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On the Powerwise Integration in the Wide Sense

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Introduction. This is a continuation of our recent works [5] and [6] on the powerwise integration. We shall be concerned with showing that this integration is capable of still further generalization. The new integration is strictly wider than the old, as will be ensured by a concrete example. The final section, which is supplementary, will deal with a few results on the Dirichlet continuity of functions.

§1. Functions powerwise continuous in the wide sense.

A function will be called *strongly Dirichlet continuous* on a compact nonconnected set of real numbers, if the function is Dirichlet continuous on this set and approximately derivable to zero at almost all points of the same set. We do not know whether this property is really more restrictive than the Dirichlet continuity.

We shall term a function $\varphi(x)$ to fulfil the *condition* (P_W) on a linear set E, if there exist two functions $F(x)$ and $G(x)$ such that

(i) we have $\varphi(x)=F(x)+G(x)$ for $x \in \boldsymbol{R}$,
(ii) $F(x)$ is absolutely continuous on E,
(iii) $G(x)$ is strongly Dirichlet continuous on some compact nonconnected set containing E (so that E must be a bounded set).

Evidently, such a function $\varphi(x)$ fulfils the condition (P_W) on every subset of E. Again, a function which fulfils the condition (P) on a linear bounded set, necessarily fulfils the condition (P_W) on the same set.

THEOREM 1. *A function $\varphi(x)$ which fulfils the condition* (P_W) *on a set E, does so also on the closure of E, provided that the function is continuous on this closure.*

PROOF. By hypothesis, there exist a function $G(x)$ and a compact nonconnected set $Q \supset E$, such that $G(x)$ is strongly Dirichlet continuous on Q and that the function $F(x)=\varphi(x)-G(x)$ is AC on E. Now $\varphi(x)$ is assumed continuous on the closure E_0 of E. On the other hand $G(x)$, which is continuous on Q, is so also on $E_0 \subset Q$. It follows that $F(x)$ is continuous on E_0. But a function which is AC on E and continuous on E_0, is necessarily AC on E_0 (see Saks [7], p. 224). Hence $F(x)$ is AC on E_0, and the proof is complete.

THEOREM 2. *The sum, or more generally, any linear combination with constant coefficients, of two functions* $\varphi_1(x)$ *and* $\varphi_2(x)$ *which fulfil the condition* (P_W) *on a set* E, *itself does so on this set.*

PROOF. It is sufficient to deal with the sum $\varphi(x)=\varphi_1(x)+\varphi_2(x)$. By hypothesis, we can write $\varphi_i(x)=F_i(x)+G_i(x)$ for $i=1$ and $i=2$, where in each case the function $F_i(x)$ is AC on the set E and $G_i(x)$ is strongly Dirichlet continuous on a compact nonconnected set $Q_i \supset E$. Then the sum $F(x)=F_1(x)+F_2(x)$ is AC on E. Let us write $Q=Q_1 \cap Q_2$, so that Q is a compact set containing E.

We have two cases to distinguish, according as the set Q is connected or not. In the former case, the function $G_1(x)$ is a constant on Q, and similarly for $G_2(x)$. Accordingly the sum $G(x)=G_1(x)+G_2(x)$ is AC on E, and hence so is also the function $\varphi(x)=F(x)+G(x)$, which therefore fulfils the condition (P_W) on E. In the latter case, Q is a compact nonconnected set, and both $G_1(x)$ and $G_2(x)$ are strongly Dirichlet continuous on Q. Hence the same is true of their sum $G(x)$. The function $\varphi(x)=F(x)+G(x)$ thus fulfils the condition (P_W) on E, and the proof is complete.

A function will be called *powerwise continuous in the wide sense*, or *continuous* (P_W), on a linear set E, if it is continuous on E and if this set is expressible as the union of a sequence (finite or enumerable) of sets on each of which the function fulfils the condition (P_W). Plainly, a function which is powerwise continuous (in particular, GAC) on a set, is necessarily continuous (P_W) on this set. Again, a function which is continuous (P_W) on a set, is so also on every subset of this set.

THEOREM 3. *Every function* $\varphi(x)$ *which is continuous* (P_W) *on a measurable set* E *is approximately derivable at almost all points of this set.*

PROOF. Consider first any linear set S on which the function $\varphi(x)$ of the theorem fulfils the condition (P_W). Then $\varphi(x)$ is expressible as the sum of two functions $F(x)$ and $G(x)$, such that $F(x)$ is AC on S and that $G(x)$ is strongly Dirichlet continuous on some compact nonconnected set containing S. $G(x)$ is then continuous on S and AD at all the points of $S \setminus T$, where T is some null subset of S. But it is known that every function which is continuous on, and AD at all the points of, a linear set, is GAC on this set (see Saks [7], p. 239). Hence $G(x)$ is GAC on the set $S\setminus T$, and it follows at once that so must also be the function $\varphi(x)$.

This being premised, the assertion may be established as follows. The set E is the union of a sequence of sets E_n on each of which $\varphi(x)$ fulfils the condition ($\mathrm{P_W}$). But E_n contains, for each n, a null set T_n such that $\varphi(x)$ is GAC on $E_n\setminus T_n$. The union T of the sets T_n being null, the set $E\setminus T$ is measurable together with E; and $\varphi(x)$ is GAC on this set, since $E\setminus T$ is the union of the sets $E_n\setminus T\subset E_n\setminus T_n$ and since $\varphi(x)$ is continuous on E. But any function which is GAC on a measurable set, is AD at almost all points of this set (see Saks [7], p. 223). It follows finally that the function $\varphi(x)$ is AD at almost all points of the set E. The proof is thus complete.

THEOREM 4. *Every linear combination, with constant coefficients, of two functions which are continuous* ($\mathrm{P_W}$) *on a linear set, is itself continuous* ($\mathrm{P_W}$) *on this set.*

This is immediate from Theorem 2.

THEOREM 5. *In order that a function which is continuous over a nonvoid closed set E, be continuous* ($\mathrm{P_W}$) *on E, it is necessary and sufficient that each nonvoid closed subset of E contain a portion on which the function fulfils the condition* ($\mathrm{P_W}$).

Making use of Theorem 1, we may establish this in quite the same way as in Theorem 11 of [5].

THEOREM 6. *Given a function $F(x)$ which is* AC *on a compact nonconnected set Q and given a function $G(x)$ which is Dirichlet continuous on the same set, suppose that $F(x)$ has a finite nonnegative approximate derivative $F'_{\mathrm{ap}}(x)$ at almost every point x of Q and further that the sum $\xi(x)=F(x)+G(x)$ has a nonnegative increment over each closed interval contiguous to Q.*

Then $\xi(x)$ is nondecreasing over Q.

PROOF. Let A be any closed interval with end points belonging to the set Q. We have to ascertain that $\xi(A)\geqq 0$. But this is obvious if $A\subset Q$; in fact, the hypothesis then implies that $F(x)$ is nondecreasing on A (see Saks [7], p. 225) and that $G(x)$ is a constant on A. We may therefore assume in the sequel that Q does not contain A.

If we write $\lambda(x)$ for the linear modification of $F(x)$ with respect to Q, then Theorem 4 of [5] shows that $\lambda(x)$ is AC on the interval I spanned by Q. Hence $\lambda(x)$ is derivable almost everywhere on I. Moreover, by Lemma 3 of [5], the function $\lambda(x)$ is AED with $F(x)$ at almost all points of Q. Consequently $\lambda(x)$ has a finite nonnegative derivative $\lambda'(x)$ at almost all points $x \in Q$. We may plainly suppose, without loss of generality, that $F(x)$ coincides with $\lambda(x)$ identically.

For definiteness, let us set $F'(x)=0$ for every point $x \in \boldsymbol{R}$ at which $F(x)$ is not derivable. $F'(x)$ is then summable on the interval I. Denoting by H a generic open interval contiguous to the compact nonconnected set $A \cap Q$, we find that $A \setminus Q$ is the union of all the H and that

$$F(A)=\int_A F'(x)\,dx=\int_{A\cap Q} F'(x)\,dx+\int_{A\setminus Q} F'(x)\,dx$$

$$=\int_{A\cap Q} F'(x)\,dx+\sum_H \int_H F'(x)\,dx=\int_{A\cap Q} F'(x)\,dx+\sum_H F(H).$$

Since $F'(x) \geqq 0$ at almost all points of Q, it follows that

$$F(A) \geqq \sum_H F(H), \quad \text{where} \quad \sum_H |F(H)| < +\infty.$$

On the other hand, writing p generically for a real number >1, we have

$$\sum_H F(H)=\lim_{p\to 1} \sum_H F(H)\Box^p \quad \text{and} \quad G(A)=\lim_{p\to 1} \sum_H G(H)\Box^p,$$

since $\sum |F(H)| < +\infty$ and since $G(x)$ is Dirichlet continuous on Q. Hence

$$\xi(A)=F(A)+G(A) \geqq \lim_{p\to 1} \sum_H \{F(H)\Box^p+G(H)\Box^p\}.$$

But we have, for any two real numbers α and β, the relation

$$|(\alpha+\beta)\Box^p-\alpha\Box^p-\beta\Box^p| \leqq 2^{p-1}(p-1)(|\alpha|^p+|\beta|^p)$$

by Lemma 1 of [5], and this ensures that

$$\xi(H)\Box^p-F(H)\Box^p-G(H)\Box^p \leqq 2^p(p-1)\{|F(H)|^p+|G(H)|^p\}$$

for every H, where $\xi(H)\Box^p \geqq 0$ since $\xi(H) \geqq 0$ by hypothesis. Thus

$$F(H)\Box^p+G(H)\Box^p \geqq -2^p(p-1)\{|F(H)|^p+|G(H)|^p\},$$

$$\sum_H \{F(H)\Box^p+G(H)\Box^p\} \geqq -2^p(p-1)\{\sum_H |F(H)|^p+\sum_H |G(H)|^p\}.$$

This inequality, together with the relations

$$\sum_H |F(H)|^p=\sum_H |F(H)|+o(1) \quad \text{and} \quad \sum_H |G(H)|^p=o\left(\frac{1}{p-1}\right),$$

implies that $\lim_{p\to 1} \sum_H \{F(H)\Box^p+G(H)\Box^p\} \geqq 0$. It follows finally that $\xi(A) \geqq 0$, which completes the proof.

THEOREM 7. *Given a function $\xi(x)$ which is continuous on a closed interval I and having a finite nonnegative approximate derivative almost everywhere on I, define a subset S of I as follows: a point of I belongs to S if and only if there exists no open interval V containing this point and such that the function $\xi(x)$ is* GAC *on the interval $I \cap V$.*

Then S is a perfect set. Moreover, if a finite interval D contained in I is disjoint with S, the function $\xi(x)$ is GAC *and nondecreasing on D.*

This proposition is incidentally established in the proof for Theorem 13 of [5]. The function $\xi(x)$ is even AC on D, as we can easily show.

THEOREM 8. *Given two functions $\varphi(x)$ and $\psi(x)$ which are continuous* $(\mathrm{P_W})$ *over a closed interval I, if we have $\varphi'_{\mathrm{ap}}(x) \leqq \psi'_{\mathrm{ap}}(x)$ at almost every point x of I at which both functions are* AD, *their difference $\psi(x)-\varphi(x)$ is both* AC *and nondecreasing, on the interval I.*

COROLLARY. *If two functions are continuous* $(\mathrm{P_W})$ *on a closed interval I and if they are approximately equiderivable almost everywhere on I, then the functions differ over I only by an additive constant.*

PROOF OF THEOREM 8. The difference $\xi(x)=\psi(x)-\varphi(x)$ is continuous $(\mathrm{P_W})$ on I by Theorem 4 and has a finite nonnegative approximate derivative almost everywhere on I (see Theorem 3). Let S be the set attached to $\xi(x)$ and I in the manner of Theorem 7. The assertion will follow if we show that S is void; for then $\xi(x)$ will be GAC and nondecreasing, on the whole interval I, and therefore, by a routine inference, also AC on I.

Suppose, if possible, that S is nonvoid. There exists, by Theorem 5, an open interval H such that the intersection $M=S\cap H$ is nonvoid and that $\xi(x)$ fulfils the condition $(\mathrm{P_W})$ on M. This set M must be infinite; for otherwise each point of M would be an isolated point of S, contradicting the fact that S is a perfect set. The function $\xi(x)$, which is continuous on the interval I, fulfils the condition $(\mathrm{P_W})$ on the closure M_0 of M, in virtue of Theorem 1. We can therefore write $\xi(x)=F(x)+G(x)$ for $x \in \boldsymbol{R}$, where the function $F(x)$ is AC on M_0 and where $G(x)$ is strongly Dirichlet continuous on some compact nonconnected set containing M_0.

We shall show that $\xi(x)$ is nondecreasing on M_0. This is obvious if M_0 is a connected set and hence a closed interval. Indeed, $G(x)$ is then a constant on M_0, so that $\xi(x)$ must be AC on M_0. But $0 \leqq \xi'_{\mathrm{ap}}(x) < +\infty$ almost everywhere on I, and the result follows at once. We may therefore assume in the sequel that M_0 is a compact nonconnected set. Then $G(x)$ is strongly Dirichlet continuous on M_0.

At almost every point x of M_0, the function $F(x)$ is AD by the Denjoy-Khintchine Theorem (see Saks [7], p. 222) and $G(x)$ is AD to zero. We thus have

$$F'_{\mathrm{ap}}(x)=F'_{\mathrm{ap}}(x)+G'_{\mathrm{ap}}(x)=\xi'_{\mathrm{ap}}(x)\in[0, +\infty)$$

at almost every point x of M_0. On the other hand, since M_0 is the closure of the set $M=S\cap H$, each open interval D contiguous to M_0 is at the same time contiguous to S. Hence $\xi(x)$ is GAC and nondecreasing on D, in virtue of Theorem 7. This together with the continuity of $\xi(x)$ on I, shows that $\xi(x)$ is nondecreasing on the closure of D and in particular that $\xi(D)\geqq 0$. Combining the above results and applying Theorem 6, where we replace the set Q by M_0, we find that $\xi(x)$ is nondecreasing over M_0, as announced.

This fact implies that the function $G(x)=\xi(x)-F(x)$ is BV on M_0. Since $G(x)$ is further Dirichlet continuous on M_0, Theorem 3 of [5] shows that $G(x)$, and hence $\xi(x)$ also, is AC on M_0. But $\xi(x)$ is GAC on each open interval contiguous to M_0, as already proved. We thus conclude that $\xi(x)$ is GAC on the closed interval spanned by M_0. This contradicts the definition of the set S, since M_0 is an infinite subset of S. This completes the proof.

THEOREM 9. *If a function $\varphi(x)$ which is continuous* ($\mathrm{P_W}$) *on a closed set E, is* GBV *on this set, then the function is necessarily* GAC *on E. If in particular the set E is compact and the function is* BV *on E, then it is* AC *on E.*

PROOF. We need only prove the first half of the theorem, the second half being an easy consequence of the first. We express the set E as the union of a sequence of bounded sets M on each of which $\varphi(x)$ fulfils the condition ($\mathrm{P_W}$). In view of Theorem 1, we may assume the sets M closed. It suffices to verify that the function is GAC on each M.

Suppose, if possible, that $\varphi(x)$ fails on some M to be GAC. Then M must be an infinite set. The function $\varphi(x)$ is expressible on $\boldsymbol{R}$ as the sum of two functions $F(x)$ and $G(x)$, such that $F(x)$ is AC on M and that $G(x)$ is strongly Dirichlet continuous on some compact nonconnected set containing M. If M is a connected set, then $G(x)$ is a constant over M, so that $\varphi(x)$ must be AC on M, which is impossible. M is therefore a compact nonconnected set, and the function $G(x)=\varphi(x)-F(x)$ is GBV as well as Dirichlet continuous, on M. Hence it follows from Theorem 6 of [5] that $\varphi(x)$ is GAC on M. This contradiction completes the proof.

REMARK. Concerning the function $G(x)$, it may be observed that the above proof utilizes only its Dirichlet continuity on the set M.

§ 2. The powerwise integration in the wide sense.

The descriptive definition and the properties of the *powerwise integration in the wide sense,* or simply the *integration* (P_W), is quite similar to those of the powerwise integration, and we think it needless to state them at great length here.

We shall be concerned in this § with showing that the integration (P_W) is strictly wider than the powerwise integration. For this purpose, we shall have recourse to the same method as used in § 4 of our paper [5].

Given a closed interval $I=[a,b]$ and an integer $m\geqq 0$, let us determine four points $c_1<c_2<c_3<c_4$ of I so as to fulfil

$$c_1-a=b-c_4 \quad\text{and}\quad c_{i+1}-c_i=2^{-m-3}(b-a),$$

where $i=1,2,3$. We attach to I two figures $I^{(m)}$ and $I^{[m]}$ defined by

$$I^{(m)}=[a,c_1]\cup[c_2,c_3]\cup[c_4,b] \quad\text{and}\quad I^{[m]}=[a,c_1]\cup[c_4,b].$$

It is obvious that

$$|I^{(m)}|=(1-2^{-m-2})|I| \quad\text{and}\quad |I\setminus I^{[m]}|<2^{-m-1}|I|.$$

Given a nonvoid figure F with the components $I_1<\cdots\cdot<I_k$, we write further by definition

$$F^{(m)}=I_1^{(m)}\cup\cdots\cdot\cup I_k^{(m)} \quad\text{and}\quad F^{[m]}=I_1^{[m]}\cup\cdots\cdot\cup I_k^{[m]}$$

for $m\in\boldsymbol{M}$, where and subsequently $\boldsymbol{M}$ denotes the set of the integers $\geqq 0$. Clearly

$$|F^{(m)}|=(1-2^{-m-2})|F| \quad\text{and}\quad |F\setminus F^{[m]}|<2^{-m-1}|F|.$$

This being so, we now construct inductively a descending infinite sequence $U_0\supset U_1\supset\cdots\cdot$ of figures by the rule:

$$U_0=[0,1] \quad\text{and}\quad U_{m+1}=U_m^{(m)} \quad\text{for}\quad m\in\boldsymbol{M}.$$

Writing $\Gamma=U_0\cap U_1\cap\cdots\cdot$, we find easily that Γ is a nonconnected perfect set spanning the interval U_0.

LEMMA 1. *With the above notation, let K be any component of the figure U_m, where $m\in\boldsymbol{M}$. Then $|K|\leqq 2^{-m}$ and $|K\cap\Gamma|>2^{-1}|K|$.*

In particular, we have $|\Gamma|>2^{-1}$.

PROOF. The first inequality being obvious, we need only prove the second. Keeping m fixed, we define inductively a descending infinite sequence $F_0 \supset F_1 \supset \cdots$ of figures, as follows:

$$F_0 = K \quad \text{and} \quad F_{i+1} = F_i^{(m+i)} \quad \text{for} \quad i \in \boldsymbol{M}.$$

Then $F_i = K \cap U_{m+i}$, so that

$$K \cap \Gamma = F_0 \cap F_1 \cap \cdots = \lim_i F_i \quad \text{and} \quad |K \cap \Gamma| = \lim_i |F_i|.$$

But $|F_{i+1}| = (1 - 2^{-m-i-2})|F_i|$, and it follows that

$$|K \cap \Gamma| = |F_0| \prod_{i=0}^{\infty} (1 - 2^{-m-i-2}) > |K| \left(1 - \sum_{i=0}^{\infty} 2^{-m-i-2}\right)$$
$$= (1 - 2^{-m-1})|K| \geqq 2^{-1}|K|,$$

which completes the proof.

The above inequality $|K| \leqq 2^{-m}$ implies that Γ is a nondense set.

Given any $m \in \boldsymbol{M}$, we now construct inductively a descending infinite sequence $W_0 \supset W_1 \supset \cdots$ of figures, writing

$$W_0 = U_m \quad \text{and} \quad W_{i+1} = W_i^{[m+i]} \quad \text{for} \quad i \in \boldsymbol{M}.$$

Then every component of W_i is a component of U_{m+i}, as is easily verified by induction on i.

Consider now the following set Γ_m:

$$\Gamma_m = W_0 \cap W_1 \cap \cdots \subset U_m \cap U_{m+1} \cap \cdots = \Gamma.$$

Evidently, Γ_m is nonvoid and compact.

LEMMA 2. *We have* $|\Gamma \setminus \Gamma_m| < 2^{-m}$ *for* $m \in \boldsymbol{M}$.

PROOF. Using the above sequence $W_0 \supset W_1 \supset \cdots$, we find successively

$$\Gamma \setminus \Gamma_m \subset U_m \setminus \Gamma_m = \bigcup_{i=0}^{\infty} (W_i \setminus W_{i+1}),$$

$$|W_i \setminus W_{i+1}| = |W_i \setminus W_i^{[m+i]}| < 2^{-m-i-1}|W_i| \leqq 2^{-m-i-1},$$

$$|\Gamma \setminus \Gamma_m| \leqq \sum_{i=0}^{\infty} |W_i \setminus W_{i+1}| < \sum_{i=0}^{\infty} 2^{-m-i-1} = 2^{-m}.$$

By a *binary sequence of numbers* we shall understand any infinite sequence $\langle a_1, a_2, \cdots \rangle$ such that $a_n \in \{0, 1\}$ for every $n \in \boldsymbol{N}$, where and subsequently $\boldsymbol{N}$ denotes the set of the positive integers.

Let x be any point of the set $\Gamma = U_0 \cap U_1 \cap \cdots$. For each $m \in \boldsymbol{M}$ there exists, among the components of the figure U_m, exactly one, say K_m, to which x belongs. It follows, from the relation $U_{m+1} = U_m^{(m)}$, that the interval K_{m+1} is then one of the three components of the figure $K_m^{(m)}$. Let a_{m+1} be 1 or 0 according as K_{m+1} is the middle one among these components (in their natural ordering) or not, respectively. We have thus defined a binary sequence $\sigma(x)=\langle a_1, a_2, \cdots\cdot\rangle$ for each point x of Γ. When occasion demands, we may write $a_n=a_n(x)$ for $n\in \boldsymbol{N}$.

From now on, let δ denote a generic positive number $<2^{-1}$. By means of the above sequence $\sigma(x)=\langle a_1, a_2, \cdots\cdot\rangle$, we now define on the set Γ a function $\Theta(x;\delta)$ with x for the variable, as follows:

$$\Theta(x;\delta)=\sum_{n=1}^{\infty}\frac{a_n(x)}{n}\delta^n \qquad \text{for} \quad x\in\Gamma.$$

We then extend the domain of definition of this function to the whole real line, in such a way that the extended function, which we shall denote still by $\Theta(x;\delta)$, is linear on each closed interval contiguous to Γ and vanishes outside the interval $U_0=[0,1]$.

LEMMA 3. *If K is any component of the figure U_m, where $m\in \boldsymbol{M}$, the function $\Theta(x)=\Theta(x;\delta)$ is a constant on the intersection $K\cap\Gamma_m$, where $\Gamma_m=W_0\cap W_1\cap\cdots\cdot$ as above.*

PROOF. The definition of the sequence $W_0\supset W_1\supset\cdots\cdot$, namely $W_0=U_m$ and $W_{i+1}=W_i^{[m+i]}$ for $i\in\boldsymbol{M}$, clearly implies that if $x\in\Gamma_m$, then the binary sequence $\sigma(x)=\langle a_1(x), a_2(x), \cdots\cdot\rangle$ has the property:

$$a_n(x)=0 \qquad \text{for all} \quad n>m.$$

On the other hand, in the case in which $m>0$, it is obvious that for each $n=1, 2, \cdots\cdot, m$ the number $a_n(x)$, as a function of x, is a constant over $K\cap\Gamma$.

It follows that the sequence $\sigma(x)$ is independent of x, when x ranges over $K\cap\Gamma_m$. The function $\Theta(x)$ must therefore be a constant over this intersection, and the proof is complete.

We shall say that a function is *locally constant* on a linear set E, if there corresponds to each point (if existent) of E an open interval J containing this point and such that the function is a constant over $E\cap J$.

LEMMA 4. *Every function $\varphi(x)$ which is locally constant on a measurable set E, is* AD *to zero at almost all points of E.*

PROOF. Assuming E nonvoid, take any point of E and consider the corresponding open interval J of the above definition. We may plainly suppose that the end points of J are rational numbers. It then suffices to show that the function $\varphi(x)$ is AD to zero at almost all points of the measurable set $E\cap J$. But this is an immediate consequence of Lemma 3 of [5]. In fact, we need only consider, alongside of $\varphi(x)$, the function which is identically equal on $\boldsymbol{R}$ to the constant value assumed by $\varphi(x)$ on $E\cap J$.

LEMMA 5. *The function* $\Theta(x)=\Theta(x;\delta)$ *is* AD *to zero at almost all points of* Γ.

PROOF. Let m be a fixed integer $\geqq 0$. On account of Lemma 3, the function $\Theta(x)$ is locally constant on the set $\Gamma_m\subset\Gamma$. This, together with Lemma 4, shows that $\Theta(x)$ is AD to zero at almost all points of Γ_m. Consequently, writing M for the set of the points $x\in\Gamma$ at which $\Theta(x)$ is not AD to zero, we have $|M\cap\Gamma_m|=0$. Combining this with Lemma 2, we get

$$|M|\leqq|M\cap\Gamma_m|+|M\setminus\Gamma_m|\leqq|\Gamma\setminus\Gamma_m|<2^{-m},$$

which completes the proof since m is arbitrary.

Let H denote generically an open interval contiguous to the set Γ, i.e. a component of the open set $U_0\setminus\Gamma$. We clearly have

$$U_0\setminus\Gamma=\bigcup_{m=0}^{\infty}(U_m\setminus U_{m+1}),$$

where the summands $U_m\setminus U_{m+1}$ are open and mutually disjoint. Each H is therefore a component of $U_m\setminus U_{m+1}$ for an $m\in\boldsymbol{M}$, where m is uniquely determined by H. This integer m will be called *order* of the interval H. H is then contained in a component, say K, of the figure U_m and hence coincides with one of the two components of the open set $K\setminus U_{m+1}$. We shall term H *ascendent* or *descendent*, according as it is, respectively, the first or the second of these components in their natural ordering.

LEMMA 6. *Let* K *be any component of the figure* U_m, *where* $m\in\boldsymbol{M}$.

(i) *The function* $\Theta(x)=\Theta(x;\delta)$, *where* $0<\delta<2^{-1}$, *assumes the same value at the end points of* K *and, denoting this value by* $\gamma(K)$, *we have*

$$\gamma(K)\leqq\Theta(x)<\gamma(K)+\frac{2\delta^{m+1}}{m+1}\qquad \textit{for}\quad x\in K;$$

this implies in particular that $\Theta(x)$ *is continuous on* Γ, *and hence on* $\boldsymbol{R}$.

(ii) *If* H *is an open interval contiguous to the figure* $K^{(m)}$, *then*

$$\Theta(H)=\frac{\delta^{m+1}}{m+1}\quad \textit{or}\quad \Theta(H)=-\frac{\delta^{m+1}}{m+1},$$

according as H *is ascendent or descendent, respectively.*

PROOF. We associated before with each point x of the set Γ a binary sequence $\sigma(x)=\langle a_1(x), a_2(x),\cdots\rangle$, by means of which the value of $\Theta(x)$ was defined as follows:

$$\Theta(x)=\Theta(x;\delta)=\sum_{n=1}^{\infty}\frac{a_n(x)}{n}\delta^n, \quad \text{where} \quad 0<\delta<\frac{1}{2}.$$

By Lemma 3 the function $\Theta(x)$ assumes the same value $\gamma(K)$ on the set $K\cap\Gamma_m$ which contains the end points of K, and the proof of that lemma shows further that

$$\Theta(x)=\gamma(K)+\sum_{n>m}\frac{a_n(x)}{n}\delta^n \qquad \text{for} \quad x\in K\cap\Gamma,$$

where

$$0\leqq\sum_{n>m}\frac{a_n(x)}{n}\delta^n<\frac{1}{m+1}\sum_{n>m}\delta^n=\frac{\delta^{m+1}}{(m+1)(1-\delta)}<\frac{2\delta^{m+1}}{m+1}.$$

This establishes part (i) of the assertion, since $\Theta(x)$ is linear on each closed interval contiguous to $K\cap\Gamma$.

Passing on to part (ii), let us assume first that the interval H is ascendent. Writing $H=(u,v)$, we have $\Theta(u)=\gamma(K)$, since $u\in K\cap\Gamma_m$. On the other hand, we find immediately that $a_{m+1}(v)=1$ and that $a_n(v)=0$ for $n>m+1$. Hence

$$\Theta(v)=\gamma(K)+\frac{\delta^{m+1}}{m+1}, \qquad \Theta(H)=\Theta(v)-\Theta(u)=\frac{\delta^{m+1}}{m+1}.$$

The case in which H is descendent may be treated similarly, and the proof is complete.

We shall denote henceforth by Δ_m the boundary of the figure U_m, where $m\in\boldsymbol{M}$, and by Δ the union $\Delta_0\cup\Delta_1\cup\cdots\cdot$. Noting that $\Delta\subset\Gamma$ and that every component of U_m has length $\leqq 2^{-m}$ by Lemma 1, we see that Γ is the closure of Δ. It follows at once that, for any function $\varphi(x)$ which is continuous on Γ, we have

$$\mathrm{V}(\varphi;\Gamma)=\mathrm{V}(\varphi;\Delta)=\lim \mathrm{V}(\varphi;\Delta_m),$$

where the symbol V signifies the weak variation (Saks [7], p. 221).

LEMMA 7. *If* $0<\delta<3^{-1}$, *the function* $\Theta(x)=\Theta(x;\delta)$ *is* BV *on the interval* $[0,1]$. *On the other hand, the function* $\Theta(x;3^{-1})$ *is not* BV *on the set* $K\cap\Gamma$ *for any component* K *of the figure* U_m, *where* $m\in\boldsymbol{M}$.

PROOF. By the preceding lemma, $\Theta(x)$ has a vanishing increment over each component of U_n, where $n\in\boldsymbol{N}$. Accordingly $\mathrm{V}(\Theta;\Delta_n)$ equals $\sum|\Theta(G)|$, where G denotes generically an open interval contiguous to U_n, i.e. a component of the open set $U_0\setminus U_n=\bigcup(U_{i-1}\setminus U_i)$, where i stands for $1,2,\cdots,n$ and where the summands are disjoint open sets. But each

$U_{i-1} \setminus U_i$ has exactly $2\cdot 3^{i-1}$ components, over each of which, by Lemma 6, $\Theta(x)$ has absolute increment $i^{-1}\delta^i$. It thus follows that

$$\mathrm{V}(\Theta ; \Delta_n)=\sum_{i=1}^{n} \frac{2}{3i}(3\delta)^i.$$

But the function $\Theta(x)$ is continuous on $\boldsymbol{R}$ by the same Lemma 6. Consequently, if $0<\delta<3^{-1}$, we have

$$\mathrm{V}(\Theta ; \Gamma)=\lim_n \mathrm{V}(\Theta ; \Delta_n)=\sum_{i=1}^{\infty} \frac{2}{3i}(3\delta)^i<+\infty .$$

On the other hand, $\Theta(x)$ is linear on each closed interval contiguous to Γ, so that $\mathrm{V}(\Theta; U_0)=\mathrm{V}(\Theta;\Gamma)$. $\Theta(x)$ is therefore BV on the interval U_0.

We proceed to deal with the second half of the assertion, writing $\Omega(x)=\Theta(x;3^{-1})$ for short. Given an $m\in \boldsymbol{M}$, let K be any component of U_m. Then $K\cap\Gamma$ is the closure of the set $K\cap\Delta$, and it follows that

$$\mathrm{V}(\Omega ; K\cap\Gamma)=\mathrm{V}(\Omega ; K\cap\Delta)=\lim_n \mathrm{V}(\Omega ; K\cap\Delta_n),$$

where and below n represents an integer $>m$. We have further

$$\mathrm{V}(\Omega ; K\cap\Delta_n)=\sum|\Omega(D)|,$$

where D denotes generically an open interval contiguous to $K\cap U_n$, i. e. a component of the open set $K\setminus U_n$. But this set is the union of the $n-m$ open sets

$$S_i=K\cap(U_{i-1}\setminus U_i), \quad \text{where } i=m+1, \cdots, n.$$

Since each S_i has precisely $2\cdot 3^{i-m-1}$ components, we find successively that

$$\mathrm{V}(\Omega ; K\cap\Delta_n)=\sum_{i=m+1}^{n} \frac{2\cdot 3^{i-m-1}}{i}\left(\frac{1}{3}\right)^i=\sum_{i=m+1}^{n} \frac{2}{3^{m+1}i},$$

$$\mathrm{V}(\Omega ; K\cap\Gamma)=\sum_{i>m+1} \frac{2}{3^{m+1}i}=+\infty,$$

which completes the proof.

LEMMA 8. *The function $\Omega(x)=\Theta(x;3^{-1})$ is not powerwise continuous on the set Γ.*

PROOF. In view of Theorem 11 of [5], it is enough to show that $\Omega(x)$ does not fulfil the condition (P) on any portion of Γ. Given a portion S of Γ, let c be a point of S. Then c belongs, for each $m\in\boldsymbol{M}$, to a component K of the figure U_m. Since $|K|\leqq 2^{-m}$ by Lemma 1, we can take m so large that $K\cap\Gamma$ lies in S. Then $|S|\geqq|K\cap\Gamma|>0$ by the same lemma. On the other hand, by Lemma 7, the function $\Omega(x)$ fails to be AC on $K\cap\Gamma$, and does so *a fortiori* on S. This completes the proof.

LEMMA 9. *If $p>1$ and if four real numbers ξ, η, ζ, θ belong to a closed interval of length l, we have*

(i) $|\xi-\zeta|^p-|\xi-\eta|^p-|\eta-\zeta|^p \leqq 2^p l^p(p-1)$,

(ii) $|\xi-\theta|^p-|\xi-\eta|^p-|\eta-\zeta|^p-|\zeta-\theta|^p \leqq 3^p l^p(p-1)$.

PROOF. *re* (i): We may clearly suppose that $\xi>\zeta$. We need only consider the case in which $\xi>\eta>\zeta$, for otherwise the inequality (i) is evident. Then Lemma 1 of [5] shows that

$$(\xi-\zeta)^p-(\xi-\eta)^p-(\eta-\zeta)^p \leqq 2^{p-1}(p-1)\{(\xi-\eta)^p+(\eta-\zeta)^p\},$$

whence (i) follows at once.

re (ii): We may suppose that $\xi>\theta$. Let us write for short

$$d=(\xi-\theta)^p-|\xi-\eta|^p-|\eta-\zeta|^p-|\zeta-\theta|^p.$$

If $\eta \geqq \xi$, we have by the inequality (i)

$$d \leqq (\eta-\theta)^p-|\eta-\zeta|^p-|\zeta-\theta|^p \leqq 2^p l^p(p-1);$$

while if $\eta \leqq \theta$, then evidently $d \leqq (\xi-\theta)^p-(\xi-\eta)^p \leqq 0$. We may therefore assume in the sequel that $\xi>\eta>\theta$, and similarly that $\xi>\zeta>\theta$.

If now $\eta \leqq \zeta$, then the inequality (i) implies that

$$d \leqq (\xi-\theta)^p-(\xi-\zeta)^p-(\zeta-\theta)^p \leqq 2^p l^p(p-1).$$

Consequently we may restrict ourselves to the case $\xi>\eta>\zeta>\theta$, and it follows from Lemma 2 of [5] that

$$d=(\xi-\theta)\square^p+(\theta-\zeta)\square^p+(\zeta-\eta)\square^p+(\eta-\xi)\square^p \leqq 3^p l^p(p-1),$$

which completes the proof.

Let $Z_0 \supset Z_1 \supset \cdots$ be a descending infinite sequence of nonvoid linear figures. Such a sequence will be called *regular*, if the following conditions are fulfilled:

(i) The first figure Z_0 is a closed interval,

(ii) the set $Z_m \setminus Z_{m+1}$ is nonvoid and open for each $m \in \boldsymbol{M}$,

(iii) the intersection $Z=Z_0 \cap Z_1 \cap \cdots$ is a nondense set.

We find easily that the above condition (iii) is equivalent to the following condition: if λ_m denotes $\max|K|$ for each $m \in \boldsymbol{M}$, where K ranges over the components of the figure Z_m, then $\lambda_m \to 0$ as $m \to +\infty$.

Let $Z_0 \supset Z_1 \supset \cdots$ be a regular sequence of figures. It is evident that, for each m, the boundary of Z_{m+1} contains that of Z_m and the component intervals of the open set $Z_m \setminus Z_{m+1}$ are finite in number. Further, the intersection Z is a nonvoid compact set spanning the interval Z_0, and the open set $Z_0 \setminus Z$ is the union of all the sets $Z_m \setminus Z_{m+1}$. It follows that $Z_0 \setminus Z$ is nonvoid and that an open interval H is contiguous to Z if and only if it is a component of $Z_m \setminus Z_{m+1}$ for an $m \in \boldsymbol{M}$. When this occurs, the index m, uniquely determined by H, will be called *order* of H with respect to the sequence $Z_0 \supset Z_1 \supset \cdots$.

Given a function $\varphi(x)$, a regular sequence $Z_0 \supset Z_1 \supset \cdots$, and an open interval H contiguous to $Z = Z_0 \cap Z_1 \cap \cdots$, let m be the order of H. Then H is contained in a component K of Z_m. We define a quantity $\omega(H)$ by

$$\omega(H) = \omega(\varphi; H) = \mathrm{O}(\varphi; K),$$

where the symbol O signifies the oscillation. Needless to say, $\omega(H)$ depends also on the sequence $Z_0 \supset Z_1 \supset \cdots$, though we omit explicit indication of this dependence. If $\varphi(x)$ is continuous on Z_0, then $\omega(H)$ is always finite.

Let Q be a compact nonconnected set and let X denote a generic open interval contiguous to Q. Given a function $\varphi(x)$ and a positive number p, we write by definition:

$$\Lambda(\varphi; p; Q) = \sum_X |\varphi(X)|^p \quad \text{and} \quad \Upsilon(\varphi; p; Q) = \sum_X \varphi(X)\square^p,$$

where in the case of $\Upsilon(\varphi; p; Q)$ we assume that $\Lambda(\varphi; p; Q) < +\infty$.

Let us agree to suppose, in the following two lemmas, that we are given

(i) a regular sequence $Z_0 \supset Z_1 \supset \cdots$ of figures,
(ii) a function $\varphi(x)$ which is continuous on the interval Z_0,
(iii) and a real number p greater than 1.

This agreement will not be repeated. Since the sequence $Z_0 \supset Z_1 \supset \cdots$ is regular, it is allowed, in the above definition of $\Lambda(\varphi; p; Q)$ and $\Upsilon(\varphi; p; Q)$, to take as Q any CT subset of $Z = Z_0 \cap Z_1 \cap \cdots$; in fact, such a set is always nonconnected, on account of the nondenseness of Z.

For each $m \in \boldsymbol{M}$, we shall denote by B_m the boundary of the figure Z_m, so that $B_m \subset Z$.

LEMMA 10. *Given in Z a CT set Q, suppose that $\Lambda(\varphi; p; Q) < +\infty$ and that $B_m \subset Q$ for an $m \in \boldsymbol{M}$. If we write $Q^* = Q \cup B_{m+1}$, then*

$$|\Upsilon(\varphi; p; Q) - \Upsilon(\varphi; p; Q^*)| \leqq 3^p (p-1) \sum_D \omega^p(D),$$

where D ranges over the components of the open set $Z_m \setminus Z_{m+1}$.

Given further a closed set R such that $R \cup B_{m+1} = Q^*$, denote generically by G an open interval contiguous to the figure Z_{m+1}, by L a component of Z_{m+1}, and by J an open interval (if existent) contiguous to R and con-

tained in Z_{m+1}. *Then*

$$|\Upsilon(\varphi;p;Q^*)-\sum_G \varphi(G)\square^p| \leqq 2\sum_L \mathrm{O}^p(\varphi;L)+\sum_J |\varphi(J)|^p,$$

where the last series means zero if it is a void one.

REMARK. Since B_{m+1} is a finite set, the hypothesis $\Lambda(\varphi;p;Q)<+\infty$ implies that $\Lambda(\varphi;p;Q^*)<+\infty$; in other words, $\Upsilon(\varphi;p;Q^*)$ is an absolutely convergent series. Consequently we have also $\sum|\varphi(J)|^p<+\infty$, on account of the condition $R\cup B_{m+1}=Q^*$.

PROOF. Consider any component, say $D=(\alpha,\beta)$, of $Z_m\setminus Z_{m+1}$. Then D is contained in a component K of the figure Z_m. Writing $Q'=Q\cup\{\alpha,\beta\}$, we shall appraise the difference $d=\Upsilon(\varphi;p;Q)-\Upsilon(\varphi;p;Q')$.

For this purpose, we express d in the form

$$d=\Upsilon(\varphi;p;Q\cap K)-\Upsilon(\varphi;p;Q'\cap K),$$

which follows at once from the fact that the end points of the interval K belong to Q on account of $B_m\subset Q$. We denote by α' the rightmost point of the set $Q\cap(-\infty,\alpha]$ and by β' the leftmost point of $Q\cap[\beta,+\infty)$, so that $\alpha'\leqq\alpha<\beta\leqq\beta'$, $\alpha'\in K$, and $\beta'\in K$. By means of α' and β', the difference d can now be written explicitly, as follows:

$$d=\varphi([\alpha',\beta'])\square^p-\varphi([\alpha',\alpha])\square^p-\varphi([\alpha,\beta])\square^p-\varphi([\beta,\beta'])\square^p,$$

where $[\alpha',\alpha]$ is singletonic if $\alpha'=\alpha$, and similarly for $[\beta,\beta']$. It follows from Lemma 2 of [5] that $|d|\leqq 3^p(p-1)\omega^p(D)$.

We now arrange all components D of $Z_m\setminus Z_{m+1}$ in a finite sequence $D_1<D_2<\cdots<D_N$ and we define inductively a sequence of $N+1$ sets. $\langle Q_0,Q_1,\cdots,Q_N\rangle$, as follows, where we write $D_n=(\alpha_n,\beta_n)$:

$$Q_0=Q \quad\text{and}\quad Q_n=Q_{n-1}\cup\{\alpha_n,\beta_n\} \quad\text{for}\quad n=1,\cdots,N.$$

By what was already proved we have, for the same values of n,

$$|\Upsilon(\varphi;p;Q_{n-1})-\Upsilon(\varphi;p;Q_n)|\leqq 3^p(p-1)\omega^p(D_n),$$

whence it follows immediately that

$$|\Upsilon(\varphi;p;Q)-\Upsilon(\varphi;p;Q^*)|\leqq 3^p(p-1)\sum_{n=1}^{N}\omega^p(D_n),$$

since Q_N coincides with the set $Q^*=Q\cup B_{m+1}$. This establishes the first of the asserted inequalities.

To prove the second inequality, we denote generically by G an open interval contiguous to Z_{m+1} and by L a component of Z_{m+1}. As $B_{m+1}\subset Q^*$, we have the relation

$$\Upsilon(\varphi;p;Q^*)=\sum_G \varphi(G)\Box^p+\sum_L \Upsilon(\varphi;p;Q^*\cap L).$$

Fixing an L, let us write $L=[u,v]$, so that

$$Q^*\cap L=(R\cap B_{m+1})\cap L=(R\cap L)\cup\{u,v\}.$$

If the set $R\cap L$ is CT, then the interval spanned by $R\cap L$ is a subinterval, say $[u',v']$, of the interval L. Denoting by J a generic open interval contiguous to R and contained in Z_{m+1}, we have in this case

$$\Upsilon(\varphi;p;Q^*\cap L)=\Upsilon(\varphi;p;R\cap L)+\varphi([u,u'])\Box^p+\varphi([v',v])\Box^p$$

$$=\sum_{J\subset L}\varphi(J)\Box^p+\varphi([u,u'])\Box^p+\varphi([v',v])\Box^p,$$

$$|\Upsilon(\varphi;p;Q^*\cap L)|\leqq 2O^p(\varphi;L)+\sum_{J\subset L}|\varphi(J)|^p.$$

Again, if $R\cap L$ is not CT, then L contains none of the intervals J, and the last inequality still holds in this case also, the void series signifying zero.

Since L is arbitrary in the above, we have

$$\left|\sum_L \Upsilon(\varphi;p;Q^*\cap L)\right|\leqq 2\sum_L O^p(\varphi;L)+\sum_J|\varphi(J)|^p.$$

This, together with the above expression for $\Upsilon(\varphi;p;Q^*)$, leads to

$$\left|\Upsilon(\varphi;p;Q^*)-\sum_G\varphi(G)\Box^p\right|\leqq 2\sum_L O^p(\varphi;L)+\sum_J|\varphi(J)|^p, \qquad \text{Q. E. D.}$$

LEMMA 11. *The notation and the hypothesis being the same as in the foregoing lemma, we have the inequalities*:

$$\Lambda(\varphi;p;Q)-\Lambda(\varphi;p;Q^*)\leqq 3^p(p-1)\sum_D\omega^p(D),$$

$$\Lambda(\varphi;p;Q^*)\leqq\sum_G|\varphi(G)|^p+2\sum_L O^p(\varphi;L)+\sum_J|\varphi(J)|^p.$$

PROOF. We need only give an outline of the proof, since the argument is the same as in Lemma 10. Let D be any component of the open set $Z_m\setminus Z_{m+1}$. Then D is contained in some component K of Z_m. Writing $D=(\alpha,\beta)$ and $Q'=Q\cup\{\alpha,\beta\}$, we shall appraise the difference

$$d=\Lambda(\varphi;p;Q)-\Lambda(\varphi;p;Q')$$

$$=\Lambda(\varphi;p;Q\cap K)-\Lambda(\varphi;p;Q'\cap K).$$

Consider the minimal closed interval $[\alpha',\beta']$ fulfilling the conditions $[\alpha,\beta]\subset[\alpha',\beta']\subset K$, $\alpha'\in Q$, $\beta'\in Q$. Then we have

$$d=|\varphi([\alpha',\beta'])|^p-|\varphi([\alpha',\alpha])|^p-|\varphi([\alpha,\beta])|^p-|\varphi([\beta,\beta'])|^p.$$

But this amounts at most to $3^p(p-1)\omega^p(D)$, on account of Lemma 9. Using this result and arguing as in Lemma 10, we obtain the first of the asserted inequalities.

The deduction of the second inequality is quite similar to that of the corresponding inequality of Lemma 10.

We now resume the sequence of figures, $U_0 \supset U_1 \supset \cdots$, whose intersection Γ was seen nondense and which is therefore regular. The boundary of U_m was denoted by Δ_m for each $m \in \boldsymbol{M}$, so that $\Delta_0 = \{0, 1\}$. The function $\Theta(x) = \Theta(x; \delta)$, where $0 < \delta < 2^{-1}$, was originally defined for $x \in \Gamma$ by

$$\Theta(x) = \sum_{n=1}^{\infty} \frac{a_n(x)}{n} \delta^n, \quad \text{where } \langle a_1(x), a_2(x), \cdots \rangle = \sigma(x).$$

We then extended its definition to the whole $\boldsymbol{R}$ in a routine way. The new function, still denoted by $\Theta(x)$, was found continuous over $\boldsymbol{R}$. The reader is requested to keep all this in his or her mind.

LEMMA 12. *Suppose given a compact set* R *such that* $\Delta_0 \subset R \subset \Gamma$. *If* $0 < \delta < 3^{-1}$ *and* $p > 1$, *then we have*

$$\Lambda(\Theta; p; R) < \left(\frac{7}{\sqrt{\delta}}\right)^p \frac{p}{\sqrt{p-1}}.$$

PROOF. Let us take, in the foregoing lemma,

$$\langle Z_0, Z_1, \cdots \rangle = \langle U_0, U_1, \cdots \rangle, \quad \varphi(x) = \Theta(x), \quad \text{and} \quad Q = R_m,$$

where $m \in \boldsymbol{M}$ and $R_m = R \cup \Delta_m$. Since by Lemma 7 the function $\Theta(x)$ is BV on the interval $U_0 = [0, 1]$, we have $\Lambda(\Theta; 1; R_m) \leqq \mathrm{V}(\Theta; U_0) < +\infty$, so that $\Lambda(\Theta; p; R_m)$ must be finite on account of $p > 1$. Thus the hypothesis $\Lambda(\varphi; p; Q) < +\infty$ of Lemma 11 is fulfilled. Hence, noting the relation $Q^* = Q \cup \Delta_{m+1} = R_{m+1}$, we obtain for every m

$$\Lambda(\Theta; p; R_m) - \Lambda(\Theta; p; R_{m+1}) \leqq 3^p (p-1) \sum_{D} \omega^p(D),$$

where D ranges over all the components of the open set $U_m \setminus U_{m+1}$. But $U_m \setminus U_{m+1}$ has exactly $2 \cdot 3^m$ components, while we have $\omega(D) \leqq 2\delta^{m+1}/(m+1)$ for every D by part (i) of Lemma 6. Consequently

$$\Lambda(\Theta; p; R_m) - \Lambda(\Theta; p; R_{m+1}) < 6^p (p-1) \frac{(3\delta^p)^{m+1}}{m+1}.$$

This, together with $R_0 = R \cup \Delta_0 = R$, leads to

$$\Lambda(\Theta; p; R) - \Lambda(\Theta; p; R_{m+1}) < 6^p (p-1) \sum_{n=1}^{\infty} \frac{(3\delta^p)^n}{n},$$

where the last series converges to $\log(1 - 3\delta^p)^{-1}$.

On the other hand, the second inequality of Lemma 11 gives

$$\Lambda(\Theta; p; R_{m+1}) \leqq \sum_{G} |\Theta(G)|^p + 2 \sum_{L} \mathrm{O}^p(\Theta; L) + \sum_{J} |\Theta(J)|^p,$$

where we denote generically by G an open interval contiguous to U_{m+1}, by L a component of U_{m+1}, and by J an open interval (if existent) contiguous to R and contained in U_{m+1}. But Lemma 6 shows that

$$\sum_G |\Theta(G)|^p + 2\sum_L O^p(\Theta; L) \leqq \sum_{i=0}^{m} \frac{2\cdot 3^i}{i+1}\delta^{p(i+1)} + \frac{2\cdot 3^{m+1}}{m+2}(2\delta^{m+2})^p$$

$$< \log\frac{1}{1-3\delta^p} + (3\delta^p)^m;$$

Combining the above results, we get

$$\Lambda(\Theta; p; R) < 6^p p \log\frac{1}{1-3\delta^p} + (3\delta^p)^m + \sum_J |\Theta(J)|^p.$$

Now each interval J is contained in U_{m+1} and hence $|J| < 2^{-m}$ by Lemma 1. Moreover, J is contiguous to R and we have $\Lambda(\Theta; p; R) < +\infty$. Consequently $\sum|\Theta(J)|^p$ tends to 0 as $m \to +\infty$. But the same is true of $(3\delta^p)^m$ also. It thus follows that

$$\Lambda(\Theta; p; R) \leqq 6^p p \log\frac{1}{1-3\delta^p}.$$

We proceed to estimate the logarithm. For this purpose, we write $\delta = 3^{-t}$, so that $t > 1$. Then, by the mean value theorem,

$$1 - 3\delta^p = 1 - 3^{1-pt} > 3^{1-pt}(pt-1) > 3\delta^p(p-1).$$

On the other hand, $\log\xi = 2\log\sqrt{\xi} < 2\sqrt{\xi}$ for $\xi > 0$. Hence

$$\log\frac{1}{1-3\delta^p} < \frac{2}{\sqrt{3\delta^p(p-1)}}.$$

We thus conclude that

$$\Lambda(\Theta; p; R) < \frac{2\times 6^p p}{\sqrt{3\delta^p(p-1)}} < \left(\frac{7}{\sqrt{\delta}}\right)^p \frac{p}{\sqrt{p-1}}.$$

The following lemma is obtained immediately from the definition of the function $\Theta(x) = \Theta(x; \delta)$ for the points $x \in \Gamma$.

LEMMA 13. *If x is a fixed point of Γ, then $\Theta(x;\delta)$ is a continuous function of δ, where we suppose that $0 < \delta < 2^{-1}$.*

LEMMA 14. *If S is any* CT *set contained in Γ, we have*

$$\Lambda(\Omega; p; S) < \frac{13^p p}{\sqrt{p-1}} \qquad \textit{for} \quad p > 1,$$

where we write $\Omega(x) = \Theta(x; 3^{-1})$ as before.

PROOF. Let $\langle H_1, \cdots, H_n\rangle$ be an arbitrary finite sequence of disjoint open intervals contiguous to the set S, and let M denote the boundary of the union $H_1 \cup \cdots \cup H_n$. Specializing the compact set R of Lemma 12 to $M \cup \varDelta_0$, we have

$$\sum_{i=1}^{n} |\varTheta(H_i)|^p \leqq \Lambda(\varTheta; p; M \cup \varDelta_0) < \left(\frac{7}{\sqrt{\delta}}\right)^p \frac{p}{\sqrt{p-1}}$$

for $p>1$, where $\varTheta(x)=\varTheta(x;\delta)$ and $0<\delta<3^{-1}$. Making now $\delta \to 3^{-1}$, we find at once by Lemma 13 that

$$\sum_{i=1}^{n} |\varOmega(H_i)|^p \leqq \frac{p(7\sqrt{3})^p}{\sqrt{p-1}}.$$

Since $\langle H_1, \cdots, H_n \rangle$ is arbitrary, this inequality gives

$$\Lambda(\varOmega; p; S) \leqq \frac{p(7\sqrt{3})^p}{\sqrt{p-1}} < \frac{13^p p}{\sqrt{p-1}}.$$

LEMMA 15. *If R is a closed set such that $\varDelta_0 \subset R \subset \varGamma$, we have*

$$|\Upsilon(\varOmega; p; R)| < 13^p \cdot \sqrt{p-1} \qquad \text{for} \quad p>1.$$

PROOF. Let us take, in Lemma 10,

$$\langle Z_0, Z_1, \cdots \rangle = \langle U_0, U_1, \cdots \rangle, \quad \varphi(x) = \varOmega(x), \quad \text{and} \quad Q = R_m,$$

where $m \in \boldsymbol{M}$ and $R_m = R \cup \varDelta_m$. The hypothesis $\Lambda(\varphi; p; Q) < +\infty$ of that lemma is then secured by the foregoing lemma. Consequently, taking note of the relation $Q^* = Q \cup \varDelta_{m+1} = R_{m+1}$ and arguing as in the proof of Lemma 12, we obtain for every $m \in \boldsymbol{M}$

$$|\Upsilon(\varOmega; p; R) - \Upsilon(\varOmega; p; R_{m+1})| < 6^p(p-1) \log \frac{1}{1-3^{1-p}}$$

$$< 6^p(p-1) \frac{2}{\sqrt{3^{1-p}(p-1)}} < 12^p \cdot \sqrt{p-1}.$$

On the other hand, the second inequality of Lemma 10 gives

$$\left| \Upsilon(\varOmega; p; R_{m+1}) - \sum_{G} \varOmega(G)\square^p \right| \leqq 2 \sum_{J} \mathrm{O}^p(\varOmega; L) + |\varOmega(J)|^p,$$

where G, L, and J range respectively over the open intervals contiguous to U_{m+1}, the components of U_{m+1}, and the open intervals contiguous to R and contained in U_{m+1}. But $\sum \varOmega(G)\square^p$ vanishes by part (ii) of Lemma 6, while part (i) of the same lemma shows that

$$2 \sum \mathrm{O}^p(\varOmega; L) \leqq \frac{2 \cdot 3^{m+1}}{m+2} 3^{-p(m+1)} < (3^{1-p})^m.$$

Combining the above results, we get

$$|\Upsilon(\varOmega; p; R) < 12^p \cdot \sqrt{p-1} + (3^{1-p})^m + \sum_{J} |\varOmega(J)|^p.$$

We now make $m \to +\infty$ here. Using $\Lambda(\Omega;p;R) < +\infty$ and arguing as in the proof of Lemma 12, we conclude that

$$|\Upsilon(\Omega;p;R)| \leqq 12^p \cdot \sqrt{p-1} < 13^p \cdot \sqrt{p-1}.$$

LEMMA 16. *The function* $\Omega(x) = \Theta(x;3^{-1})$ *is strongly Dirichlet continuous, without being powerwise continuous, on the set* Γ.

PROOF. To prove the strong Dirichlet continuity of $\Omega(x)$ on Γ, it is enough, in view of Lemma 5, to show that it is Dirichlet continuous on Γ.

For this purpose, consider any compact nonconnected set $R \subset \Gamma$. We have to ascertain that $\Omega(x)$ fulfilis the Dirichlet condition on R. Let $p > 1$ and denote by H a generic open interval contiguous to R. Lemma 14 then shows that

$$\sum_H |\Omega(H)|^p = \Lambda(\Omega;p;R) = o\left(\frac{1}{p-1}\right) \quad \text{as} \quad p \to 1\,.$$

We shall verify next that if the end points of a closed interval A belong to R, while A itself is not contained in R, we have

$$\sum_{H \subset A} \Omega(H)\Box^p = \Upsilon(\Omega;p;A \cap R) = \Omega(A) + o(1)$$

as $p \to 1$. To see this we may, without loss of generality, suppose first that R spans the interval A and secondly that $A = [0, 1]$. Then this assertion reduces to $\Upsilon(\Omega;p;R) = o(1)$ as $p \to 1$, which is ensured by Lemma 15.

The function $\Omega(x)$ thus fulfils the Dirichlet condition on every compact nonconnected set $R \subset \Gamma$. Since $\Omega(x)$ is further continuous on Γ by Lemma 6, we conclude that $\Omega(x)$ is Dirichlet continuous over Γ. This, together with Lemma 8, completes the proof.

THEOREM 10. *The integration* ($\mathrm{P_W}$) *is strictly wider than the powerwise integration.*

The proof is almost the same as in Theorem 22 of [5].

§ 3. Supplements on the Dirichlet continuity of functions.

It was stated in § 1 of [5] that we do not know whether the following assertion is true: *If a function* $\varphi(x)$ *is Dirichlet continuous on a compact nonconnected set* Q, *then* $|\varphi[Q]| = 0$ *and the function is* AD *to zero at almost all points of* Q. If this is true, then the strong and the ordinary Dirichlet continuity are equivalent.

We are unable, at present, to prove or disprove this assertion. We can only establish a few results related to it, as will be shown in the rest of the paper.

LEMMA 17. *If a function $\varphi(x)$ is derivable to zero at all the points of a linear set E, then necessarily* $|\varphi[E]|=0$ (see Saks [7], p. 226).

THEOREM 11. *Suppose that a function $\varphi(x)$ is Dirichlet continuous on a compact nonconnected set Q and linear on each closed interval contiguous to Q. If D denotes the set of all the points of Q at which the function is not derivable, we have* $|\varphi[Q]|=|\varphi[D]|$.

PROOF. The function $\varphi(x)$ is plainly continuous on the closed interval I spanned by the set Q. This being premised, we attach to each point y of the set $\varphi[Q]$ the leftmost point, say $f(y)$, of the nonvoid compact set $\varphi^{-1}(y)\cap Q$. If we write M for the set of all the points $f(y)$ at which $\varphi(x)$ is derivable, then clearly $\varphi[Q]=\varphi[D]\cup\varphi[M]$, so that

$$|\varphi[D]|\leqq|\varphi[Q]|\leqq|\varphi[D]|+|\varphi[M]|.$$

It thus suffices to prove that $|\varphi[M]|=0$.

Denoting by $H=(\alpha,\beta)$ a generic open interval contiguous to Q, we define a subset R of M as follows: a point s of M belongs to R if and only if there is an infinity of intervals H such that $\beta<s$ and $\varphi(s)\in\varphi[H]$. Given any point s of R, let us choose, from each interval H of this definition, a point s_H such that $\varphi(s_H)=\varphi(s)$. The set of all the points s_H is infinite and hence has an accumulation point, say σ. We find at once that $\sigma\leqq s$ and $\sigma\in Q$. Furthermore, the continuity of $\varphi(x)$ on the interval I (spanned by Q) implies that $\varphi(\sigma)=\varphi(s)$. It follows that $\sigma=s$; in fact, the relation $s\in R\subset M$ and the definition of the set M together imply that s is the leftmost of all the points $x\in Q$ at which $\varphi(x)=\varphi(s)$.

At every point s of the set R, the function $\varphi(x)$ is derivable and the coincidence of σ and s necessitates that $\varphi'(s)=0$. It accordingly follows from Lemma 17 that $|\varphi[R]|=0$.

In order to ascertain the validity of $|\varphi[M]|=0$, it is thus sufficient to show that $|\varphi[M\setminus R]|=0$. For this purpose, we may conveniently suppose that $\varphi(l)<\varphi(x)$ whenever $l<x\in Q$, where l is the left extremity of the interval I. Plainly there will arise no loss of generality from doing so.

As above, let $H(\alpha,\beta)$ stand for a generic open interval contiguous to the set Q. Writing $W=M\setminus R$, let us consider the set of all the points $x\in W$ for each of which there exists no interval H such that $\beta<x$ and $\varphi(x)\in\varphi[H]$. This set, which we denote by E, will be shown to fulfil $|\varphi[E]|=0$. Obviously, we need only consider the case in which E is an infinite set.

We shall verify first that the function $\varphi(x)$ is increasing on E. For this purpose, let us take any two points $s_1<s_2$ of E. These points belong to the set M, and hence $\varphi(s_1)\neq\varphi(s_2)$. Suppose, if possible, that $\varphi(s_2)<\varphi(s_1)$.

This inequality, combined with $\varphi(l)<\varphi(s_2)$ which is true since $l\leqq s_1<s_2\in Q$, ensures the existence of a point s_0 such that $l<s_0<s_1$ and $\varphi(s_0)=\varphi(s_2)$. Since $s_2\in M$, the point s_0 cannot belong to Q, so that there exists an $H=(\alpha, \beta)$ containing s_0. We then have $\beta\leqq s_1<s_2$ and $\varphi(s_2)=\varphi(s_0)\in\varphi[H]$, where $s_2\in E$. This contradicts the definition of the set E. It thus follows that $\varphi(s_1)<\varphi(s_2)$, which shows $\varphi(x)$ to be increasing on E.

Supposing as above that E is infinite, consider now its closure C. Since $E\subset Q$, we have $C\subset Q$ also. By hypothesis, $\varphi(x)$ is Dirichlet continuous on Q. We thus distinguish two cases, according as C is connected or not. In the former case, C is a closed interval, so that $\varphi(x)$ is a constant over C and hence over E. On the other hand, if C is nonconnected, then $\varphi(x)$ is Dirichlet continuous on C. But $\varphi(x)$ is nondecreasing on C, since it is increasing on E. It follows from Theorem 3 of [5] that $\varphi(x)$ is AC on C. Then Theorem 5 of [5] requires that $|\varphi[C]|=0$ and *a fortiori* that $|\varphi[E]|=0$, Q. E. D.

It remains to show that $|\varphi[W_0]|=0$, where we write $W_0=W\setminus E$ for short. Take any point s of W_0 and consider all the intervals $H=(\alpha, \beta)$ contiguous to Q, such that $\beta<s$ and $\varphi(s)\in\varphi[H]$. Since s does not belong to $E\cup R$, we can arrange these intervals H in a sequence $H_1<\cdots<H_n$, where $n\in\boldsymbol{N}$. The last interval H_n, which is uniquely determined by the point s, will be written $H_n=H(s)$.

There corresponds, to each open interval H contiguous to Q, the set of all the points s of W_0 such that $H(s)=H$. Denoting this set by E_H, we find at once that W_0 is the union of all the sets E_H. But there evidently exists at most a countable infinity of the intervals H. Consequently the relation $|\varphi[W_0]|=0$ will follow if we prove that $|\varphi[E_H]|=0$ for every H.

Keeping an $H=(\alpha, \beta)$ fixed, let us assume, as we may for this purpose, that the set E_H is infinite. Consider any point s of E_H, so that $\beta<s$ and further there is a point $s'\in H$ at which $\varphi(s')=\varphi(s)$. Since $s\in M$, we cannot have $\varphi(\beta)=\varphi(s)$. On the other hand, by hypothesis, $\varphi(x)$ is linear on the interval $[\alpha, \beta]$. It thus follows that $\varphi(\alpha)\neq\varphi(\beta)$ and that

$$\varphi(\alpha)<\varphi(s)<\varphi(\beta) \quad \text{or} \quad \varphi(\alpha)>\varphi(s)>\varphi(\beta),$$

according as $\varphi(\alpha)<\varphi(\beta)$ or $\varphi(\alpha)>\varphi(\beta)$, respectively.

Making use of this last fact, we can prove first that $\varphi(x)$ is strictly monotone on the set E_H and then that $|\varphi[E_H]|=0$. The argument will be quite the same as in the above, where we dealt with the set E. This completes the proof.

LEMMA 18. *Let I be the closed interval spanned by a compact nonconnected set Q and let K denote generically a closed interval contiguous* *to Q. Suppose that a function $\varphi(x)$ is linear on each K, that a function $\psi(x)$ is continuous on I, and that $\varphi(x)=\psi(x)$ unless $x\in I\setminus Q$.*

Given a point ξ of Q, if no interval K has ξ for one of its end points and if the function $\psi(x)$ is derivable at this point ξ, then so is also the function $\varphi(x)$ at ξ and we have $\varphi'(\xi)=\psi'(\xi)$.

PROOF. We shall make use of two functions $\Phi(x)$ and $\Psi(x)$ defined by

$$\Phi(x)=\frac{\varphi(x)-\varphi(\xi)}{x-\xi} \quad \text{and} \quad \Psi(x)=\frac{\psi(x)-\psi(\xi)}{x-\xi}$$

for every real number $x\neq\xi$, and by $\Phi(\xi)=\Psi(\xi)=\psi'(\xi)$ for $x=\xi$. The assertion amounts to showing that the function $\Phi(x)$ is continuous at $x=\xi$.

By hypothesis, given any $\varepsilon>0$ there exists an open interval H containing the point ξ and such that $|\Psi(x)-\Psi(\xi)|<\varepsilon$ whenever $x\in H$. As we find easily, this interval H can be so chosen that every closed interval K (if existent) contiguous to Q and intersecting H is entirely contained in H.

It is enough to show that $|\Phi(x)-\Phi(\xi)|<\varepsilon$ for $x\in H$. This will follow at once, if we prove that $\Phi(x)\in\Psi[H]$ for $x\in H$. To see this, let x be any point of H. Unless $x\in I\setminus Q$, we have $\varphi(x)=\psi(x)$ by hypothesis, and in view of $\xi\in Q$ this equality implies that $\Phi(x)=\Psi(x)\in\Psi[H]$, the case $x=\xi$ being inclusive. We may therefore suppose that $x\in I\setminus Q$. Then the closed interval K contiguous to Q and containing x, must be contained in H. Moreover, the point ξ does not belong to K, since $\xi\in Q$ and since ξ is neither of the end points of K. Now, by hypothesis, the function $\varphi(x)$ is linear on K. Hence, writing $K=[\alpha,\beta]$, we find at once that if $\Phi(\alpha)=\Phi(\beta)$, then $\Phi(x)=\Phi(\alpha)=\Psi(\alpha)\in\Psi[H]$, since $\alpha\in Q$. On the other hand, if we have $\Phi(\alpha)<\Phi(\beta)$, the same linearity of $\Phi(x)$ necessitates that

$$\Phi[K]=[\Phi(\alpha),\Phi(\beta)]=[\Psi(\alpha),\Psi(\beta)].$$

But this interval is contained in $\Psi[K]$, since $\Psi(x)$ is continuous on K together with $\psi(x)$. It follows that $\Phi(x)\in\Psi[H]$.

Needless to say, this last result is valid also in the case in which we have $\Phi(\alpha)>\Phi(\beta)$. This completes the proof.

Given a function $\varphi(x)$ and a linear set E (which may be void), the function $\varphi(x)$ will be called AC *superposable* on E, if it is expressible on E in the form $\varphi(x)=G\circ F(x)$, by means of two functions $F(x)$ and $G(y)$ the former of which is AC on E and the latter AC on the set $F[E]$. When this is the case, $\varphi(x)$ is necessarily continuous on E.

It appears to the author that the notion of the superposition of two AC functions, as introduced on p. 286 of Saks [7], lacks clearness. Let us understand this notion as follows: when he says that a function is expressible on a closed interval as a superposition of two AC functions, this in reality means that the function is AC superposable on this interval. With this precise interpretation, everything goes well in reading the Saks treatise [7].

LEMMA 19. *If a function* $\varphi(x)$ *is* AC *superposable on a compact nonconnected set* Q*, there exists a function* $\psi(x)$ *which coincides with* $\varphi(x)$ *on* Q *and which is* AC *superposable on the closed interval* I *spanned by* Q.

PROOF. We can write $\varphi(x)=G\circ F(x)$ for $x\in Q$, where $F(x)$ is AC on Q and $G(y)$ is AC on the set $F[Q]$. The assertion will be established by choosing these two functions suitably and then writing $\psi(x)=G\circ F(x)$ for $x\in \boldsymbol{R}$.

We may suppose that $F(x)$ is AC on the interval I and linear on each closed interval contiguous to Q; for otherwise we have only to replace $F(x)$ by its linear modification $\lambda(x)$ with respect to Q and to utilize Theorem 4 of [5], according to which the function $\lambda(x)$ is AC on the whole interval I.

If the set $F[Q]$ is singletonic, then so is also the set $F[I]$, since this coincides with $F[Q]$ in this case; and we have nothing more to prove. We may therefore assume $F[Q]$ to contain at least two points. Noting that $F[Q]$ is a compact set, let us denote by A the closed interval spanned by $F[Q]$. The function $F(x)$ being linear on each closed interval contiguous to Q, we find easily that $F[I]$ coincides with A.

It may happen that $F[Q]=A$, in which case we have nothing more to prove. When this does not happen, however, by replacing, if necessary, the function $G(y)$ by its linear modification with respect to $F[Q]$, we may still suppose $G(y)$ to be AC on the whole interval $A=F[I]$. This completes the proof.

LEMMA 20. *In order that a function* $\varphi(x)$ *which is continuous on a closed interval* I *be* AC *superposable on this interval, it is necessary and sufficient that the set of the values assumed by the function* $\varphi(x)$ *at the points of* I *at which* $\varphi(x)$ *fails to be derivable, be of measure zero* (see Saks [7], p. 289).

THEOREM 12. *Let* $\varphi(x)$ *be a function which is continuous on a compact nonconnected set* Q. *In order that* $|\varphi[Q]|=0$*, it is necessary that the function* $\varphi(x)$ *be* AC *superposable on* Q*, and it is sufficient that* $\varphi(x)$ *be both Dirichlet continuous and* AC *superposable, on* Q.

PROOF. Without loss of generality we may assume that $\varphi(x)$ is linear on each closed interval contiguous to Q, so that $\varphi(x)$ is found continuous on the closed interval I spanned by Q.

(i) The necessity part. On account of the assumption just made, the function $\varphi(x)$ is derivable at all points of the open set $I \setminus Q$. It thus follows from Lemma 20 that if $|\varphi[Q]|=0$, then $\varphi(x)$ must necessarily be AC superposable on Q.

(ii) The sufficiency part. Suppose that $\varphi(x)$ is both Dirichlet continuous and AC superposable, on the set Q. There exists, by Lemma 19, a function $\psi(x)$ coinciding with $\varphi(x)$ on Q and AC superposable on I. Then $\psi(x)$ is of itself continuous on I. Moreover, $\psi(x)$ may plainly be assumed to coincide with $\varphi(x)$ for the points x outside I, so that we have $\varphi(x)=\psi(x)$ unless $x \in I \setminus Q$.

This being so, let us denote by D [or by Δ] the set of the points of Q at which the function $\varphi(x)$ [or $\psi(x)$] is not derivable, and by S the subset of Q defined as follows: a point of Q belongs to S if and only if it is an end point of some closed interval contiguous to Q. We then have $D \subset \Delta \cup S$ by Lemma 18. It follows that $\varphi[D] \subset \varphi[\Delta] \cup \varphi[S] = \psi[\Delta] \cup \varphi[S]$, since $\varphi(x)=\psi(x)$ for $x \in Q$. Consequently $|\varphi[D]| \leqq |\psi[\Delta]| + |\varphi[S]|$. But both $\psi[\Delta]$ and $\varphi[S]$ are null sets, the former fact being ensured by Lemma 20 (where we replace the letter φ by ψ) and the latter by the countability of the set S. We therefore find that $|\varphi[D]|=0$, which combined with Theorem 11 leads to the required relation $|\varphi[Q]|=0$. This completes the proof.

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