)$ is necessarily AD to zero at almost all points of Q, since it is approximately equiderivable with $\Psi(x)$ at almost all points of Q on account of Lemma 3 of [1]. On the other hand, the function $F(x)-\lambda(x)$, which coincides with $\Phi(x)$ on Q, is AC on Q. Moreover, Q is clearly a Dirichlet set for $\lambda(x)$. Consequently we may suppose without loss of genceralty that

(e) $\Psi(x)$ is continuous on I and linear on each closed interval (if existent) contiguous to Q.

This condition and the continuity of $F(x)$ on I together imply that $\Phi(x)$ is continuous on I.

The function $G(x)$ is continuous on I by part (i) of Lemma 2. In order to verify the above assertion (1), it is thus enough to show that to each set Q considered above there correspond two functions $U(x)$ and $V(x)$ subject to the conditions:

(a*) we have $G(x)=U(x)+V(x)$ for $x\in\boldsymbol{R}$,
(b*) $U(x)$ is absolutely continuous on Q,
(c*) $V(x)$ is AD to zero at almost all points of Q,
(d*) Q is a Dirichlet set for $V(x)$.

Let $V(x)$ be any function which is expressed by

$$V(x)=M(x)\Psi(x)-\int_a^x \Psi(t)\,dM(t) \qquad \text{for} \quad x\in I$$

and let us write $U(x)=G(x)-V(x)$ for $x\in \boldsymbol{R}$, so that by condition (a)

$$U(x)=M(x)\Phi(x)-\int_a^x \Phi(t)\,dM(t) \qquad \text{for} \quad x\in I.$$

The proof will be complete if we show that these two functions fulfil the conditions (a*) to (d*), of which the first one is evident.

re (b*): This follows from condition (b) and part (ii) of Lemma 2.

re (c*): This follows from condition (c) and part (iii) of Lemma 2.

re (d*): Integrating by parts the following integral, we find that

$$V(x)-V(a)=\int_a^x M(t)\,d\Psi(t) \qquad \text{for} \quad x\in I.$$

The assertion follows from this combined with Lemma 4, in view of conditions (d) and (e).

From the theorem established just now we deduce at once the following second mean value theorem for the Dirichlet total. The proof is the same as on p. 247 of Saks [5] and may be omitted.

THEOREM 11. *If $M(x)$ is a nondecreaing function and if $f(x)$ is a function Dirichlet totalizable on a closed interval $I=[a,b]$,, there necessarily exists a point $\xi\in I$ such that*

$$\int_a^b M(x)f(x)\,dx=M(a)\int_a^{\xi} f(x)\,dx+M(b)\int_{\xi}^b f(x)\,dx,$$

where each integral is a Dirichlet total, the function $M(x)f(x)$ being Dirichlet totalizable on I by the foregoing theorem.

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