Product Type Cylindrical Measures and Rotational Quasi-Invariance

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1. Introduction and preliminaries

The author [1] introduced the concept of the rotationally quasi-invariant (RQI) cylindrical measure as follows:

Let H be a real separable Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|=\sqrt{\langle\cdot,\cdot\rangle}$, $\mathcal{C}_H$ the class of all cylindrical sets on H and μ a cylindrical measure on H. We denote by $U(H)$ the class of all unitary operators of H. If for every $\varepsilon>0$ there exists a $\delta>0$ such that $\mu(u(A))<\varepsilon$ for all $u\in U(H)$ and all $A\in\mathcal{C}_H$ for which $\mu(A)<\delta$, then μ is said to be rotationally quasi-invariant.

However the above definition contains a certain uniform property with respect to $u\in U(H)$. Therefore, from now on we call it a strongly rotationally quasi-invariant (SRQI) cylindrical measure. And we shall define rotationally quasi-invariant cylindrical measures without any strong senses. Let μ and ν be two cylindrical measures on H. We say that ν is cylindrically absolutely continuous with respect to μ, in symbols $\nu\ll_C\mu$, if for every $\varepsilon>0$, there exists a $\delta>0$ such that $\nu(A)<\varepsilon$ for all $A\in\mathcal{C}_H$ for which $\mu(A)<\delta$. Two cylindrical measures μ and ν for which both $\nu\ll_C\mu$ and $\mu\ll_C\nu$ are called cylindrically equivalent, in symbols $\mu\sim_C\nu$.

DEFINITION. If for each $u\in U(H)$ the relation $\mu\sim_C\mu\circ u^{-1}$ holds, then μ is said to be a rotationally quasi-invariant (RQI) cylindrical measure.

It is still open whether the above two concepts coincide with each other.*

In this short note we investigate rotationally quasi-invariant cylindrical measures in the latter sense.

Let $\mathfrak{S}_H$ be the collection of all closed balls of H and μ be a cylindrical measure on H. We say that μ is of type 0 if for any $\varepsilon>0$ there exists

* During the preparation of this article, the author knows the result of H. Shimomura [4], which shows the equivalence of these two concepts. Our results will be contained in it. However, we decide to publish this short note as the different proof for the special case.

a ball $A\in\mathfrak{S}_H$ such that $\mu\circ\xi^{-1}(\xi(A))\geqq 1-\varepsilon$ for all $\xi\in H$.

Let $\{e_n\}_{n=1,2,\cdots}$ be a complete orthonormal system (CONS) of H and for each n, μ_n be a probability measure defined on R. Any k-dimensional space R^k has a probability measure $\mu_{n_1}\otimes\cdots\otimes\mu_{n_k}$ on it. If μ is a cylindrical measure on H satisfying that $\mu\circ(e_{n_1},\cdots,e_{n_k})^{-1}=\mu_{n_1}\otimes\cdots\otimes\mu_{n_k}$ for any finite system $\{e_{n_1},\cdots,e_{n_k}\}$, where $(e_{n_1},\cdots,e_{n_k})$ means the mapping from H into R^k such that

$$x\in H\longmapsto(\langle x, e_{n_1}\rangle,\cdots,\langle x, e_{n_k}\rangle)\in R^k,$$

then μ is called *a product type* (PT) *cylindrical measure with respect to* $\{e_n\}_{n=1,2,\cdots}$.

It is obvious that for any system of probability measures $\{\mu_n\}_{n=1,2,\cdots}$ there exist many PT-cylindrical measures on H satisfying that $\mu\circ(e_{n_1},\cdots,e_{n_k})^{-1}=\mu_{n_1}\otimes\cdots\otimes\mu_{n_k}$ for every finite system $\{e_{n_1},\cdots,e_{n_k}\}$.

For each positive integer n, we denote by $U_{e_1\cdots,e_n}$ the subset of $U(H)$ consisting of u satisfying the following two conditions:

(1) Restrict u to the space H_n, where H_n is the subspace of H generated by $\{e_1,\cdots,e_n\}$. Then such a restriction is a unitary operator of H_n.

(2) Restrict u to the space $H_n^\perp$. Then it is the identity operator on $H_n^\perp$.

2. *PT*-cylindrical measures and rotational quasi-invariance

We have already had the result that every SRQI-cylindrical measure is of type 0. Here we consider the case of RQI-cylindrical measures.

PROPOSITION ([5]). *Let* $\{e_n\}_{n=1,2,\cdots}$ *be a CONS of* H *and for each* n, μ_n *be a probability measure on* R *such that* $\mu_n=f_n(x)dx$, *where* dx *is the Lebesque measure on* R. *Suppose that for each* n, $x\sqrt{f_n(x)}\in L^2(dx)$, $\sup_n\|x\sqrt{f_n(x)}\|_{L^2}<+\infty$ *and*

$$\sum_{n=1}^{\infty}\left\{\int_{-\infty}^{\infty}xf_n(x)dx\right\}^2<+\infty.$$

Then there uniquely exists a PT-cylindrical measure μ *with respect to* $\{e_n\}_{n=1,2,\cdots}$ *which is of type* 0 *and satisfying that* $\mu\circ(e_{n_1},\cdots,e_{n_k})^{-1}=\mu_{n_1}\otimes\cdots\otimes\mu_{n_k}$ *for every finite system* $\{e_{n_1},\cdots,e_{n_k}\}$.

This proposition shows that there are many PT-cylindrical measures which are of type 0.

Let μ be a PT-cylindrical measure with respect to a CONS $\{e_n\}_{n=1,2,\cdots}$ on H. If $\mu\circ e_1^{-1}=\mu\circ e_2^{-1}=\cdots=\mu\circ e_n^{-1}=\cdots$, then we call μ *a stationary product type* (SPT) *cylindrical measure.*

We start with the following lemma.

LEMMA. *Let μ be an RQI-cylindrical measure on H and E a subspace of H containing some CONS $\{e_n\}_{n=1,2,\dots}$ of H. Suppose that for any $\varepsilon>0$ there exists a ball $A\in\mathfrak{S}_H$ such that $\mu\circ\xi^{-1}(\xi(A))\geqq 1-\varepsilon$ for all $\xi\in E$. Then μ is of type 0.*

PROOF. The above assumption induces the following result (see [3], p. 265): There exists a linear operator f of H into $L^0(\Omega, m)$ satisfying that the restriction of f on E is continuous, where (Ω, m) is a Radon probability measure space and L^0 is the linear space of all real valued random variables.

Suppose that f is not continuous on H. Then there exists a positive number $\varepsilon_0>0$ and a sequence $\{x_n\}\subset H$ such that $0<\|x_n\|<\min\{\|x_{n-1}\|, 1/n\}$ for $n=1, 2, \cdots$ and $d(f(x_n), f(0))\geqq\varepsilon_0$, where $d(\cdot,\cdot)$ is the metric of L^0, i.e.

$$d(g, h)=\int_\Omega \frac{|g-h|}{1+|g-h|}dm.$$

Using the Schmidt's orthogonalization, we have an orthonormal system $\{\tilde{x}_n\}$ from $\{x_n\}$. Then we can make a CONS $\{c_n\}_{n=1,2,\dots}$ containing $\{\tilde{x}_n\}_{n=1,2,\dots}$. There exists a unitary operator u such that $u(e_n)=c_n$ for $n=1, 2, \cdots$. Since μ is rotationally quasi-invariant, the restriction of f on $u(E)$ is continuous. Therefore it follows that for $\varepsilon_0>0$ there exists a $\delta_0>0$ such that $d(f(x), f(0))<\varepsilon_0$ for every $\|x\|<\delta_0$ for which $x\in u(E)$. We have a number n such that $1/n<\delta_0$ and also $\{x_n\}_{n=1,2,\dots}\subset u(E)$. Hence we get the contradiction. Thus it follows that f is continuous on H, i.e., μ is of type 0. □

THEOREM. *Let μ be a stationary product type cylindrical measure on H with respect to $\{e_n\}_{n=1,2,\dots}$, where $\{e_n\}_{n=1,2,\dots}$ is a CONS of H. If μ is rotationally quasi-invariant, then μ is of type 0.*

PROOF. Assume that $H=l^2$ and $e_n=\{0, \cdots, 0, \overset{n\text{-}th}{1}, 0, \cdots\}$ for $n=1, 2, \cdots$. We have the same consideration for other general cases. It is clear that $R_0^\infty\subset l^2\subset R^\infty$. First we shall show that

(2.1) For any $\varepsilon>0$, there exists an $r>0$ such that

$$\mu(\{y\in l^2 : \langle y, x\rangle\geqq r\})<\varepsilon$$

for all $x\in R_0^\infty$ satisfying that $\|x\|=1$.

Suppose that (2.1) is not correct. Let $\{r_k\}_{k=1,2,\dots}$ be a sequence of positive numbers satisfying that r_k tends to ∞ as $k\to\infty$. Then we have a sequence $\{x_k\}_{k=1,2,\dots}\subset R_0^\infty$ such that $\|x_k\|=1$ and $\mu(\{y\in l^2 : \langle y, x_k\rangle\geqq r_k\})\geqq\varepsilon_0$, where ε_0 is a certain positive number. Let $x_k=\{x_1^k, \cdots, x_{n_k}^k, 0, 0, \cdots\}$ for

$k=1,2,\cdots$. We construct a sequence $\{\tilde{x}^k\}_{k=1,2,\cdots}$ of elements of l^2 as follows:

$$\begin{aligned}
\tilde{x}^1 &= \{x_1^1, \cdots, x_{n_1}^1, 0, 0, \cdots\cdots\cdots\cdots\} \\
\tilde{x}^2 &= \{0, \cdots, \overset{n_1\text{-}th}{0}, x_1^2, \cdots, x_{n_2}^2, 0, 0, \cdots\cdots\} \\
&\vdots \\
\tilde{x}^k &= \{0, \cdots\cdots\cdots \overset{(n_1+\cdots+n_{k-1})\text{-}th}{0}, x_1^k, \cdots, x_{n_k}^k, 0, 0, \cdots\} \\
&\vdots
\end{aligned}$$

Then we can find another sequence $\{y^n\}_{n=1,2,\cdots}\subset l^2$ such that $\{\tilde{x}^1, y^1, \tilde{x}^2, y^2, \cdots, \tilde{x}^n, y^n, \cdots\}$ is a CONS of l^2. Let u be a unitary operator of l^2 satisfying that $u(e_{2k-1})=\tilde{x}^k$ and $u(e_{2k})=y^k$ for $k=1,2,\cdots$. Since that μ is an SPT-cylindrical measure and $\mu(\{y\in l^2 : \langle y, x_k\rangle \geqq r_k\})\geqq \varepsilon_0$, we have

$$\mu(\{y\in l^2 : \langle y, \tilde{x}^k\rangle \geqq r_k\})\geqq \varepsilon_0 \quad \text{for} \quad k=1,2,\cdots. \tag{2.2}$$

For every k, $\mu\circ e_k^{-1}$ is the same probability measure on R. Therefore it follows that for any $\varepsilon>0$ there exists a number n_0 such that

$$\mu(\{y\in l^2 : \langle y, e_k\rangle \geqq r_n\})<\varepsilon \quad \text{for all} \quad n\geqq n_0 \text{ and all } k. \tag{2.3}$$

Thus (2.2) and (2.3) contradict the result $\mu\sim_C \mu\circ u^{-1}$. Then we have the desired result (2.1). Since R_0^∞ is a subspace of l^2 containing $\{e_n\}_{n=1,2,\cdots}$, Lemma says that μ is of type 0. □

COROLLARY. *Let $\{e_n\}_{n=1,2,\cdots}$ be a CONS of H. There exists a cylindrical measure which is rotationally invariant with respect to $\bigcup_{n=1}^{\infty} U_{e_1,\cdots,e_n}$ and not rotationally quasi-invariant with respect to $U(H)$.*

PROOF. Let $I=\{e_\iota\}_{\iota\in I}$ be an algebraic basis of H containing $\{e_n\}_{n=1,2,\cdots}$. I is an uncountable index set.

First we define some cylindrical measure on R^I. The dual of R^I is the direct sum $R^{(I)}$, and the product topology on R^I coincides with the weak topology $\sigma(R^I, R^{(I)})$. For each $\iota\in I$ let λ_ι be the canonical Gauss measure γ on R, i.e., $\lambda_\iota=\gamma=(\sqrt{2\pi})^{-1}\exp(-x^2/2)dx$. For each finite subset J of I let λ_J be the probability measure $\bigotimes_{\iota\in J}\lambda_\iota$ on R^J. Thus we have a cylindrical measure λ on R^I (cf. [3]).

Let H^* be the algebraic dual of H. Then H^* equipped with the weak topology $\sigma(H^*, H)$ is isomorphic to R^I equipped with the product topology. Therefore, using the above considerations, we can define a cylindrical measure on H^*, denote it by μ. Also, we can consider every cylindrical measure on H^* equipped with $\sigma(H^*, H)$ to be the same thing as a cylindrical measure on H. Thus we have a cylindrical measure μ on H([2]).

Obviously μ is rotationally invariant with respect to $\bigcup_{n=1}^{\infty} U_{e_1,\dots,e_n}$, and also of stationary product type with respect to $\{e_n\}_{n=1,2,\dots}$. However μ is not of type 0 ([2]). Therefore μ is not rotationally quasi-invariant with respect to $U(H)$. $\square$

References

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