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On the Dirichlet Continuity of Functions

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This paper, which is a supplement to our recent work [5] on the powerwise integration, consists of three mutually independent sections, each being concerned, respectively, with elucidating a doubtful point contained explicitly or implicitly in the paper [5].

As defined in [5], a function is called Dirichlet continuous on a compact nonconnected set Q, if it is continuous on Q and if it fulfils the Dirichlet condition on every compact nonconnected set contained in Q. Now the definition of the Dirichlet condition consists of three items. We are interested in examining whether or not the second item is superfluous. The answer is in the negative, as will be shown in § 1.

As defined in [5], a function is said to fulfil the condition (P) on a linear set E, if either the function is AC on E, or else if there exists a CT null set which contains E and on which the function is Dirichlet continuous. It is the object of § 2 to show that the Dirichlet continuity cannot be replaced here by the Dirichlet condition, in the sense that the theory of the powerwise integration would collapse if we did so. We thus find that the Dirichlet continuity is essentially stronger than the Dirichlet condition.

We proposed in [5] the following problem: *To decide whether a function which is Dirichlet continuous on a compact nonconnected set, is necessarily powerwise continuous on this set.* This problem will be solved in the negative in § 3.

As in our previous papers, a *function*, by itself, will always mean a mapping of the real line $\boldsymbol{R}$ into itself, unless another meaning is obvious from the context. We shall also continue denoting by $\boldsymbol{N}$ the set of the positive integers and by $\boldsymbol{M}$ that of the nonnegative integers.

§ 1. Concerning the definition of the Dirichlet condition.

The letter p will represent a real number >1 in this section. Given a compact nonconnected set Q, let H denote a generic open interval contiguous to Q. A function $\varphi(x)$ will, for the nonce, be termed to fulfil the *condition* (D_0) on Q, if $\sum|\varphi(H)|^p<+\infty$ for every p and if

$$\sum_{H\subset J}\varphi(H)\square^p=\varphi(J)+o(1) \qquad \text{as} \quad p\to 1\,,$$

where J is an arbitrary closed interval such that the set Q contains the end points of J without containing the whole of J.

Let Q_0 be the compact set consisting of 1 and of the numbers $1-n^{-1}$, where $n\in\boldsymbol{N}$. We shall show that the class of the functions which fulfil the condition (D_0) on Q_0 is not closed under addition.

Given an infinite sequence of real numbers, $\sigma=\langle a_1, a_2, \cdots\cdot\rangle$, with the series $\sum a_n$ converging, let s_m be for each $m\in\boldsymbol{M}$ the mth partial sum of this series, i.e. let $s_0=0$ and $s_n=a_1+\cdots\cdot+a_n$ for $n\in\boldsymbol{N}$. We associate with the sequence σ a continuous function $F(x;\sigma)$ defined as follows, where we write $S(\sigma)=\sum a_n$.

(i) $F(x;\sigma)=0$ for $x\leqq 0$ and $F(x;\sigma)=S(\sigma)$ for $x\geqq 1$;
(ii) $F(1-n^{-1};\sigma)=s_{n-1}$ for $n\in\boldsymbol{N}$;
(iii) $F(x;\sigma)$ is linear on each closed interval contiguous to Q_0.

Let us consider two special cases of this function $F(x;\sigma)$, taking σ to be respectively

$$\sigma_1=\left\langle 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \cdots\cdot\right\rangle \quad \text{and} \quad \sigma_2=\left\langle 0, \frac{1}{2}, 0, -\frac{1}{4}, 0, \frac{1}{6}, \cdots\cdot\right\rangle,$$

so that $S(\sigma_1)=\log 2$ and $S(\sigma_2)=2^{-1}\log 2$. Writing for short

$$f_1(x)=F(x;\sigma_1), \quad f_2(x)=F(x;\sigma_2), \quad f(x)=F(x;\sigma)\,,$$

where $\sigma=\sigma_1+\sigma_2=\left\langle 1, 0, \frac{1}{3}, -\frac{1}{2}, \frac{1}{5}, 0, \frac{1}{7}, -\frac{1}{4}, \cdots\cdot\right\rangle$, we have

$$S(\sigma)=S(\sigma_1)+S(\sigma_2)=\frac{3}{2}\log 2 \quad \text{and} \quad f(x)=f_1(x)+f_2(x)\,.$$

It is easy to see that the functions $f_1(x)$ and $f_2(x)$ both fulfil the condition (D_0) on the set Q_0. On the other hand, from

$$\sum_D f(D)\square^p=1^p+\left(\frac{1}{3}\right)^p-\left(\frac{1}{2}\right)^p+\left(\frac{1}{5}\right)^p+\left(\frac{1}{7}\right)^p-\left(\frac{1}{4}\right)^p+\cdots\cdot=\sum_D f_1(D)\square^p\,,$$

where D means a generic open interval contiguous to Q_0, it follows that

$$\lim_{p\to 1}\sum_D f(D)\square^p=\lim_{p\to 1}\sum_D f_1(D)\square^p=f_1(1)-f_1(0)$$
$$=S(\sigma_1)<S(\sigma)=f(1)-f(0)\,.$$

This shows that the function $f(x)=f_1(x)+f_2(x)$ fails to fulfil the condition (D_0) on Q_0. Consequently, as stated above, fulfilment of the condition (D_0) is not preserved under addition of functions.

In our paper [5] a function $\varphi(x)$ was called to fulfil the Dirichlet condition on a compact nonconnected set Q, if it fulfils the condition (D_0) on Q and if we have

$$\sum_H |\varphi(H)|^p = o\left(\frac{1}{p-1}\right) \quad \text{as} \quad p \to 1,$$

where H ranges as above over the open intervals contiguous to Q. If the set Q is fixed, the class of all such functions $\varphi(x)$ is closed under linear combination, as asserted by Theorem 1 of [5]; and this theorem is fundamental to the whole theory of the powerwise integration. We find now, from what was proved already, that the requirement $\sum|\varphi(H)|^p = o\{(p-1)^{-1}\}$ is not superfluous in the above definition.

As we may observe in passing, the failure of the functions $f_1(x)$ and $f_2(x)$ to fulfil the Dirichlet condition is seen at once by means of the relation

$$\left(1-\frac{2}{2^p}\right)\sum_{n=1}^{\infty}\frac{1}{n^p} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^p},$$

which together with the mean value theorem implies that

$$(p-1)\sum_{n=1}^{\infty}\frac{1}{n^p} > \frac{1}{\log 2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^p}.$$

§ 2. Concerning the definition of the condition (P).

Let us begin by recalling the following basic symbols used in § 4 of [5]: Given a closed interval I, we write by definition

$$E_0 = I, \quad E_{m+1} = E_m(3; 3^{2m}) \quad \text{for } m \in \boldsymbol{M}, \quad \Gamma = E_0 \cap E_1 \cap \cdots.$$

This being premised, let H denote a generic open interval contiguous to the set Γ. There exist for each H an $m \in \boldsymbol{M}$ and a component K of the figure E_m such that $H \subset K \setminus E_{m+1}$. Plainly m and K are each uniquely determined by H, and we shall call the index m the *order* of the interval H. Denoting the components of the ramification $K(3)$ by $K_1 < K_2 < K_3$, consider the case in which the interval H is contiguous to $K(3)$. H will be termed *ascendent* or *descendent*, according as $K_1 < H < K_2$ or $K_2 < H < K_3$, respectively. On the other hand, we shall call H *horizontal*, when it is not contiguous to $K(3)$.

The function $\Xi(x) = \Psi(x; I; 3^{-1})$ of [5] can now be completely characterized by the following conditions, where H means the same as above.

(i) If the order of H is m, we have respectively

$$\Xi(H) = 3^{-m^2} \text{ or } -3^{-m^2} \text{ or } 0,$$

according as H is ascendent, descendent, or horizontal;

(ii) $\Xi(x)$ is Dirichlet continuous on the set Γ;
(iii) $\Xi(x)$ is linear on the closure of each H;
(iv) $\Xi(x)$ vanishes outside the interior of I.

The letters A and δ will represent generically an ascendent open interval and a positive number, respectively, in the sequel. In connection with the function $\Xi(x)$, we define a quantity $f(\delta;A)$ as follows:

$$f(\delta;A)=3^{-m^2}\{3^{-\delta m^2}-3^{-\delta(m+1)^2}\},$$

where m is the order of the interval A. Clearly $0<f(\delta;A)<3^{-m^2}$.

LEMMA 1. *If K is a component of the figure E_m, then*

$$\sum_{A\subset K} f(\delta;A)=3^{-(1+\delta)m^2} \qquad \textit{for} \quad \delta>0.$$

In particular, we have $\sum_A f(\delta;A)=1$.

PROOF. For each integer $k\geqq m$, the interval K contains exactly $3^{k^2-m^2}$ ascendent intervals of order k, since K contains exactly $3^{k^2-m^2}$ components of the figure E_k. Consequently

$$\sum_{A\subset K} f(\delta;A)=\sum_{k\geqq m} 3^{k^2-m^2}\cdot 3^{-k^2}\{3^{-\delta k^2}-3^{-\delta(k+1)^2}\}$$

$$=3^{-m^2}\sum_{k\geqq m}\{3^{-\delta k^2}-3^{-\delta(k+1)^2}\}=3^{-(1+\delta)m^2}.$$

In the special case in which $m=0$, we have $\sum_A f(\delta;A)=1$.

Let us write $I=[a,b]$ henceforth. Given a $\delta>0$, we define on Γ a function $F(x;\delta)$ of the variable x as follows. If $a<x\in\Gamma$, then

$$F(x;\delta)=\sum_{A\subset X} f(\delta;A), \quad \text{where} \quad X=[a,x];$$

while if $x=a$, we set $F(a;\delta)=0$. We find at once that

$$0\leqq F(x;\delta)\leqq\sum_A f(\delta;A)=1.$$

It is also obvious that $F(x;\delta)$ is a nondecreasing function of x.

LEMMA 2. *If c is a fixed point of Γ, then $F(c;\delta)$ tends to a finite limit $F(c;0)$, as $\delta\to 0+$. In particular, we have the values $F(a;0)=0$ and $F(b;0)=1$.*

PROOF. If c is a boundary point of an E_m, the assertion follows immediately from Lemma 1 and the definition of the function $f(\delta;A)$.

This being premised, consider the general case. Given any $\varepsilon>0$, let us take an $m\in\boldsymbol{M}$ such that $3^{-m^2}<\varepsilon$. The point c belongs to a component, say $K=[r,s]$, of the figure E_m. Lemma 1 then shows that

$$0 \leqq F(c;\delta) - F(r;\delta) \leqq \sum_{A \subset K} f(\delta;A) \leqq 3^{-m^2} < \varepsilon.$$

On the other hand, $F(r;\delta)$ tends to $F(r;0)$ as $\delta \to 0+$, since r is a boundary point of E_m. Hence there exists a number $\delta_0 > 0$ such that

$$-\varepsilon < F(r;\delta) - F(r;0) < \varepsilon \qquad \text{whenever} \quad 0 < \delta < \delta_0.$$

It follows that, for the same values of δ,

$$-\varepsilon < F(c;\delta) - F(r;0) < 2\varepsilon.$$

This shows that the oscillation of the quantity $F(c;\delta)$ for $0 < \delta < \delta_0$ is at most 3ε. $F(c;\delta)$ therefore tends to a limit $F(c;0)$ as $\delta \to 0$.

We have finally $F(a;0) = 0$ and $F(b;0) = 1$, since $F(a;\delta) = 0$ by definition and since $F(b;\delta) = \sum_A f(\delta;A) = 1$ by Lemma 1.

LEMMA 3. *The function $F(x;0)$, thus defined on the set Γ, is nondecreasing and continuous. Moreover, this function has a vanishing increment over each closed interval contiguous to Γ.*

PROOF. Given any two points $x_1 < x_2$ of Γ, we have $F(x_1;\delta) \leqq F(x_2;\delta)$ for every $\delta > 0$. Making $\delta \to 0$, we get at once $F(x_1;0) \leqq F(x_2;0)$, which shows that $F(x;0)$ is nondecreasing.

To prove the continuity of $F(x;0)$, suppose given any $\varepsilon > 0$ and choose an $m \in \boldsymbol{M}$ such that $3^{-m^2} < \varepsilon$. Writing K for a generic component of the figure E_m, we take any two points $x_1 < x_2$ of $K \cap \Gamma$. Then

$$0 \leqq F(x_2;\delta) - F(x_1;\delta) = \sum_{A \subset X} f(\delta;A), \quad \text{where} \quad X = [x_1, x_2].$$

But this sum is appraised by Lemma 1 as follows:

$$\sum_{A \subset X} f(\delta;A) \leqq \sum_{A \subset K} f(\delta;A) = 3^{-(1+\delta)m^2} < \varepsilon.$$

Consequently $0 \leqq F(x_2;\delta) - F(x_1;\delta) < \varepsilon$, whence we get

$$0 \leqq F(x_2;0) - F(x_1;0) \leqq \varepsilon.$$

This plainly implies the continuity of $F(x;0)$.

Let H be any open interval contiguous to Γ. Then the increment, written $F(H;\delta)$, of $F(x;\delta)$ over the closure of H is expressed by

$$F(H;\delta) = 3^{-m^2}\{3^{-\delta m^2} - 3^{-\delta(m+1)^2}\},$$

provided that H is an ascendent interval of order m; while $F(H;\delta)$ vanishes if H is descendent or horizontal. Hence, making $\delta \to 0$, we obtain $F(H;0) = 0$, which completes the proof.

On account of Lemma 2 and Lemma 3, we can extend the domain of definition of the function $F(x;0)$ to the whole real line, in such a way that the resulting function, which we denote by $F(x)$, fulfils the following conditions, where H means the same as hitherto:

(i) $F(x)$ is a constant on the closure of each interval H;
(ii) $F(x)=0$ for $x\leqq a$ and $F(x)=1$ for $x\geqq b$;
(iii) $F(x)$ is continous and nondecreasing over $\boldsymbol{R}$.

The function $F(x)$ is thus singular on every closed interval (see Saks [6], p. 96).

Let us write $\Omega(x)=\Xi(x)-F(x)$ for $x\in\boldsymbol{R}$ by definition. The following properties of the function $\Omega(x)$ are obvious:

(i) $\Omega(x)$ is linear on the closure of each H;
(ii) we have $\Omega(H)=\Xi(H)$ for each H;
(iii) $\Omega(x)$ is continuous on the real line.

Consider now any ascendent interval $A=(u, v)$. We associate with A a finite set $A^*\subset A$ defined by

$$A^*=\{u+3^{-2m-1}i(v-u)\ ;\ i\in\boldsymbol{N},\ i<3^{2m+1}\},$$

where m is the order of A. Let M be the union of all the sets A^* and let us write $\Gamma^*=\Gamma\cup M$. Clearly Γ^* is a CT null set, together with Γ.

As in § 1, the letter p will denote generically a real number >1, in the rest of this section.

LEMMA 4. *The fuuction $\Omega(x)$ fulfils the Dirichlet condition on the set Γ^*.*

PROOF. Let D be a generic open interval contiguous to Γ^*. If A is any ascendent interval, then $\Omega(x)$ increases strictly on the closure of A and we have

$$\sum_{D\subset A}\Omega^p(D)<\Omega^p(A) \qquad \text{for every } p;$$

this is a particular case of the evident inequality

$$\xi_1^p+\xi_2^p+\cdots\cdots+\xi_n^p<(\xi_1+\xi_2+\cdots\cdots+\xi_n)^p,$$

where n is any integer >1 and where $\xi_1, \xi_2, \cdots\cdot, \xi_n$ are n arbitrary positive numbers. It follows that

$$\sum_D|\Omega(D)|^p\leqq\sum_H|\Omega(H)|^p=\sum_H|\Xi(H)|^p=o\left(\frac{1}{p-1}\right),$$

since the function $\Xi(x)$ is Dirichlet continuous on Γ.

Let us fix a point c such that $a<c\in\Gamma^*$ and let us write $L=[a, c]$ for brevity. It remains to show that

$$\sum_{D\subset L}\Omega(D)\square^p=\Omega(c)+o(1) \quad \text{as} \quad p\to 1.$$

For this purpose, we may assume c to belong to Γ, without loss of generality.

This being so, write $\delta=p-1$ and consider any ascendent interval A of order m. Then $\Omega(A)=\Xi(A)=3^{-m^2}$, and hence

$$\sum_{D\subset A}\Omega^p(D)=3^{2m+1}\{3^{-(m+1)^2}\}^p=3^{-m^2}\cdot 3^{-\delta(m+1)^2}$$

$$=3^{-(1+\delta)m^2}-f(\delta;A)=\Xi^p(A)-f(\delta;A).$$

It accordingly follows that

$$\sum_{H\subset L}\Xi(H)\square^p-\sum_{D\subset L}\Omega(D)\square^p=\sum_{A\subset L}\{\Xi^p(A)-\sum_{D\subset A}\Omega^p(D)\}$$

$$=\sum_{A\subset L}f(\delta;A)=F(c;\delta).$$

Combining this with $\sum_{H\subset L}\Xi(H)\square^p=\Xi(c)+o(1)$, we get

$$\sum_{D\subset L}\Omega(D)\square^p=\Xi(c)-F(c;\delta)+o(1)$$

$$=\Xi(c)-F(c)+o(1)=\Omega(c)-o(1),$$

as $p\to 1$ (i. e. as $\delta\to 0$). This completes the proof.

Let us introduce now two temporary definitions. A function will be said to fulfil the *condition* (P*) on a linear set E, if either the function is AC on E, or else if there exists a CT null set which contains E and on which the function fulfils the Dirichlet condition. Again, we shall term a function to be *continuous* (P*) on a linear set E, if the function is continuous on E and if this set is expressible as the union of a sequence of sets on each of which the function fulfils the condition (P*).

Returning to the functions $\Xi(x)$ and $\Omega(x)$, we find at once that both of them are continuous (P*) on the underlying interval $I=[a, b]$. Moreover, the two functions are AED (approximately equiderivable) at almost all points of I, since they are linear on every closed interval contiguous to Γ and since they have equal increments over every such interval. However, their difference $F(x)=\Xi(x)-\Omega(x)$ is nonconstant on I. This shows that the theory of the powerwise integration breaks down if we replace the Dirichlet continuity by the Dirichlet condition in the definition of the condition (P). The Dirichlet continuity is thus essentially stronger than the Dirichlet condition.

§ 3. The Dirichlet continuity does not necessarily include the powerwise continuity.

LEMMA 5. *If $f(x)$ and $g(x)$ are continuous nondecreasing functions and if C is a linear compact set, then*

$$|\theta[C]|=|f[C]|+|g[C]|,$$

where we write $\theta(x)=f(x)+g(x)$.

PROOF. Assuming the set C nonvoid as we may, we express C as the limit of a descending infinite sequence $C_1\supset C_2\supset\cdots$ of elementary figures. For this purpose, let $n\in\boldsymbol{N}$ and denote by V_n the union of the open intervals $(x-n^{-1}, x+n^{-1})$, where the point x ranges over C. Then V_n is a bounded open set and each of its components has length $\geqq 2n^{-1}$. It follows that the components of V_n must be finite in number. Consequently, writing C_n for the closure of V_n, we find that C_n is a figure containing C. Moreover, we have $C_n\supset C_{n+1}$ for each $n\in\boldsymbol{N}$, since evidently $V_n\supset V_{n+1}$. The sequence of figures $\langle C_1, C_2, \cdots\rangle$ has the required property, the set C plainly being the intersection of this sequence.

The images $f[C], f[C_1], f[C_2], \cdots$ are all measurable sets since they are compact; and we find at once that $f[C]$ is the limit of the descending sequence $f[C_1]\supset f[C_2]\supset\cdots$. It follows that

$$|f[C]|=\lim_n |f[C_n]|.$$

The same relation holds of course for the functions $g(x)$ and $\theta(x)$ also. The lemma is therefore reduced to the assertion that

$$|\theta[C_n]|=|f[C_n]|+|g[C_n]| \qquad \text{for} \quad n\in\boldsymbol{N}.$$

Given any function $h(x)$ and any nonvoid figure L, let $h(L)$ denote the sum of the increments $h(J)$, where J ranges over all the component intervals of L. If, in particular, the function $h(x)$ is continuous and nondecreasing, we have the relation $|h[L]|=h(L)$. To see this, we write $J=[a_J, b_J]$ for each J and we find successively that

$$h[J]=[h(a_J), h(b_J)], \quad |h[J]|=h(b_J)-h(a_J)=h(J),$$

$$|h[L]|=|\bigcup_J h[J]|=\sum_J |h[J]|=\sum_J h(J)=h(L).$$

Let it be remarked that the interval $[h(a_J), h(b_J)]$ is degenerate (singletonic) in the case in which $h(a_J)=h(b_J)$.

It follows from the above that

$$|\theta[C_n]|=\theta(C_n)=f(C_n)+g(C_n)=|f[C_n]|+|g[C_n]|$$

for $n \in \boldsymbol{N}$, which completes the proof.

LEMMA 6. *Given any nonvoid linear compact set S which is perfect and nondense, there exists a continuous increasing function $\theta(x)$ fulfilling $\theta[\boldsymbol{R}]=\boldsymbol{R}$ and which maps each portion of S onto a set of positive measure.*

PROOF. A closed interval I will be called *admissible*, in this proof, if each end point of I is an accumulation point of the intersection $I \cap S$ (and hence belongs to S).

By hypothesis, the open intervals contiguous to S cannot be finite in number. Let us arrange all of them in an infinite sequence $\langle D_1, D_2, \cdots\cdot \rangle$ without repetitions. Given an admissible interval I, let k be the least of the indices n such that $D_n \subset I$. Writing by definition

$$I^* = I \setminus D_k,$$

we see at once that I^* is a figure with exactly two components which are themselves admissible intervals.

By an *admissible figure* we understand any nonvoid figure all whose components are admissible intervals. Given an admissible figure L, we denote by L^* the union of the figures J^*, where J ranges over all the components of L. It is evident that L^* is itself an admissible figure.

This being so, we now define a descending infinite sequence of admissible figures, $A_0 \supset A_1 \supset \cdots\cdot$, inductively as follows: A_0 is the closed interval spanned by the compact set S and

$$A_{m+1} = A_m^* \qquad \text{for} \quad m \in \boldsymbol{M}.$$

Writing $E = A_0 \cap A_1 \cap \cdots\cdot$, we obtain the relation

$$A_0 \setminus E = \bigcup_{m=0}^{\infty} (A_m \setminus A_{m+1}),$$

on account of the descent $A_0 \supset A_1 \supset \cdots\cdot$. But the union on the right will be identified below with the open set $\Delta = D_1 \cup D_2 \cup \cdots\cdot$. We thus have

$$S = A_0 \setminus \Delta = E = A_0 \cap A_1 \cap \cdots\cdot.$$

The set $A_m \setminus A_{m+1}$ is, for every $m \in \boldsymbol{M}$, plainly the union of exactly 2^m intervals of the sequence $\langle D_1, D_2, \cdots\cdot \rangle$ and hence is contained in Δ. It therefore suffices to show that each D_n is contained in $A_m \setminus A_{m+1}$ for some $m \in \boldsymbol{M}$.

Suppose, if possible, that there is a D_q not contained in any $A_m \setminus A_{m+1}$. Then D_q must be disjoint with every $A_m \setminus A_{m+1}$, and we find at once by induction that $D_q \subset A_m$ for every $m \in \boldsymbol{M}$. Consequently there is for each $m \in \boldsymbol{M}$ a component I_m of A_m such that $D_q \subset I_m$. We then have $I_{m+1} \subset I_m^*$

for every $m\in\boldsymbol{M}$, and it follows that the sequence $\langle I_m\setminus I_m^*; m\in\boldsymbol{M}\rangle$ is a subsequence of $\langle D_1, D_2, \cdots\rangle$. Hence there necessarily exist an $m\in\boldsymbol{M}$ and an $n\in\boldsymbol{N}$ such that

$$I_m\setminus I_m^*=D_n \quad \text{and} \quad n>q.$$

But the definition of I_m^*, together with the inclusion $D_q\subset I_m$, implies that $n\leqq q$, which is a contradiction.

Using the relation $S=A_0\cap A_1\cap\cdots$ thus established, we proceed to construct on S a continuous nondecreasing function $f(x)$, as follows.

Let ξ be any fixed point of the set S. Then ξ belongs, for each $m\in\boldsymbol{M}$, to a uniquely determined component K_m of the figure A_m. For each $n\in\boldsymbol{N}$, the interval K_n is a component of the figure K_{n-1}^*. We define ξ_n to be 0 or 1, according as K_n is the left-hand or the right-hand component of K_{n-1}^*, respectively. Using the sequence $\langle\xi_1, \xi_2, \cdots\rangle$ thus constructed, we write by definition

$$f(\xi)=\sum_{n=1}^{\infty} 2^{-n}\xi_n=(0.\,\xi_1\xi_2\cdots)_2,$$

where the symbol $(\cdots)_2$ instructs that the decimal be understood in the binary scale of notation.

We have thus defined on S a function $f(x)$. It is obvious that $f[S]$ is the interval $[0, 1]$.

The interval K_m associated with $m\in\boldsymbol{M}$ depends not only on m, but also on the point ξ. In case we want to show this dependence explicitly, we shall write $K_m(\xi)$. Similarly, the sequence $\langle\xi_1, \xi_2, \cdots\rangle$ may be written $\sigma(\xi)$, as occasion demands.

Besides the above point ξ, consider another point η of S. Then

$$f(\eta)=(0.\,\eta_1\eta_2\cdots)_2, \quad \text{where } \langle\eta_1, \eta_2, \cdots\rangle=\sigma(\eta).$$

Assuming that $\xi<\eta$, we shall now show that $f(\xi)\leqq f(\eta)$, so that the function $f(x)$ will be found nondecreasing.

The perfect set S being nondense, the interval $[\xi, \eta]$ contains an open interval D contiguous to S. But $D\subset A_{n-1}\setminus A_n$ for an $n\in\boldsymbol{N}$, as we proved already. It follows that $K_n(\xi)\neq K_n(\eta)$; for otherwise we must have

$$[\xi, \eta]\subset K_n(\xi)\subset A_n\subset\boldsymbol{R}\setminus D,$$

which contradicts the inclusion $D\subset[\xi, \eta]$.

Denoting by q the smallest of the indices $i\in\boldsymbol{N}$ fulfilling $K_i(\xi)\neq K_i(\eta)$, we find that q is also the smallest of the indices $j\in\boldsymbol{N}$ fulfilling $\xi_j\neq\eta_j$, where

$$\langle\xi_1, \xi_2, \cdots\rangle=\sigma(\xi) \quad \text{and} \quad \langle\eta_1, \eta_2, \cdots\rangle=\sigma(\eta).$$

Moreover, the inequality $\xi<\eta$ clearly implies that $\xi_q=0$ and $\eta_q=1$. It accordingly follows that

$$f(\xi)=(0.\,\xi_1\xi_2\cdots)_2\leqq(0.\,\eta_1\eta_2\cdots)_2=f(\eta), \qquad \text{Q. E. D.}$$

The above argument shows in passing that the mapping $\sigma(x)$, where $x\in S$, is biunique.

Let us examine the case in which we have $f(\xi)=f(\eta)$ in the last inequality. This occurs only if

$$\xi_i=1 \quad \text{and} \quad \eta_i=0 \qquad \text{for all} \quad i>q,$$

i. e. only if $\sigma(\xi)=\sigma(a)$ and $\sigma(\eta)=\sigma(b)$ simultaneously, where we write

$$(a, b)=K_{q-1}(\xi)\setminus K_{q-1}^*(\xi).$$

This simultaneous condition implies, on account of the biuniqueness of $\sigma(x)$, that $\xi=a$ and $\eta=b$.

Given two points $\xi<\eta$ of S, suppose that the interval $J=[\xi, \eta]$ is not contiguous to S. Then $f(\xi)<f(\eta)$ by what we proved just now. But $f(x)$ is nondecreasing on S and further $f[S]=[0, 1]$, whence $f[J\cap S]=[f(\xi), f(\eta)]$. It follows that $|f[J\cap S]|>0$.

It is obvious that if a function, with a nonvoid linear set for its domain of definition, is nondecreasing and maps this set onto a connected set, then the function is necessarily continuous. The function $f(x)$, which is defined on S, is therefore continuous, and hence so is also the function $\theta(x)=f(x)+x$, where $x\in S$. Evidently $\theta(x)$ is an increasing function.

Let P be any portion of the set S. Since S is perfect, there exist in P two points $\xi<\eta$ such that the interval $J=[\xi, \eta]$ is not contiguous to S. We then have $|f[J\cap S]|>0$, as shown above. But $|\theta[J\cap S]|\geqq|f[J\cap S]|$ by Lemma 5, where we take $C=J\cap S$ and $g(x)\equiv x$. Noting the relation $J\cap S\subset P$, we find that

$$|\theta[P]|\geqq|\theta[J\cap S]|\geqq|f[J\cap S]|>0.$$

Since the portion P is a set $(\mathfrak{F}_\sigma)$, so is also its continuous image $\theta[P]$, which is therefore a measurable set. Thus $\theta[P]$ is a set of positive measure.

We now extend, as we easily can, the function $\theta(x)$ defined on S to one which is defined, continuous, and increasing on the whole $\boldsymbol{R}$ and which maps $\boldsymbol{R}$ onto itself. This completes the proof.

We shall also use the following lemma, whose proof is quite immediate and may be omitted.

LEMMA 7. *If a function $\varphi(x)$ is Dirichlet continuous on a compact nonconnected set Q and if $\theta(x)$ is a continuous increasing function which* *maps* $\boldsymbol{R}$ *onto itself, then the composite function* $\varphi\circ\theta^{-1}(t)$, *defined on* $\boldsymbol{R}$, *is Dirichlet continuous on* $\theta[Q]$ *which is itself a compact nonconnected set together with* Q.

THEOREM. *Given a nonvoid linear compact set* S *which is perfect and nondense, suppose that a function* $\varphi(x)$ *is Dirichlet continuous on* S *and not* BV *on any portion of* S.

Let $\theta(x)$ *be any continuous increasing function with* $\theta[\boldsymbol{R}]=\boldsymbol{R}$ *and which maps each portion of* S *onto a set of positive measure, the existence of such a function* $\theta(x)$ *being ensured by Lemma* 6. *Then the composite function* $\varphi\circ\theta^{-1}(t)$, *defined on* $\boldsymbol{R}$, *is Dirichlet continuous, without being powerwise continuous, on the compact set* $\theta[S]$ *which is perfect and nondense together with* S.

REMARK. Let I be a closed interval and let us resume the set $\Gamma=E_0\cap E_1\cap\cdots$ and the function $\Xi(x)=\Psi(x;I;3^{-1})$ that we have already used in § 2. By Lemma 12 and Theorem 21 of [5], this function is Dirichlet continuous on Γ, but not GBV on the intersection $\Gamma\cap K$, where K is any component of the figure E_m $(m\in\boldsymbol{M})$. It follows at once that $\Xi(x)$ cannot be GBV on any portion of Γ. Noting that Γ is a perfect nondense set, we see that the hypothesis of the theorem is certainly fulfilled if we take $S=\Gamma$ and $\varphi(x)=\Xi(x)$.

PROOF. On account of Lemma 7, the function $\psi(t)=\varphi\circ\theta^{-1}(t)$ is Dirichlet continuous on the set $T=\theta[S]$. To show that this function is not powerwise continuous on T, consider any portion T_0 of T and express T_0 in the form $T_0=T\cap D$, where D is an open interval. Let t_0 be any point of T_0, so that $\theta^{-1}(t_0)\in S$. We can take an open interval J, with $\theta^{-1}(t_0)$ for its centre, in such a way that $\theta[J]\subset D$. Then

$$\theta[S\cap J]=\theta[S]\cap\theta[J]\subset T\cap D=T_0,$$

where $\theta[S\cap J]$, and hence T_0 also, has positive measure, since $S\cap J$ is a portion of S. On the other hand, the function $\varphi(x)$ is not BV on this portion, so that the function $\psi(t)$ cannot be BV on $\theta[S\cap J]$ either. It follows that $\psi(t)$ is not AC on T_0. This, combined with $|T_0|>0$, shows that $\psi(t)$ does not fulfil the condition (P) on the portion T_0 of T. We thus find, by Theorem 11 of [5], that $\psi(t)$ is not powerwise continuous on T. This completes the proof.

The above Theorem and Remark together imply that a function which is Dirichlet continuous on a compact nonconnected set, is not necessarily powerwise continuous on this set. We have thus solved in the negative the problem proposed at the end of § 2 of [5].

In the particular case considered in the above Remark, it is possible to construct a function $\theta(x)$ which conforms to the requirement of the Theorem and whose way of construction is simpler and more concrete than in the proof of Lemma 6. For this purpose, we need only modify slightly the sequence $\langle E_0, E_1, \cdots\rangle$ attached to the interval I, so as to make each portion of the intersection of the modified sequence have positive measure.

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