On Conformally Flat Submanifolds of Codimension 2 with Parallel Mean Curvature Vector in a Euclidean Space

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0. Introduction. A Riemannian manifold (M, g) is said to be conformally flat, if it is locally conformally equivalent to a Euclidean space.

The study of conformally flat hypersurfaces in a conformally flat space arose from Schouten [12], who showed that an $n(\geqq 4)$-dimensional hypersurface M in a Euclidean space is conformally flat if and only if M is quasi-umbilic. Afterward this subject has been investigated by many authors, for example, [3], [4], and [10]. However, Cecil and Ryan [1] showed the counter example against [10].

Conformally flat submanifolds of codimension 2 is treated by Chen and Yano [5]. On the other hand, Moore [8] and Moore and Morvan [9] studied conformally flat submanifolds whose codimension is greater than 2. Especially, the latter studied the second fundamental form of such submanifold. And Kitagawa [7] studied the shape of such submanifolds.

The purpose of this paper is to study locally the shape of a conformally flat manifold M^n $(n\geqq 5)$ which is isometrically immersed into a Euclidean space with parallel mean curvature vector. Then we obtain the next theorem:

THEOREM. *Let M^n $(n\geqq 5)$ be a conformally flat manifold which is isometrically immersed into an $(n+2)$-dimensional Euclidean space E^{n+2}. Assume that the mean curvature vector is parallel and it is non-trivial on M. Then if we take off points of constant sectional curvature and a certain subset of measure 0 from M, the rest satisfies locally one of the following two conditions, where $(x^1, \cdots, x^{n+2})$ denotes the natural coodinate of E^{n+2}:*

(1) *There exist a curve $(x^1(t), x^2(t), 0, \cdots, 0)$ in E^{n+2} and a C^∞-function $\gamma(t)$ (>0) such that M is denoted by*

$$(x^1(t), x^2(t), x^3, \cdots, x^{n+2}),$$

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where

$$(x^3)^2+\cdots\cdots+(x^{n+2})^2=\gamma(t)^2.$$

(2) *There exist a surface* $(x^1(u,v), x^2(u,v), x^3(u,v), 0, \cdots\cdots, 0)$ *in* E^{n+2} *and a* C^∞*-function* $\gamma(u,v)$ (>0) *such that* M *is denoted by*

$$(x^1(u,v), x^2(u,v), x^3(u,v), x^4, \cdots\cdots, x^{n+2}),$$

where

$$(x^4)^2+\cdots\cdots+(x^{n+2})^2=\gamma(u,v)^2.$$

In Section 1, the notations are explained and some properties of the manifold M which we will seek are prepared for. Section 2 is devoted to the study of the situation of the second fundamental form of M, which is devided into different three cases (Theorem 2.1). In particular, M is totally umbilic in a Euclidean space in the case of III in Theorem 2.1. In Section 3, we study the shape of M in the case of II in Theorem 2.1 and obtain that it becomes as the statement (1). In Section 4, we study the shape of M in the case of I in Theorem 2.1 and obtain that it becomes as the statement (2).

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1. Preliminaries. Let M be an $n(\geqq 2)$-dimensional connected Riemannian manifold with the Riemannian metric g. Let f be an isometric immersion of M into an $(n+p)$-dimensional Euclidean space E^{n+p}. Since the argument is only considered in the local version, M needs not to be distinguished from $f(M)$. So, in order to simplify the discussion, we identify a point q in M with the point $f(q)$ and a tangent vector X at q with the tangent vector df_qX.

Now, we choose an orthonormal local frame field $\{e_1, \cdots\cdots, e_n, e_{n+1}, \cdots\cdots, e_{n+p}\}$ on E^{n+p} in such a way that, restricted to M, the vectors $e_1, \cdots\cdots, e_n$ are tangent to M, and hence the others are normal to M. With respect to the fields of frame of E^{n+p}, let $\{\tilde{\omega}^1, \cdots\cdots, \tilde{\omega}^{n+p}\}$ be the dual forms. Here and in the sequel, the following convention on the range of the indices is adopted, unless otherwise stated:

$$A, B, \cdots\cdots=1, \cdots\cdots, n, n+1, \cdots\cdots, n+p,$$

$$i, j, \cdots\cdots=1, \cdots\cdots, n,$$

$$\alpha, \beta, \cdots\cdots=n+1, \cdots\cdots, n+p.$$

Then, associated to the frame field $\{e_A\}$, there exist 1-forms $\tilde{\omega}^A_B$ on E^{n+p} so that they satisfy the following structure equations on E^{n+p}:

$$d\tilde{\omega}^A+\tilde{\omega}_B^A\wedge\tilde{\omega}^B=0\,,$$

$$\text{(1.1)}\qquad \tilde{\omega}_B^A+\tilde{\omega}_A^B=0\,,$$

$$d\tilde{\omega}_B^A+\tilde{\omega}_C^A\wedge\tilde{\omega}_B^C=0\,,$$

where the Einstein convention for the summation is adopted. Let D denote the connection on E^{n+p}. Then we have

$$D_X e_A=\tilde{\omega}_A^B(X)e_B$$

for any tangent vector X on E^{n+p}. And $\{\tilde{\omega}_B^A\}$ are called *connection forms* on E^{n+p}.

Restricting $\{\tilde{\omega}^A\}$ and $\{\tilde{\omega}_B^A\}$ on E^{n+p} to the submanifold M, we denote them by ω^A and ω_B^A respectively, that is,

$$\tilde{\omega}^A|_M=\omega^A\,,$$

$$\tilde{\omega}_B^A|_M=\omega_B^A\,.$$

It then yields

$$\text{(1.2)}\qquad \omega^\alpha=0\,.$$

Let ∇ denote the connection of M. Then $\nabla_X Y$ is equal to the tangential component of $D_X Y$ to M, where X and Y are any tangent vector fields on M. The metric g in M induced from the standard metric in the ambient space E^{n+p} is given by

$$g=\sum_{i=1}^n \omega^i\otimes\omega^i$$

in other words,

$$g=\sum_{i=1}^n (\omega^i)^2\,.$$

Then $\{e_1,\cdots,e_n\}$ are also the orthonormal frame field with respect to g, and $\{\omega^1,\cdots,\omega^n\}$ are the dual fields with respect to $\{e_1,\cdots,e_n\}$. It follows from (1.1) and the Cartan's lemma that

$$\omega_i^\alpha=h_{ij}^\alpha\omega^j\,,$$

$$\text{(1.3)}$$

$$h_{ij}^\alpha=h_{ij}^\alpha\,.$$

The quadratic form $h_{ij}^\alpha\omega^i\otimes\omega^j$ is called the *second fundamental form* of the immersion f on M in the direction of e_α. The second fundamental form α of M can be written as

$$\text{(1.4)}\qquad \alpha(X,Y)=h_{ij}^\beta\omega^i(X)\omega^j(Y)e_\beta$$

for any tangent vectors X and Y on M.

From the structure equations (1.1) of the ambient space the following structure equations on the submanifold M are given:

$$
\begin{aligned}
& d\omega^i + \omega^i_j \wedge \omega^j = 0 \,, \\
(1.5) \qquad & d\omega^i_j + \omega^i_k \wedge \omega^k_j = \Omega^i_j \,, \\
& \Omega^i_j = -\frac{1}{2} R^i_{jkh} \omega^k \wedge \omega^h \,,
\end{aligned}
$$

where ω^i_j and Ω^i_j denote the connection form and the curvature form on the submanifold M, respectively. Moreover they yield

$$
\begin{aligned}
& d\omega^\alpha_\beta + \omega^\alpha_\gamma \wedge \omega^\gamma_\beta = \Omega^\alpha_\beta \\
(1.6) \qquad & \Omega^\alpha_\beta = -\frac{1}{2} R^{N\alpha}{}_{\beta kh} \omega^k \wedge \omega^h \,,
\end{aligned}
$$

where ω^α_β defines the connection form induced in the normal bundle $N(M)$ of M and Ω^α_β is called the *normal curvature form* of M and R^N is called the *normal curvature.* If $\Omega^\alpha_\beta = 0$ for any α, β, the normal connection is said to be *flat.*

Taking the exterior derivative of (1.3), we have

$$(1.7) \qquad (dh^\alpha_{ij} - h^\alpha_{ik}\omega^k_j - h^\alpha_{kj}\omega^k_i + h^\beta_{ij}\omega^\alpha_\beta) \wedge \omega^j = 0 \,.$$

So, we define the covariant derivative h^α_{ijk} of h^α_{ij} by

$$(1.8) \qquad h^\alpha_{ijk}\omega^k = dh^\alpha_{ij} - h^\alpha_{ik}\omega^k_j - h^\alpha_{kj}\omega^k_i + h^\beta_{ij}\omega^\alpha_\beta \,.$$

It turns out that (1.7) says that

$$h^\alpha_{ijk}\omega^j \wedge \omega^k = 0 \,,$$

which is equivalent to

$$h^\alpha_{ijk} = h^\alpha_{ikj} \,.$$

The mean curvature vector η on the submanifold M is defined by

$$\eta = \frac{1}{n} \sum_i \alpha(e_i, e_i) = \frac{1}{n} \sum_i h^\beta_{ii} e_\beta \,,$$

which is independent of the choice of the orthonormal frame field $\{e_i\}$. The mean curvature vector η is said to be *parallel* if $D_X\eta$ is tangent to M for any tangent vector X on M. It is equivalent that

$$\sum_i h^\alpha_{iik} = 0$$

for all indices k and α.

For each normal vector ξ at x, a linear transformation A_ξ on the tangent space $T_x(M)$ is defined by

$$g(A_\xi X, Y)=g(\alpha(X, Y), \xi)$$

for any X and $Y\in T_x(M)$. A_ξ is symmetric with respect to the metric g and it is called the *shape operator* or the *second fundamental tensor* with respect to the normal vector ξ. In particular if $\xi=e_\beta$, then

$$g(A_\beta X, Y)=h_{ij}^\beta\omega^i(X)\omega^j(Y),$$

where $A_\beta=A_{e_\beta}$.

Now, a Riemannian manifold is said to be *conformally flat* if each point of M has a neighborhood where there exists a conformal diffeomorphism onto a subset in a Euclidean space.

For a conformally flat submanifold immersed into a Euclidean space, Moore and Morvan [9] proved the following property:

LEMMA 1.1. *Let M be an n-dimensional conformally flat submanifold in E^{n+p}. If $p\leqq 4$ and $p\leqq n-3$, then at each point x of M there exists a normal vector ξ such that*

$$\langle\beta(X, Y), \beta(Z, W)\rangle=\langle\beta(X, W), \beta(Z, Y)\rangle$$

for any vectors X, Y, Z and W at x, where

$$\beta(X, Y)=\alpha(X, Y)-\langle X, Y\rangle\xi$$

and α is the second fundamental form of the submanifold and $\langle\,,\rangle$ is the metric of E^{n+p}, and hence of M.

DEFINITION 1.2. M is said to be *quasi-umbilic* in the sence of Moore and Morvan if at each point of M there exist orthonormal normal vectors e_α $(\alpha=n+1, \cdots, n+p)$ such that the second fundamental tensor A_α with respect to each normal vector e_α has only two distinct eigenvalues with multiplicity $n-1$ and 1 or n and 0.

THEOREM 1.3. (Moore and Morvan [9]) *Let M be an n-dimensional submanifold in E^{n+p}. If $p\leqq 4$ and $p\leqq n-3$ and if M is conformally flat, then M is quasi-umbilic in the sence of Moore and Morvan.*

We must remark that in Lemma 1.1. it is not proved for the normal vector ξ to be smooth and in Theorem 1.3. it is also not proved for e_α $(\alpha=n+1, \cdots, n+p)$ to be smooth. For the smoothness of ξ Kitagawa [7] asserted the followings:

LEMMA 1.4. (Kitagawa [7]) *If the same assumption as Lemma 1.1. is satisfied, then there exists an open dense set M^* of M on which ξ is smooth.*

THEOREM 1.5. (Kitagawa [7]) *Suppose* $\mathcal{D}$ *be a distribution on* M^* *defined by*

$$\mathcal{D}(q)=\{X\in T_q(M) : \alpha(X, Y)-\langle X, Y\rangle\xi=0 \quad \textit{for all} \ \ Y\in T_q(M)\} .$$

Then $\mathcal{D}$ *is completely integrable and its integral manifold is umbilic in* E^{n+p}.

From now on, we study M^* in place of M. Lemma 1.4. implies that the complement of M^* in M is of measure 0.

2. Conformally flat submanifolds of codimension 2 with parallel mean curvature vector in a Euclidean space.

From now on, we assume M^n is isometrically immersed into $(n+2)$-dimensional Euclidean space E^{n+2}. Let η be the mean curvature vector on the submanifold.

THEOREM 2.1. *Let* M *be a conformally flat submanifold in* E^{n+2}. *If the mean curvature vector* η *is parallel and non-trivial on* M, *then there exists an open dense subset* M^* *in* M *as follows*:

At each point p *of* M^*, *there are a neighborhood* U *of* p, *a* C^∞ *orthonormal frame field* $e_1, \cdots, e_n$ *and* C^∞ *orthonormal normal vector fields* e_{n+1}, e_{n+2} *such that the connection form with respect to* $e_1, \cdots, e_n$, e_{n+1}, e_{n+2} *satisfies one of the following three conditions* I, II, III *on* U:

$$\text{I}\ \begin{cases} \omega_s^{n+1}=\lambda\omega^s , & \omega_s^{n+2}=\tau\omega^s , \\ \omega_{n-1}^{n+1}=\mu\omega^{n-1} , & \omega_{n-1}^{n+1}=\tau\omega^{n-1} , \\ \omega_n^{n+1}=\lambda\omega^n , & \omega_n^{n+2}=\rho\omega^n , \\ \textit{where } \lambda\neq\mu, \tau\neq\rho \textit{ and } s=1, \cdots, n-2 . \end{cases}$$

$$\text{II}\ \begin{cases} \omega_a^{n+1}=\lambda\omega^a , & \omega_a^{n+2}=\tau\omega^a , \\ \omega_n^{n+1}=\lambda\omega^n , & \omega_n^{n+2}=\rho\omega^n , \\ \textit{where } \tau\neq\rho \textit{ and } a=1, \cdots, n-1 . \end{cases}$$

$$\text{III}\ \begin{cases} \omega_i^{n+1}=\lambda\omega^i , & \omega_i^{n+2}=\tau\omega^i , \\ \textit{where } i=1, \cdots, n . \end{cases}$$

PROOF. Choose arbitrary C^∞ orthonormal normal vector fields $\tilde{e}_{n+1}$, $\tilde{e}_{n+2}$. Since the codimension is 2 and the mean curvature vector η is parallel and non-trivial, which implies $\eta/\|\eta\|$ is parallel, it turned out that the unit normal vector which is orthonormal to η must be also parallel.

This yields that the normal connection is flat. It follows from the flatness of the normal connection that the second fundamental tensors of $\tilde{e}_{n+1}$ and $\tilde{e}_{n+2}$ are simultaneously diagonalizable. Therefore, at each point p of M there exist a neighborhood V of p and an orthonormal C^∞ frame $e_1, \cdots, e_n$ on V such that the second fundamental tensors $\tilde{A}_{n+1}, \tilde{A}_{n+2}$ with respect to $\tilde{e}_{n+1}, \tilde{e}_{n+2}$ are represented as follows:

$$(\langle \tilde{A}_{n+1}(e_i), e_j\rangle)=\begin{pmatrix} \nu_1 & & & \\ & \nu_2 & & \\ & & \ddots & \\ & & & \nu_n \end{pmatrix}$$

$$(\langle \tilde{A}_{n+2}(e_i), e_j\rangle)=\begin{pmatrix} \pi_1 & & & \\ & \pi_2 & & \\ & & \ddots & \\ & & & \pi_n \end{pmatrix}$$

where ν_i, π_i $(i=1, \cdots, n)$ are C^∞-functions on V.

Let α be the secoed fundamental form. Then,

$$\begin{aligned}\alpha(e_i, e_j) &= \langle \tilde{A}_{n+1}(e_i), e_j\rangle e_{n+1} + \langle \tilde{A}_{n+2}(e_i), e_j\rangle e_{n+2} \\ &= \nu_i \delta_{ij} e_{n+1} + \pi_i \delta_{ij} e_{n+2} .\end{aligned}$$

Therefore

$$\beta(e_i, e_j) = \nu_i \delta_{ij} e_{n+1} + \pi_i \delta_{ij} e_{n+2} - \delta_{ij} \xi ,$$

where β and ξ are those defined in Lemma 1.1. Clearly,

$$\beta(e_i, e_j)=0 \quad \text{if } i \neq j . \tag{2.1}$$

Replacing the indices if necessary, we may consider

$$\begin{aligned}&\beta(e_1, e_1)=0, \cdots, \beta(e_l, e_l)=0 , \\ &\beta(e_{l+1}, e_{l+1}) \neq 0, \cdots, \beta(e_n, e_n) \neq 0\end{aligned} \tag{2.2}$$

at each point of V. We set $r=n-l$, and consider r as a function on V. If $r(q)>0$ for a point $q \in V$, then there exists a neighborhood $U(\subset V)$ of q where r is equal to $r(q)$.

For each t $(t=l+1, \cdots, n)$ we define a real-valued function c_t and a normal vector field ζ_t by

$$\begin{aligned}&\beta(e_t, e_t)=c_t \zeta_t , \\ &\langle \zeta_t, \zeta_t \rangle = 1 .\end{aligned} \tag{2.3}$$

From (2.2), we find

$$c_t \neq 0 \quad \text{for } t=l+1, \cdots, n ; \tag{2.4}$$

Making use of (2.3), (2.4) and Lemma 1.1., we get

$$\langle \zeta_u, \zeta_t \rangle = \frac{1}{c_u c_t} \langle \beta(e_u, e_t), \beta(e_u, e_t) \rangle$$

for $u, t = l+1, \cdots, n$. Therefore

$$\langle \zeta_u, \zeta_t \rangle = 0 \quad \text{if } u \neq t \tag{2.5}$$

because of (2.1). By (2.3) and (2.5), we find ζ_t $(t = l+1, \cdots, n)$ are orthonormal normal vector fields.

From definition,

$$\zeta_t = \frac{\beta(e_t, e_t)}{\|\beta(e_t, e_t)\|},$$

where $\| \ \|$ denotes the norm with resrect to the metric $\langle \ , \ \rangle$. Since e_t is a C^∞ unit vector field on $U \subset V$, ζ_t is a C^∞ vector field on U. By the assumption that the codimension is 2, r $(= n - l)$ is at most 2.

First we consider the case when $r = 2$ on U. Using (2.1), (2.2) and (2.3), we get

$$(\beta(e_i, e_j))_{i,j=1,\cdots,n} = \begin{pmatrix} 0 & & & & \\ & 0 & \ddots & & \\ & & 0 & & \\ & & & c_{n-1}\zeta_{n-1} & \\ & & & & c_n\zeta_n \end{pmatrix}$$

where $c_{n-1} \neq 0$ and $c_n \neq 0$. By the definition of β in Lemma 1.1., we find

$$\alpha(e_i, e_j) = \beta(e_i, e_j) + \langle e_i, e_j \rangle \xi .$$

Considering ζ_{n-1}, ζ_n as e_{n+1}, e_{n+2}, we have

$$\omega_s^{n+1} = \langle \xi, \zeta_{n-1} \rangle \omega^s ,$$

$$\omega_{n-1}^{n+1} = (c_{n-1} + \langle \xi, \zeta_{n-1} \rangle) \omega^{n-1} ,$$

$$\omega_n^{n+1} = \langle \xi, \zeta_{n-1} \rangle \omega^n ,$$

$$\omega_s^{n+2} = \langle \xi, \zeta_n \rangle \omega^s ,$$

$$\omega_{n-1}^{n+1} = \langle \xi, \zeta_n \rangle \omega^{n-1} ,$$

$$\omega_n^{n+2} = (c_n + \langle \xi, \zeta_n \rangle) \omega^n ,$$

where $s = 1, \cdots, n-2$. Therefore, we find the condition I is satisfied on U.

Next, we study the case where $r = 1$ on U. Since ζ_n only exists, we define the unit normal vector ζ_{n-1} such that ζ_{n-1} is orthogonal to ζ_n. By (2.1), (2.2) and (2.3), we get

$$(\beta(e_i, e_j))_{i,j=1\cdots,n}=\begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & c_n\zeta_n \end{pmatrix},$$

where $c_n \neq 0$. By definition of β, we find

$$\alpha(e_i, e_j)=\beta(e_i, e_j)+\langle e_i, e_j\rangle\xi .$$

Considering ζ_{n-1}, ζ_n as e_{n+1}, e_{n+2} respectively, we have

$$\omega_a^{n+1}=\langle\xi, \zeta_{n-1}\rangle\omega^a ,$$

$$\omega_n^{n+1}=\langle\xi, \zeta_{n-1}\rangle\omega^n ,$$

$$\omega_a^{n+2}=\langle\xi, \zeta_n\rangle\omega^a ,$$

$$\omega_n^{n+2}=(c_n+\langle\xi, \zeta_n\rangle)\omega^n ,$$

where $a=1, \cdots, n-1$. Therefore we find the condition II is satisfied on U.

Now, we define a subset $\tilde{V}$ in V by

$$\tilde{V}=\{q\in V : r(q)=0\}^o ,$$

where A^o denote the set of all inner points of a set A. On $\tilde{V}$

$$\beta(e_i, e_j)=0 .$$

Let e_{n+1}, e_{n+2} be arbitrary C^∞ orthonormal normal vector fields. Then we get

$$(\langle\alpha(e_i, e_j), e_{n+1}\rangle)_{i,j=1,\cdots,n}=\begin{pmatrix} \langle\xi, e_{n+1}\rangle & & \\ & \ddots & \\ & & \langle\xi, e_{n+1}\rangle \end{pmatrix}$$

$$(\langle\alpha(e_i, e_j), e_{n+2}\rangle)=\begin{pmatrix} \langle\xi, e_{n+2}\rangle & & \\ & \ddots & \\ & & \langle\xi, e_{n+2}\rangle \end{pmatrix}$$

on $\tilde{V}$. From which it follows that the condition III is satisfied, where we consider $\tilde{V}$ as U. Q. E. D.

REMARK 2.2. In the case of III, U is totally umbilic in E^{n+2}. It is then known that this implies that there exists a totally geodesic $(n+1)$-dimensional plane L in E^{n+2} such that U is a part of n-dimensional sphere of L.

REMARK. 2.3. We call $p\in M$ a *point of constant sectional curvature* if all sectional curvatures at p have the same value. When we study the shape of U which satisfies I or II, we avoid the points of constant sectional curvature.

3. The case of II in Theorem 2.1.

In this section we study the shape of U on which the condition II in Theorem 2.1. is satisfied.

In this case there exist C^∞ orthonormal normal vector fields e_{n+1}, e_{n+2} and a C^∞ orthonormal frame $e_1, \cdots, e_n$ on U such that

$$\omega_a^{n+1}=\lambda\omega^a\,, \qquad \omega_a^{n+2}=\tau\omega^a\,,$$
$$\omega_n^{n+1}=\lambda\omega^n\,, \qquad \omega_n^{n+2}=\rho\omega^n\,, \tag{3.1}$$

where λ, τ and ρ are C^∞-functions on $U, \tau\neq\rho$ on U and $a=1, \cdots, n-1$.

In this section indices $A, B, \cdots$ run over the range $\{1, \cdots, n+2\}$ and $a, b, \cdots$ move from 1 to $n-1$. For indices $a, b, \cdots$ the Einstein's convention is used.

LEMMA 3.1. *A point p in M is of constant sectional curvature if and only if $\tau=0$ at p.*

PROOF. Let $K(X, Y)$ denote the sectional curvature of the plane spanned by $X, Y\in T_p(M)$. Then

$$K(e_a, e_b)=\lambda^2+\tau^2\,,$$

$$K(e_a, e_n)=\lambda^2+\tau\rho\,,$$

which imply that if p is a point of constant sectional curvature, then $\tau=0$.

Conversely let $\tau=0$. Let X, Y be any unit vectors which are mutually orthogonal. If we set

$$X=\sum_{i=1}^n X^i e_i\,,$$

$$Y=\sum_{j=1}^n Y^j e_j\,,$$

then

$$\sum_{i=1}^n (X^i)^2=1\,,$$

$$\sum_{j=1}^n (Y^j)^2=1\,,$$

$$\sum_{i=1}^{n} X^i Y^i = 0 ,$$

Therefore

$$K(X, Y) = \lambda^2 . \qquad \text{Q. E. D.}$$

Since we have assumed that U has no point of constant sectional curvature, we may consider $\tau \neq 0$ on U.

We set

$$
\begin{aligned}
d\lambda &= \lambda_a \omega^a + \lambda_n \omega^n . \\
d\tau &= \tau_a \omega^a + \tau_n \omega^n . \qquad (3.2) \\
d\rho &= \rho_a \omega^a + \rho_n \omega^n .
\end{aligned}
$$

LEMMA 3.2.

$$
\begin{aligned}
\omega_n^a &= -\frac{\tau_n}{\tau - \rho} \omega^a , \qquad (3.3) \\
\omega_{n+2}^{n+1} &= 0 ,
\end{aligned}
$$

$$\lambda_a = 0 , \quad \tau_a = 0 , \quad \rho_a = 0 , \quad \lambda_n = 0 . \qquad (3.4)$$

PROOF. We set

$$\omega_n^a = A_b^a \omega^b + A_n^a \omega^n , \qquad (3.5)$$

$$\omega_{n+2}^{n+1} = C_b \omega^b + C_n \omega^n . \qquad (3.6)$$

Taking the exterior derivative of the first equation o (3.1) and using (3.2), (3.5), (3.1) and (3.6), we get

$$d(\lambda \omega^a) = (\lambda_b \omega^b + \lambda_n \omega^n) \wedge \omega^a + \lambda \omega^b \wedge \omega_b^a + \lambda \omega^n \wedge A_b^a \omega^b ,$$

$$d\omega_a^{n+1} = \lambda \omega^b \wedge \omega_b^a - \lambda A_b^a \omega^b \wedge \omega^n + \tau C_b \omega^a \wedge \omega^b + \tau C_n \omega^a \wedge \omega^n .$$

This implies that

$$\lambda_b + \tau C_b = 0 ,$$

$$\lambda_n + \tau C_n = 0 .$$

Similarly taking the exterior derivative of the other equations of (3.1), we obtain

$$\lambda_b + \rho C_b = 0 ,$$

$$\tau_b - \lambda C_b = 0 ,$$

$$\tau_n \delta_b^a + (\tau - \rho) A_b^a - \lambda C_n \delta_b^a = 0 ,$$

$$-\rho_b + (\rho - \tau) A_n^b + \rho C_b = 0 .$$

Since $\tau \neq \rho$ and $\tau \neq 0$, we obtain

$$C_b=0\,,\quad \lambda_b=0\,,\quad \tau_b=0\,,\quad C_n=-\frac{\lambda_n}{\tau}\,,$$

$$A_n^b=-\frac{\rho_b}{\tau-\rho}\,,\qquad A_b^a=\frac{\lambda C_n-\tau_n}{\tau-\rho}\delta_b^a\,.$$

Therefore

$$\omega_n^a=-\frac{\lambda\lambda_n+\tau\tau_n}{\tau(\tau-\rho)}\omega^a-\frac{\rho_a}{\tau-\rho}\omega^n\,,$$

$$\omega_{n+2}^{n+1}=-\frac{\lambda_n}{\tau}\omega^n\,,$$

$$\lambda_b=0\,,\qquad \tau_b=0\,.$$

On the other hand, by (3.1), we get

$$\eta=\lambda e_{n+1}+\frac{1}{n}\{(n-1)\tau+\rho\}e_{n+2}\,,$$

where η is the mean curvature vector. The condition that the mean curvature vector η is parallel implies

$$\lambda_a+\frac{1}{n}\{(n-1)\tau+\rho\}C_a=0\,,$$

$$-\lambda_a C+\frac{1}{n}\{(n-1)\tau_a+\rho_a\}=0\,,$$

$$\lambda_n+\frac{1}{n}\{(n-1)\tau+\rho\}C_n=0\,,$$

$$-\lambda C_n+\frac{1}{n}\{(n-1)\tau_n+\rho_n\}=0\,.$$

Using the fact that

$$\lambda_a=0\,,\quad C_a=0\,,\quad \tau_a=0\,,$$

$$\lambda_n+\tau C_n=0\,,$$

we obtain

$$\rho_a=0\,,\quad C_n=0\,,\quad \lambda_n=0\,.$$

Therefore

$$\omega_n^a=-\frac{\tau}{\tau-\rho}\omega^a\,,$$

$$\omega_{n+2}^{n+1}=0\,,$$

$$\lambda_a=0\,,\quad \tau_a=0\,,\quad \rho_n=-(n-1)\tau_n\,.$$

Q. E. D.

We set

$$d\tau_n=\tau_{na}\omega^a+\tau_{nn}\omega^n\,,$$

$$d\rho_n = \rho_{na}\omega^a + \rho_{nn}\omega^n .$$

Because of $d^2=0$, we obtain

$$\tau_{na}=0 , \qquad \rho_{na}=0 . \tag{3.7}$$

By (3.3), we get

$$-(\tau-\rho)\omega_n^a = \tau_n\omega^a .$$

By taking the exterior derivative of the above, we get

$$\tau_{nn}(\tau-\rho) = \tau_n(2\tau_n-\rho_n)+(\tau-\rho)^2(\lambda^2+\tau\rho) . \tag{3.8}$$

Let $\mathcal{D}$ be the distribution on U defined by

$$\mathcal{D}(q)=\{X\in T_q(M) \ : \ \omega^n(X)=0\}$$

for each point $q\in U$. Since

$$d\omega^n = \omega^a \wedge \omega_a^n = 0 ,$$

$\mathcal{D}$ is completely integrable. Therefore at each point q of M there exists a local coodinate $(y^1, \cdots, y^n)$ such that each slice $y^n=$constant, say t, is an integral manifold of $\mathcal{D}$. We denote by $N(t)$ this slice. Restricting the orthonormal frame field $e_1, \cdots, e_n, e_{n+1}, e_{n+2}$ to the slice $N(t)$, $\{e_a\}_{a=1,\cdots,n-1}$ can be regarded as C^∞ orthonormal vector fields tangent to $N(t)$ and e_n, e_{n+1}, e_{n+2} can be also regarded as C^∞ orthonormal normal vector fields on $N(t)$ in E^{n+1}. Furthermore, restricting the dual frame ω^A and the connection form ω_B^A to the slice $N(t)$, we denote them by the same notation $\omega^i, \omega^\alpha, \omega_j^i, \cdots$ as those of the submanifold M in E^{n+2}. Then by (3.1) and Lemma 3.2, we see

$$\omega_a^n = \frac{\tau_n}{\tau-\rho}\omega^a ,$$

$$\omega_a^{n+1} = \lambda\omega^a ,$$

$$\omega_a^{n+2} = \tau\omega^a ,$$

which mean that $N(t)$ is totally umbilic in E^{n+2}. Hence there exists an n-dimensional plane $L(t)$ in E^{n+2} such that $N(t)$ is a part of an $(n-1)$-dimensional sphere in $L(t)$.

We denote by X the position vector of E^{n+2} and by H the mean curvature vector of $N(t)$ in E^{n+2}. It is then seen that the vector field

$$X+\frac{H}{\|H\|^2} ,$$

where we take X an arbitrary point of $N(t)$, is a constant vector in E^{n+2} and lies in $L(t)$, so it is called the *center* of $N(t)$. Moreover the tangent space of $L(t)$ is spanned by $e_1, \cdots, e_{n-1}$ and $\frac{H}{\|H\|^2}$. And the tangent space of $N(t)$ is spanned by $e_1, \cdots, e_{n-1}$. Precisely describing, the mean curvature vector H is expressed as

$$H=\frac{\tau_n}{\tau-\rho}e_n+\lambda e_{n+1}+\tau e_{n+2}\,.$$

Hence

$$\|H\|^2=\frac{\tau_n^2+(\lambda^2+\tau^2)(\tau-\rho)^2}{(\tau-\rho)^2}\,.$$

Since $\tau\neq 0$ and $\tau-\rho\neq 0$, we have

$$\|H\|>0\,.$$

We set

$$F=\tau_n^2+(\lambda^2+\tau^2)(\tau-\rho)^2\,.$$

Then

$$\frac{H}{\|H\|^2}=\frac{1}{F}\{(\tau-\rho)\tau_n e_n+\lambda(\tau-\rho)^2 e_{n+1}+\tau(\tau-\rho)^2 e_{n+2}\}$$

Next we study the curve consisting of the centers of the slices. This curve is parametrized by t. We denote by $\bar{c}(t)$ this curve.

LEMMA 3.3. *$\bar{c}(t)$ is orthogonal to $e_1, \cdots, e_{n-1}$, $\frac{H}{\|H\|^2}$.*

PROOF. Fix an arbitrary point q of M. Let $\tilde{c}(s)$ be the integral curve of e_n with the initial point q. We have only to show that

$$X+\frac{H}{\|H\|^2}\quad\left(\text{i.e. }\tilde{c}(s)+\frac{H}{\|H\|^2}\bigg|_{\tilde{c}(s)}\right)$$

is orthogonal to $e_1, \cdots, e_{n-1}$ and $\frac{H}{\|H\|^2}$ when X moves on $\tilde{c}(s)$, because $\bar{c}(t)$ is the same curve as $\tilde{c}(s)+\frac{H}{\|H\|^2}\Big|_{\tilde{c}(s)}$ if we do not care the difference of the parameter.

Let $(x^1, \cdots, x^{n+2})$ denote the natural coordinate of E^{n+2}. We set

$$\tilde{c}(s)=(\tilde{c}^1(s), \cdots, \tilde{c}^{n+2}(s))\,,$$

$$\frac{H}{\|H\|^2}\bigg|_{\tilde{c}(s)}=(h^1(s), \cdots, h^{n+2}(s))\,,$$

in E^{n+2}. Then,

$$\frac{d}{ds}\left(\tilde{c}(s)+\frac{H}{\|H\|^2}\bigg|_{\tilde{c}(s)}\right)=\left(\frac{d(\tilde{c}^1(s)+h^1(s))}{ds}, \cdots, \frac{d(\tilde{c}^{n+2}(s)+h^{n+2}(s))}{ds}\right)$$

$$=D_{e_n}\left(X+\frac{H}{\|H\|^2}\right),$$

where D denotes the connection of E^{n+2}. It is sufficient to show that $D_{e_n}\left(X+\frac{H}{\|H\|^2}\right)$ is orthogonal to $e_1, \cdots, e_{n-1}$ and $\frac{H}{\|H\|^2}$.

Calculating straightforwardly, we get

$$D_{e_n}F=2\tau_{nn}\tau_n+2\tau\tau_n(\tau-\rho)^2+2(\lambda^2+\tau^2)(\tau-\rho)(\tau_n-\rho_n),$$

where

$$F=\tau_n^2+(\lambda^2+\tau^2)(\tau-\rho)^2.$$

By (3.8) we obtain

$$(\tau-\rho)D_{e_n}(F)=2F(2\tau_n-\rho_n). \tag{3.9}$$

Using (3.8) and (3.9), we get

$$D_{e_n}\frac{H}{\|H\|^2}=-\frac{1}{F}\{\tau_n^2e_n+\lambda(\tau-\rho)\tau_ne_{n+1}+(\tau-\rho)\tau\tau_ne_{n+2}\}. \tag{3.10}$$

Therefore

$$\begin{aligned}D_{e_n}\left(X+\frac{H}{\|H\|^2}\right)&=e_n+D_{e_n}\left(\frac{H}{\|H\|^2}\right)\\&=\frac{1}{F}\{(\lambda^2+\tau^2)(\tau-\rho)^2e_n\\&\quad-\lambda(\tau-\rho)\tau_ne_{n+1}\\&\quad-(\tau-\rho)\tau\tau_ne_{n+1}\}.\end{aligned}$$

This implies

$$\left\langle D_{e_n}\left(X+\frac{H}{\|H\|^2}\right), \frac{H}{\|H\|^2}\right\rangle=0,$$

$$\left\langle D_{e_n}\left(X+\frac{H}{\|H\|^2}\right), e_a\right\rangle=0.$$

for $a=1, \cdots, n-1$. Q.E.D.

LEMMA 3.4. *Let $\tilde{\mathfrak{D}}$ be the distribution spanned by $e_1, \cdots, e_{n-1}$ and $\frac{H}{\|H\|^2}$. Then $\tilde{\mathfrak{D}}$ is parallel on M.*

PROOF. By the straightforward computation, we obtain

$$D_{e_a}(e_b)=\omega_b^c(e_a)e_c+\frac{F}{(\tau-\rho)^2}\delta_{ab}\frac{H}{\|H\|^2}\in\tilde{\mathfrak{D}},$$

$$D_{e_n}(e_b)=\omega_b^c(e_n)e_c\in\tilde{\mathfrak{D}},$$

$$D_{e_a}\left(\frac{H}{\|H\|^2}\right)=D_{e_a}\left(X+\frac{H}{\|H\|^2}\right)-D_{e_a}X=-e_a\in\tilde{\mathfrak{D}},$$

$$D_{e_n}\left(\frac{H}{\|H\|^2}\right)=-\frac{\tau_n}{\tau-\rho}\frac{H}{\|H\|^2}\in\tilde{\mathfrak{D}}\,. \qquad \text{Q.E.D.}$$

PROPOSITION 3.5. *Let the same assumption as Theorem 2.1 be satisfied. Suppose the condition II in Theorem 2.1 is satisfied on U and U has no point of constant sectional curvature. Then the shape of U becomes locally as follows:*

Let $(x^1, \cdots, x^{n+2})$ *denote the natural coordinate of* E^{n+2}. *There exist a curve* $(x^1(t), x^2(t), 0, \cdots, 0)$ *in* E^{n+2} *and a* C^∞*-function* $\gamma(t)>0$ *such that U is represented locally by*

$$(x^1(t), x^2(t), x^3, \cdots, x^{n+2}),$$

where

$$(x^3)^2+\cdots+(x^{n+2})^2=\gamma(t)^2.$$

PROOF. Lemma 3.3 implies that the curve $\bar{c}(t)$, which consists of the centers of the slices, is orthogonal to $e_1, \cdots, e_{n-1}$ and $\frac{H}{\|H\|^2}$. Furthermore, Lemma 3.4 implies that the distribution $\tilde{\mathfrak{D}}$, which spanned by $e_1, \cdots, e_{n-1}$ and $\frac{H}{\|H\|^2}$, is parallel on M. From these two facts, it follows that $\bar{c}(t)$ must be a plane curve. Let this plane be x^1x^2-plane, and let $(x^1(t), x^2(t), 0, \cdots, 0)$ denote the curve $\bar{c}(t)$. On the other hand, each slice $N(t)$ is a part of $(n-1)$-dimensional sphere of an n-dimensional plane $L(t)$ in E^{n+2}. Moreover the tangent space of $L(t)$ is spanned by $e_1, \cdots, e_{n-1}$ and $\frac{H}{\|H\|^2}$. Therefore each $L(t)$ is parallel mutually since $\tilde{\mathfrak{D}}$ is parallel on U. Clearly the x^1x^2-plane and $L(t)$ are orthogonal. Therefore we may take $x^3 \cdots x^{n+2}$-space to be parallel with each $L(t)$. Then U is represented as the situation mentioned above. Q.E.D.

4. The case of I in Theorem 2.1. In this section we shall study the shape of U on which the condition I in Theorem 2.1 is satisfied.

In this case there exist C^∞ orthonormal normal vector fields e_{n+1}, e_{n+2} and a C^∞ orthonormal frame field $e_1 \cdots, e_n$ on U such that

$$\text{(4.1)}\qquad \begin{aligned} &\omega_s^{n+1}=\lambda\omega^s, && \omega_s^{n+2}=\tau\omega^s,\\ &\omega_{n-1}^{n+1}=\mu\omega^{n-1}, && \omega_{n-1}^{n+2}=\tau\omega^{n-1},\\ &\omega_n^{n+1}=\lambda\omega^n, && \omega_n^{n+2}=\rho\omega^n, \end{aligned}$$

where λ, τ, μ and ρ are C^∞-functions on U and $\lambda\neq\mu, \tau\neq\rho$ on U.

In this section indices $A, B, \cdots$ run over the range $\{1, \cdots, n+2\}$ and

$s, t, \cdots$ move from 1 to $n-2$. For indices $s, t, \cdots$, the Einstein's convention is used.

LEMMA 4.1. *A point q in M is of constant sectional curvature if and only if $\lambda=0$ and $\tau=0$ at q.*

Using the same method as Lemma 3.1, we will soon prove Lemma 4.1.

Since we have assumed that U has no point of constant sectional curvature, we may set $\lambda\neq 0$ or $\tau\neq 0$ on U. Now we assume $\tau\neq 0$.

We set

$$d\lambda=\lambda_s\omega^s+\lambda_{n-1}\omega^{n-1}+\lambda_n\omega^n,$$

$$d\tau=\tau_s\omega^s+\tau_{n-1}\omega^{n-1}+\tau_n\omega^n,$$

$$d\mu=\mu_s\omega^s+\mu_{n-1}\omega^{n-1}+\mu_n\omega^n,$$

$$d\rho=\rho_s\omega^s+\rho_{n-1}\omega^{n-1}+\rho_n\omega^n.$$

Using the same method getting Lemma 3.2, we obtain Lemma 4.2.

LEMMA 4.2.

$$\omega^s_{n-1}=-\frac{\lambda_{n-1}+\tau C_{n-1}}{\lambda-\mu}\omega^s,$$

$$\omega^s_n=-\frac{\lambda\lambda_n+\tau\tau_n}{\tau(\tau-\rho)}\omega^s, \tag{4.3}$$

$$\omega^{n+1}_{n+2}=C_{n-1}\omega^{n-1}-\frac{\lambda_n}{\tau}\omega^n,$$

$$\omega^{n-1}_n=\frac{-\lambda_n+\mu_n}{\lambda-\mu}\omega^{n-1}+\frac{\tau_{n-1}-\rho_{n-1}}{\tau-\rho}\omega^n,$$

$$\lambda_s=0,\quad \tau_s=0,\quad \mu_s=0,\quad \rho_s=0, \tag{4.4}$$

$$\lambda C_{n-1}-\tau_{n-1}=0, \tag{4.5}$$

where we set

$$\omega^{n+1}_{n+2}(e_{n-1})=C_{n-1}.$$

Now, we set

$$d\lambda_{n-1}=\lambda_{n-1,t}\omega^t+\lambda_{n-1,n-1}\omega^{n-1}+\lambda_{n-1,n}\omega^n,$$

$$d\lambda_n=\lambda_{n,t}\omega^t+\lambda_{n,n-1}\omega^{n-1}+\lambda_{n,n}\omega^n,$$

$$d\tau_{n-1}=\tau_{n-1,t}\omega^t+\tau_{n-1,n-1}\omega^{n-1}+\tau_{n-1,n}\omega^n,$$

$$d\tau_n=\tau_{n,t}\omega^t+\tau_{n,n-1}\omega^{n-1}+\tau_{n,n}\omega^n,$$

$$d\mu_{n-1}=\mu_{n-1,t}\omega^t+\mu_{n-1,n-1}\omega^{n-1}+\mu_{n-1,n}\omega^n,$$

$$d\mu_n=\mu_{n,t}\omega^t+\mu_{n,n-1}\omega^{n-1}+\mu_{n,n}\omega^n\,,$$

$$d\rho_{n-1}=\rho_{n-1,t}\omega^t+\rho_{n-1,n-1}\omega^{n-1}+\rho_{n-1,n}\omega^n\,,$$

$$d\rho_n=\rho_{n,t}\omega^t+\rho_{n,n-1}\omega^{n-1}+\rho_{n,n}\omega^n\,.$$

Because of $d^2=0$, we obtain

$$\begin{aligned}&\lambda_{n-1,t}=0\,,\quad \lambda_{n,t}=0\,,\quad \tau_{n-1,t}=0\,,\quad \tau_{n,t}=0\,,\\ &\mu_{n-1,t}=0\,,\quad \mu_{n,t}=0\,,\quad \rho_{n-1,t}=0\,,\quad \rho_{n,t}=0\,.\end{aligned}\tag{4.7}$$

We set

$$dC_{n-1}(e_s)=C_{n-1,s}\,.$$

By taking the exterior derivative of (4.3), we get

$$C_{n-1,s}=0\,,\tag{4.8}$$

$$\begin{aligned}&(\lambda-\mu)\tau(\tau-\rho)\{e_{n-1}(\lambda_{n-1}+\tau C_{n-1})\}\\ &=\tau(\tau-\rho)(\lambda_{n-1}-\mu_{n-1})(\lambda_{n-1}+\tau C_{n-1})\\ &\quad-(\lambda-\mu)(-\lambda_n+\mu_n)(\lambda\lambda_n+\tau\tau_n)\\ &\quad+(\lambda-\mu)^2\tau(\tau-\rho)(\lambda\mu+\tau^2)\\ &\quad+\tau(\tau-\rho)(\lambda_{n-1}+\tau C_{n-1})^2\,,\end{aligned}\tag{4.9}$$

$$\begin{aligned}&(\lambda-\mu)\tau(\tau-\rho)^2\{e_n(\lambda_{n-1}+\tau C_{n-1})\}\\ &=\tau(\tau-\rho)^2(\lambda_n-\mu_n)(\lambda_{n-1}+\tau C_{n-1})\\ &\quad-(\lambda-\mu)^2(\lambda\lambda_n+\tau\tau_n)(\tau_{n-1}-\rho_{n-1})\\ &\quad+(\lambda-\mu)(\tau-\rho)(\lambda_{n-1}+\tau C_{n-1})(\lambda\lambda_n+\tau\tau_n)\,,\end{aligned}\tag{4.10}$$

$$\begin{aligned}&\tau(\tau-\rho)(\lambda-\mu)^2\{e_{n-1}(\lambda\lambda_n+\tau\tau_n)\}\\ &=(\lambda-\mu)^2(\lambda\lambda_n+\tau\tau_n)\{\tau_{n-1}(\tau-\rho)+\tau(\tau_{n-1}-\rho_{n-1})\}\\ &\quad+\tau^2(\tau-\rho)^2(-\lambda_n+\mu_n)(\lambda_{n-1}+\tau C_{n-1})\\ &\quad+\tau(\tau-\rho)(\lambda-\mu)(\lambda\lambda_n+\tau\tau_n)(\lambda_{n-1}+\tau C_{n-1})\,,\end{aligned}\tag{4.11}$$

$$\begin{aligned}&\tau(\tau-\rho)(\lambda-\mu)\{e_n(\lambda\lambda_n+\tau\tau_n)\}\\ &=(\lambda-\mu)(\lambda\lambda_n+\tau\tau_n)\{\tau_n(\tau-\rho)+\tau(\tau_n-\rho_n)\}\\ &\quad+\tau^2(\tau-\rho)(\tau_{n-1}-\rho_{n-1})(\lambda_{n-1}+\tau C_{n-1})\\ &\quad+\tau^2(\tau-\rho)^2(\lambda-\mu)(\lambda^2+\rho\tau)\\ &\quad+(\lambda\lambda_n+\tau\tau_n)^2(\lambda-\mu)\,.\end{aligned}\tag{4.12}$$

Let $\mathscr{D}_1$ be the distribution on U defined by

$$\mathcal{D}_1(q)=\{X\in T_q(M)\ :\ \omega^{n-1}(X)=0,\ \omega^n(X)=0\}\ .$$

Since $d\omega^{n-1}=\omega^n\wedge\omega_n^{n-1}$ and $d\omega^n=\omega^{n-1}\wedge\omega_{n-1}^n$, $\mathcal{D}_1$ is completely integrable.

On the other hand, let $\mathcal{D}_2$ be the distribution on U defined by

$$\mathcal{D}_2(q)=\{X\in T_q(M)\ :\ \omega^s(X)=0\ \ (s=1,\cdots,n-2)\}\ .$$

(4.3) implies that $\mathcal{D}_2$ is completely integrable.

From above two facts it is obtained that at each point q on M there exists a local coordinate $(y^1,\cdots,y^n)$ such that the slice $y^{n-1}=$constant, $y^n=$constant is an integral manifold of $\mathcal{D}_1$ and the slice $y^s=$constant $(s=1,\cdots,n-2)$ is an integral manifold of $\mathcal{D}_2$. Now, we denote by $N(u,v)$ the slice $y^{n-1}=$constant$=u$ and $y^n=$constant$=v$, and denote by S the slice $y^s=0$ $(s=1,\cdots,n-2)$.

Restricting the orthogonal frame field $e_1,\cdots,e_n,e_{n+1}$ and e_{n+2} to the slice $N(u,v)$, $\{e_s\}_{s=1,\cdots,n-2}$ can be regarded as C^∞ orthogonal vector fields tangent to $N(u,v)$ and $e_{n-1},e_n,e_{n+1},e_{n+2}$ can be also regarded as C^∞-orthonormal normal vector fields on $N(u,v)$ in E^{n+2}. Furthermore, restricting the dual frame ω^A and the connection form ω_A^B to the slice $N(u,v)$, we denote them by the same notation $\omega^i,\omega^\alpha,\omega_j^i,\cdots$ as those of the submanifold M in E^{n+2}. By (4.1) and Lemma 4.2, we see

$$\omega_s^{n-1}=\frac{\lambda_{n-1}+\tau C_{n-1}}{\lambda-\mu}\omega^s\ ,$$

$$\omega_s^n=\frac{\lambda\lambda_n+\tau\tau_n}{\tau(\tau-\rho)}\omega^s\ ,$$

$$\omega_s^{n+2}=\lambda\omega^s\ ,$$

$$\omega_s^{n+2}=\tau\omega^s\ ,$$

which mean that $N(u,v)$ is totally umbilic in E^{n+2}. Hence there exists $(n-1)$-dimensional plane $L(u,v)$ in E^{n+2} such that $N(u,v)$ is a part of an $(n-2)$-dimensional sphere in $L(u,v)$. Let X denote the position vector of E^{n+2}, and H denote the mean curvature vector of $N(u,v)$ in E^{n+2}. It is seen that the vector field

$$X+\frac{H}{\|H\|^2}\ ,$$

where we take X an arbitrary point of $N(u,v)$, is a constant vector in E^{n+2} and lies in $L(u,v)$, so it is called the *center* of $N(u,v)$. Moreover the tangent space of $L(u,v)$ is spanned by $e_1,\cdots,e_{n-2}$ and $\dfrac{H}{\|H\|^2}$.

LEMMA 4.3. *Let $\mathcal{D}_3$ be the distribution spanned by $e,\cdots,e_{n-2}$ and*

$\frac{H}{\|H\|^2}$. *Then* $D_{e_{n-1}}\left(X+\frac{H}{\|H\|^2}\right)$ *and* $D_{e_n}\left(X+\frac{H}{\|H\|^2}\right)$ *are orthogonal to* $\mathscr{D}_3$.

PROOF. Precisely describing,

$$H=\frac{\lambda_{n-1}+\tau C_{n-1}}{\lambda-\mu}e_{n-1}+\frac{\lambda\lambda_n+\tau\tau_n}{\tau(\tau-\rho)}e_n+\lambda e_{n+1}+\tau e_{n+2}\,.$$

Since $\tau\neq 0$, $\tau\neq\rho$ and $\lambda\neq\mu$, we get

$$\|H\|^2>0\,.$$

We set

$$F=(\lambda_{n-1}+\tau C_{n-1})^2\tau^2(\tau-\rho)^2+(\lambda\lambda_n+\tau\tau_n)^2(\lambda-\mu)^2$$
$$+(\lambda^2+\tau^2)(\lambda-\mu)^2\tau^2(\tau-\rho)^2\,.$$

Then

$$\frac{H}{\|H\|^2}=\frac{1}{F}\{(\lambda-\mu)\tau^2(\tau-\rho)^2(\lambda_{n-1}+\tau C_{n-1})e_{n-1}$$
$$+(\lambda-\mu)^2\tau(\tau-\rho)(\lambda\lambda_n+\tau\tau_n)e_n$$
$$+\lambda(\lambda-\mu)^2\tau^2(\tau-\rho)^2e_{n+1}$$
$$+(\lambda-\mu)^2\tau^3(\tau-\rho)^2e_{n+2}\}\,.$$

From (4.5), (4.9) and (4.11), we get

(4.13)
$$\tau(\tau-\rho)(\lambda-\mu)D_{e_{n-1}}F=2F\{\tau(\tau-\rho)(\lambda_{n-1}+\tau C_{n-1})$$
$$+\tau_{n-1}(\tau-\rho)(\lambda-\mu)$$
$$+\tau(\tau_{n-1}-\rho_{n-1})(\lambda-\mu)$$
$$+\tau(\tau-\rho)(\lambda_{n-1}-\mu_{n-1})\}\,.$$

By (4.13), (4.9) and (4.13), we get

(4.14)
$$D_{e_{n-1}}\frac{H}{\|H\|^2}$$
$$=-\frac{1}{F}\{\tau^2(\tau-\rho)^2(\lambda_{n-1}+\tau C_{n-1})^2e_{n-1}$$
$$+(\lambda-\mu)\tau(\tau-\rho)(\lambda\lambda_n+\tau\tau_n)(\lambda\lambda_{n-1}+\tau\tau_{n-1})e_n$$
$$+\lambda(\lambda-\mu)\tau^2(\tau-\rho)^2(\lambda_{n-1}+\tau C_{n-1})e_{n+1}$$
$$+(\lambda-\mu)\tau^3(\tau-\rho)^2(\lambda_{n-1}+\tau C_{n-1})e_{n+2}\}\,.$$

Hence

(4.15)
$$\begin{aligned}
&D_{e_{n-1}}\left(X+\frac{H}{\|H\|^2}\right)\\
&=e_{n-1}+D_{e_{n-1}}\frac{H}{\|H\|^2}\\
&=\frac{1}{F}(\{(\lambda-\mu)^2(\lambda\lambda_n+\tau\tau_n)^2\\
&\quad+(\lambda^2+\tau^2)(\lambda-\mu)^2\tau^2(\tau-\rho)^2\}e_{n-1}\\
&\quad-(\lambda-\mu)\tau(\tau-\rho)(\lambda\lambda_n+\tau\tau_n)(\lambda_{n-1}+\tau C_{n-1})e_n\\
&\quad-\lambda(\lambda-\mu)\tau^2(\tau-\rho)^2(\lambda_{n-1}+\tau C_{n-1})e_{n+1}\\
&\quad-(\lambda-\mu)\tau^3(\tau-\rho)^2(\lambda_{n-1}+\tau C_{n-1})e_{n+2})\,.
\end{aligned}$$

This implies

$$\left\langle D_{e_{n-1}}\left(X+\frac{H}{\|H\|^2}\right),\frac{H}{\|H\|^2}\right\rangle=0\,,$$

$$\left\langle D_{e_{n-1}}\left(X+\frac{H}{\|H\|^2}\right),e_s\right\rangle=0\,,$$

where $s=1,\cdots,n-2$. Therefore $D_{e_{n-1}}\left(X+\frac{H}{\|H\|^2}\right)$ is orthogonal to $\mathscr{D}_3$.

By (4.10) and (4.12), we get

$$\begin{aligned}
\tau(\tau-\rho)(\lambda-\mu)D_{e_n}F=2F\{&\tau_n(\tau-\rho)(\lambda-\mu)+(\tau_n-\rho_n)\tau(\lambda-\mu)\\
&+(\lambda_n-\mu_n)\tau(\tau-\rho)+(\lambda\lambda_n+\tau\tau_n)(\lambda-\mu)\}\,.
\end{aligned}$$

Therefore we get

$$\begin{aligned}
&D_{e_n}\left(X+\frac{H}{\|H\|^2}\right)\\
&=\frac{1}{F}(-(\lambda-\mu)\tau(\tau-\rho)(\lambda_{n-1}+\tau C_{n-1})(\lambda\lambda_n+\tau\tau_n)e_{n-1}\\
&\quad+\{\tau^2(\tau-\rho)^2(\lambda_{n-1}+\tau C_{n-1})^2\\
&\quad+(\lambda^2+\tau^2)(\lambda-\mu)^2\tau^2(\tau-\rho)^2\}e_n\\
&\quad-\lambda(\lambda-\mu)^2\tau(\tau-\rho)(\lambda\lambda_n+\tau\tau_n)e_{n+1}\\
&\quad-(\lambda-\mu)^2\tau^2(\tau-\rho)(\lambda\lambda_n+\tau\tau_n)e_{n+2})\,.
\end{aligned}$$

This implies

$$\left\langle D_{e_n}\left(X+\frac{H}{\|H\|^2}\right),\frac{H}{\|H\|^2}\right\rangle=0\,,$$

$$\left\langle D_{e_n}\left(X+\frac{H}{\|H\|^2}\right),e_s\right\rangle=0\,,$$

where $s=1, \cdots, n-2$. Therefore $D_{e_n}\left(X+\frac{H}{\|H\|^2}\right)$ is orthogonal to $\mathscr{D}_3$.

Q. E. D.

LEMMA 4.4. *$\mathscr{D}_3$, which is spanned by $e_1, \cdots, e_{n-2}$ and $\frac{H}{\|H\|^2}$ is parallel on M.*

PROOF. Straightforwardly computing, we get

$$D_{e_s}e_t=\omega_t^r(e_s)e_r+\frac{\delta_{st}F}{(\lambda-\mu)^2\tau^2(\tau-\rho)^2}\frac{H}{\|H\|^2}\in\mathscr{D}_3,$$

$$D_{e_{n-1}}e_t=\omega_t^s(e_{n-1})e_s\in\mathscr{D}_3,$$

$$D_{e_n}e_t=\omega_t^s(e_n)e_s\in\mathscr{D}_3,$$

$$D_{e_s}\frac{H}{\|H\|^2}=D_{e_s}\left(X+\frac{H}{\|H\|^2}\right)-e_s=-e_s\in\mathscr{D}_3.$$

$$D_{e_{n-1}}\frac{H}{\|H\|^2}=-\frac{\lambda_{n-1}+\tau C_{n-1}}{\lambda-\mu}\frac{H}{\|H\|^2}\in\mathscr{D}_3.$$

$$D_{e_n}\frac{H}{\|H\|^2}=-\frac{\lambda\lambda_n+\tau\tau_n}{\tau(\tau-\rho)}\frac{H}{\|H\|^2}\in\mathscr{D}_3.$$

Q. E. D.

LEMMA 4.5. *Let Ψ be a map from S to E^{n+2} defined by $\Psi(q)=X_q+\left.\frac{H}{\|H\|^2}\right|_q$ for $q\in S$, where S is the slice defined by $y^s=0$ $(s=1, \cdots, n-2)$. Then Ψ is an immersion.*

PROOF. We have only to show that $d\Psi(e_{n-1})$ and $d\Psi(e_n)$ are linearly independent. Now,

$$\begin{aligned}d\Psi(e_{n-1})=&D_{e_{n-1}}\left(X+\frac{H}{\|H\|^2}\right)\\ =&\frac{1}{F}(\{\lambda\lambda_n+\tau\tau_n)^2(\lambda-\mu)^2\\ &+(\lambda^2+\tau^2)(\lambda-\mu)^2\tau^2(\tau-\rho)^2\}e_{n-1}\\ &-(\lambda-\mu)\tau(\tau-\rho)(\lambda\lambda_n+\tau\tau_n)(\lambda_{n-1}+\tau C_{n-1})e_n\\ &-\lambda(\lambda-\mu)\tau^2(\tau-\rho)^2(\lambda_{n-1}+\tau C_{n-1})e_{n+1}\\ &-(\lambda-\mu)\tau^3(\tau-\rho)^2(\lambda_{n-1}+\tau C_{n-1})e_{n+2}).\end{aligned}$$

$$\begin{aligned}d\Psi(e_n)=&\frac{1}{F}(-(\lambda-\mu)\tau(\tau-\rho)(\lambda_{n-1}+\tau C_{n-1})(\lambda\lambda_n+\tau\tau_n)e_{n-1}\\ &+\{\tau^2(\tau-\rho)^2(\lambda_{n-1}+\tau C_{n-1})^2\end{aligned}$$

$$+(\lambda^2+\tau^2)(\lambda-\mu)^2\tau^2(\tau-\rho)^2\}e_n$$
$$-\lambda(\lambda-\mu)^2\tau(\tau-\rho)(\lambda\lambda_n+\tau\tau_n)e_{n+1}$$
$$-(\lambda-\mu)^2\tau^2(\tau-\rho)(\lambda\lambda_n+\tau\tau_n)e_{n+2}) .$$

Since $\lambda\neq\mu$, $\tau\neq\rho$ and $\tau\neq 0$, it is clear that $d\Psi(e_{n-1})$ and $d\Psi(e_n)$ are linearly independent. Q. E. D.

Lemma 4.5 implies that $\Psi(S)=\left\{X+\frac{H}{\|H\|^2} : X\in S\right\}$, which consists of the centers, becomes a 2-dimensional submanifold of E^{n+2}.

PROPOSITION 4.6. *Let the same assumpsion as Theorem* 2.1 *be satisfied. Suppose the condition I in Theorem* 2.1 *is satisfied on U and U has no point of constant sectional curvature. Then the shape of U becomes as follows:*

Let $(x^1, \cdots, x^{n+2})$ *denote the natural coordinate of* E^{n+2}. *There exist a surface* $(x^1(u, v), x^2(u, v), x^3(u, v), 0, \cdots, 0)$ *in* $x^1x^2x^3$*-space in* E^{n+2} *and a* C^∞*-function* $\gamma(u, v)$ (>0) *such that U is locally represented by*

$$(x^1(u, v), x^2(u, v), x^3(u, v), x^4, \cdots, x^{n+2}),$$

where

$$(x^4)^2+\cdots+(x^{n+2})^2=\gamma(u, v)^2 .$$

PROOF. The tangent space of $\Psi(S)$ is spanned by $D_{e_{n-1}}\left(X+\frac{H}{\|H\|^2}\right)$ and $D_{e_n}\left(X+\frac{H}{\|H\|^2}\right)$. On the other hand, Lemma 4.3 shows that both $D_{e_{n-1}}\left(X+\frac{H}{\|H\|^2}\right)$ and $D_{e_n}\left(X+\frac{H}{\|H\|^2}\right)$ are orthogonal to $e_1, \cdots, e_{n-2}$ and $\frac{H}{\|H\|^2}$. Moreover Lemma 4.4 implies that the distribution spanned by $e_1, \cdots, e_{n-2}$ and $\frac{H}{\|H\|^2}$ is parallel on M. Therefore $D_{e_{n-1}}\left(X+\frac{H}{\|H\|^2}\right)$ and $D_{e_n}\left(X+\frac{H}{\|H\|^2}\right)$ are contained in a fixed 3-dimensional space. Let this space be $x^1x^2x^3$-space and let $(x^1(u, v), x^2(u, v), x^3(u, v), 0, \cdots, 0)$ represent the surface $\Psi(S)$. It has been shown that each slice $N(u, v)$ is a part of an $(n-2)$-dimensional sphere in $L(u, v)$, which is an $(n-1)$-dimensional plane in E^{n+2}. Since the tangent space of $L(u, v)$ is spanned by $e_1, \cdots, e_{n-2}$, $\frac{H}{\|H\|^2}$ and the distribution which is spanned by $e_1, \cdots, e_{n-2}$, $\frac{H}{\|H\|^2}$ is parallel on M, each $L(u, v)$ is parallel. Clearly the $x^1x^2x^3$-space is ortho gonal to $L(u, v)$. Therefore we may choose $x^4 \cdots x^{n+2}$-space to be paralle with $L(u, v)$. Then U is represented as the situation mentioned above.

Q.E.D.

It is easy to show that the Theorem 2.1, Proposition 3.5 and Proposition 4.6 lead the Theorem in the introduction.

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