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On the Powerwise Integration

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Introduction. This is a continuation of our series of papers [1] to [4]. We shall be concerned with originating, for functions of one real variable, an integration method which generalizes that of Denjoy and which will be given the name of powerwise integration. By constructing concrete functions powerwise integrable without being Denjoy integrable, it will be shown that the new integration is actually wider than that of Denjoy.

We have hitherto introduced three kinds of integration: the quasi-Denjoy, the Luzin, and the (M) integration. As concerns the relation of the powerwise integration to these three, no more is known at present than the fact that the Luzin integration does not include the powerwise integration.

§1. Dirichlet continuous functions.

Throughout this section, the letter p will denote a real number >1. If α is a real number, the symbol $\alpha\square^p$ will mean α^p or $-|\alpha|^p$, according as $\alpha \geqq 0$ or $\alpha < 0$, respectively. Thus $\alpha\square^p$ is short for $|\alpha|^p \operatorname{sgn} \alpha$.

The sets considered in this paper will usually be linear, i. e. contained in the real line $\boldsymbol{R}$. We shall, on occasion, make use of the terminology, notation, and results of the papers mentioned above. Thus, by a CT *set* we shall mean any linear compact set consisting of two points at least. Again, a *function*, by itself, will always signify one defined on the whole real line and assuming finite real values, unless another meaning is obvious from the context.

Let Q be a compact nonconnected set, i. e. a CT set which is not an interval. We shall denote generically by H an open interval contiguous to this set. A function $\varphi(x)$ will be termed to fulfil the *Dirichlet condition* on the set Q, if the following three items are fulfilled:

(i) We have $\sum|\varphi(H)|^p < +\infty$ for every p (subject, of course, to the limitation $p>1$ mentioned above), where the summation extends over all the intervals H considered above and where $\varphi(H)$ means the increment of $\varphi(x)$ over the closure of H;

(ii) $\lim_{p\to 1}(p-1)\sum|\varphi(H)|^p=0$, i. e. $\sum|\varphi(H)|^p=o\left(\frac{1}{p-1}\right)$ as $p\to 1$,

where o is the Landau symbol;

(iii) if the end points of a closed interval A belong to the set Q, while A itself is not contained in Q, then

$$\lim_{p\to 1}\sum_{H\subset A}\varphi(H)\Box^p=\varphi(A), \quad \text{i. e.} \sum_{H\subset A}\varphi(H)\Box^p=\varphi(A)+o(1) \quad \text{as } p\to 1.$$

REMARKS. (a) The series which appears in item (iii) is absolutely convergent on account of item (i). (b) As we find easily, the two series appearing in the above items can be interpreted as Dirichlet series with p for the variable; this explains our term "Dirichlet condition". (c) A function which fulfils the Dirichlet condition on the set Q is necessarily a constant on each closed interval (if existent) contained in Q.

LEMMA 1. *For any two real numbers α and β, we have*

(i) $|\alpha+\beta|^p\leqq 2^{p-1}(|\alpha|^p+|\beta|^p)$,

(ii) $|(\alpha+\beta)\Box^p-\alpha\Box^p-\beta\Box^p|\leqq 2^{p-1}(p-1)(|\alpha|^p+|\beta|^p)$.

PROOF. We may clearly suppose that $\alpha\beta\neq 0$. As we find easily,

$$2^{1-p}\leqq t^p+(1-t)^p\leqq 1 \qquad \text{if } 0\leqq t\leqq 1.$$

It follows that if $\alpha>0$ and $\beta>0$, then

$$2^{1-p}(\alpha+\beta)^p\leqq\alpha^p+\beta^p\leqq(\alpha+\beta)^p,$$

or what amounts to the same thing,

$$\alpha^p+\beta^p\leqq(\alpha+\beta)^p\leqq 2^{p-1}(\alpha^p+\beta^p).$$

From this the assertion is deduced as follows.

re (i): Noting that $|\alpha|>0$ and $|\beta|>0$, we have

$$|\alpha+\beta|^p\leqq(|\alpha|+|\beta|)^p\leqq 2^{p-1}(|\alpha|^p+|\beta|^p).$$

re (ii): Since $2^{p-1}-1<2^{p-1}(p-1)$ by the mean value theorem, we get

$$|(\alpha+\beta)^p-\alpha^p-\beta^p|\leqq 2^{p-1}(p-1)(\alpha^p+\beta^p) \qquad \text{for } \alpha>0 \text{ and } \beta>0.$$

The inequality (ii) follows from this in the case in which $\alpha\beta>0$.

We pass on to the case $\alpha\beta<0$, where we assume, as we plainly may, that $\alpha>0>\beta$. Writing $\gamma=-\beta>0$, we distinguish two cases according as $\alpha\geqq\gamma$ or not. If $\alpha\geqq\gamma$, it follows from what we proved already that

$$|\alpha^p-\gamma^p-(\alpha-\gamma)^p|\leqq 2^{p-1}(p-1)\{\gamma^p+(\alpha-\gamma)^p\}\leqq 2^{p-1}(p-1)\alpha^p;$$

hence $$|(\alpha+\beta)\Box^p-\alpha\Box^p-\beta\Box^p|\leqq 2^{p-1}(p-1)|\alpha|^p,$$

which is stronger than (ii). If $\alpha<\gamma$, we have similarly

$$|\gamma^p-\alpha^p-(\gamma-\alpha)^p|\leqq 2^{p-1}(p-1)\gamma^p\,,$$

i. e.

$$|(\alpha+\beta)\square^{p}-\alpha\square^{p}-\beta\square^{p}|\leqq 2^{p-1}(p-1)|\beta|^p\,.$$

LEMMA 2. *If the sum of four real numbers* ξ_1, ξ_2, ξ_3, ξ_4 *is* 0, *then*

$$\left|\sum_{i=1}^{4}\xi_i\square^{p}\right|\leqq 3^p(p-1)\max\,(|\xi_1|^p,\,\cdots,\,|\xi_4|^p)\,.$$

PROOF. It is readily verified that if $u, v, w\geqq 0$ and $u+v+w=1$, then $3^{1-p}\leqq u^p+v^p+w^p\leqq 1$. From this we find at once that

$$\alpha^p+\beta^p+\gamma^p\leqq(\alpha+\beta+\gamma)^p\leqq 3^{p-1}(\alpha^p+\beta^p+\gamma^p)\,,$$

whenever α, β, γ are nonnegative real numbers.

This being permised, let us write for short

$$S=\xi_1\square^{p}+\cdots\cdot+\xi_4\square^{p} \quad\text{and}\quad \mu=\max\,(|\xi_1|^p,\cdots\cdot,|\xi_4|^p)\,.$$

To prove the asserted inequality, it clearly suffices to consider the following two cases, where $\alpha, \beta, \gamma, \delta$ are nonnegative.

(a) $\xi_1=\alpha+\beta+\gamma$, $\xi_2=-\alpha$, $\xi_3=-\beta$, $\xi_4=-\gamma$. In this case, it follows from what we premised above that

$$\begin{aligned}0\leqq S&=(\alpha+\beta+\gamma)^p-\alpha^p-\beta^p-\gamma^p\leqq(3^{p-1}-1)(\alpha^p+\beta^p+\gamma^p)\\&\leqq(3^{p-1}-1)(\alpha+\beta+\gamma)^p\leqq 3^p(p-1)\mu\,,\end{aligned}$$

since we have $3^{p-1}-1<3^{p-1}(p-1)\log 3<3^p(p-1)$.

(b) $\xi_1=\alpha$, $\xi_2=\beta$, $\xi_3=-\gamma$, $\xi_4=-\delta$. In this case we have

$$\alpha+\beta=\gamma+\delta,\quad S=\alpha^p+\beta^p-\gamma^p-\delta^p\,.$$

Lemma 1 and the obvious inequalities $\alpha^p+\beta^p\leqq(\alpha+\beta)^p$, $\gamma^p+\delta^p\leqq(\gamma+\delta)^p$ show together that

$$0\leqq(\alpha+\beta)^p-\alpha^p-\beta^p\leqq 2^{p-1}(p-1)(\alpha^p+\beta^p)\,,$$

$$0\leqq(\gamma+\delta)^p-\gamma^p-\delta^p\leqq 2^{p-1}(p-1)(\gamma^p+\delta^p)\,.$$

Hence

$$|S|\leqq 2^p(p-1)\mu\leqq 3^p(p-1)\mu\,.$$

THEOREM 1. *If two functions* $\varphi(x)$ *and* $\psi(x)$ *fulfil the Dirichlet condition on a compact nonconnected set* Q, *so does also any linear combination, with constant coefficients, of these functions.*

PROOF. We may restrict ourselves to the sum $\theta(x)=\varphi(x)+\psi(x)$. Let us denote by H a generic open interval contiguous to the set Q. Noting that both series $\sum|\varphi(H)|^p$ and $\sum|\psi(H)|^p$ are convergent, we obtain, by part (i) of Lemma 1, the appraisal

$$\sum_H |\theta(H)|^p \leqq 2^{p-1} \sum_H \{|\varphi(H)|^p + |\psi(H)|^p\} < +\infty .$$

But
$$\sum_H \{|\varphi(H)|^p + |\psi(H)|^p\} = o\left(\frac{1}{p-1}\right) + o\left(\frac{1}{p-1}\right) = o\left(\frac{1}{p-1}\right).$$

Hence
$$\sum_H |\theta(H)|^p = o\left(\frac{1}{p-1}\right).$$

This being so, let A be any closed interval such that the set Q contains the end points of A without containing the whole of A. For simplicity, we agree to denote by Σ' a summation extended over the intervals $H \subset A$. Using part (ii) of Lemma 1, we deduce that

$$\begin{aligned} |\Sigma'\theta(H)\square^p - \theta(A)| &\leqq |\Sigma'\theta(H)\square^p - \Sigma'\varphi(H)\square^p - \Sigma'\psi(H)\square^p| \\ &\quad + |\Sigma'\varphi(H)\square^p - \varphi(A)| + |\Sigma'\psi(H)\square^p - \psi(A)| \\ &\leqq \Sigma'|\theta(H)\square^p - \varphi(H)\square^p - \psi(H)\square^p| + o(1) + o(1) \\ &\leqq 2^p(p-1)\Sigma'\{|\varphi(H)|^p + |\psi(H)|^p\} + o(1) \\ &= 2^p(p-1)\cdot o\left(\frac{1}{p-1}\right) + o(1) = o(1) . \end{aligned}$$

Hence $\Sigma'\theta(H)\square^p = \theta(A) + o(1)$, which completes the proof.

Given a compact nonconnected set Q, let R denote generically a compact nonconnected set contained in Q. A function will be called *Dirichlet continuous* on Q, if it is continuous on Q and if it fulfils the Dirichlet condition on all the sets R. When this is the case, the function is plainly Dirichlet continuous on each R.

It is easy to see that, in the above definition, we may restrict the sets R to those which span the same closed interval as the set Q does. This remark will be useful in § 4.

It follows at once from Theorem 1 that if two functions are Dirichlet continuous on a compact nonconnected set, the same is true of any linear combination of these functions.

Given a compact nonconnected set Q, let us write H for a generic open interval contiguous to Q, and let A be an arbitrary closed interval such that the set Q contains the end points of A without containing the whole of A. A function $\varphi(x)$ will be said to fulfil the *condition* (B) on Q, if $\Sigma|\varphi(H)| < +\infty$ and if we have $\sum_{H \subset A} \varphi(H) = \varphi(A)$ for every A.

THEOREM 2. *Given a compact nonconnected set Q, let H denote a generic open interval contiguous to this set. Suppose that there corresponds to each H a real number $\rho(H)$ in such a way that $\Sigma|\rho(H)| < +\infty$.*

(i) *Let $f(x)$ be the function defined on the real line* $\boldsymbol{R}$ *as follows: if x belongs to an H, then $f(x)=|H|^{-1}\cdot\rho(H)$; otherwise we set simply $f(x)=0$. This function $f(x)$ is summable on* $\boldsymbol{R}$, *and any indefinite integral, say $\theta(x)$, of $f(x)$ fulfils the condition* (B) *on Q. Moreover, the function $\theta(x)$ is* AC *on Q as well as derivable to zero at almost all points of Q, and we have $\theta(H)=\rho(H)$ for every H.*

(ii) *If $\varphi(x)$ is a function fulfilling the condition* (B) *on Q and if we have $\varphi(H)=\rho(H)$ for every H, then the difference $\varphi(x)-\theta(x)$ is a constant over Q.*

PROOF. *re* (i): The function $f(x)$ is summabe on the union D of all the intervals H, since evidently

$$\int_H |f(x)|\,dx=|\rho(H)| \quad \text{and} \quad \int_D |f(x)|\,dx=\sum_H |\rho(H)|<+\infty .$$

But $f(x)=0$ on the set $\boldsymbol{R}\setminus D$, and it follows that $f(x)$ is summable on $\boldsymbol{R}$. Consequently, its indefinite integral $\theta(x)$ is AC on $\boldsymbol{R}$, and hence on Q. Moreover, $\theta(x)$ is derivable to $f(x)$ almost everywhere on $\boldsymbol{R}$, so that we have $\theta'(x)=0$ at almost all points of Q. Obviously $\theta(H)=\int_H f(x)\,dx=\rho(H)$ for every H.

We shall show that $\theta(x)$ fulfils the condition (B) on Q. We have in the first place $\Sigma|\theta(H)|=\Sigma|\rho(H)|<+\infty$. If A is a closed interval such that Q contains the end points of A without containitg A itself, then

$$\begin{aligned}\theta(A)&=\int_A f(x)\,dx=\int_{A\cap D} f(x)\,dx+\int_{A\setminus D} f(x)\,dx\\ &=\sum_{H\subset A}\int_H f(x)\,dx+0=\sum_{H\subset A}\theta(H).\end{aligned}$$

This completes the proof of part (i).

re (ii): Write $\psi(x)=\varphi(x)-\theta(x)$ and let A be any interval considered above. We then have

$$\psi(A)=\varphi(A)-\theta(A)=\sum_{H\subset A}\varphi(H)-\sum_{H\subset A}\theta(H)=\sum_{H\subset A}\{\varphi(H)-\rho(H)\}=0 .$$

This plainly implies that $\psi(x)$ is a constant over the set Q.

We introduced in our paper [4] the notion of approximate equiderivability. However, we feel at present that the notion cannot be regarded as quite adequate. We now want to replace it with one simpler and less restrictive: two functions will be termed *approximately equiderivable,* or briefly AED, at a point of $\boldsymbol{R}$ if at this point the functions are both AD (approximately derivable) and have coinciding approximate derivatives.

LEMMA 3. *If two functions of a real variable coincide on a measurable set and if one of them is* AD *at almost all points of this set, then the functions are* AED *at almost all points of the same set.*

This follows directly from Theorem (3.3) on p. 220 of Saks [5].

THEOREM 3. *Let Q be a compact nonconnected set.*

(i) *If, on the set Q, a function $\varphi(x)$ is* BV *and fulfils the Dirichlet condition, then it fulfils the condition* (B) *on this set.*

(ii) *If a function $\varphi(x)$ fulfils the condition* (B) *on Q, then it does so on every compact nonconnected set contained in Q, and the function is both* AC *and Dirichlet continuous, on this set Q. Moreover, the function is* AD *to zero at almost every point of Q.*

PROOF. *re* (i): Let H be a generic open interval contiguous to Q and let A be any closed interval such that Q contains the end points of A without containing A itself. The function $\varphi(x)$ being BV on Q, we evidently have $\sum|\varphi(H)|<+\infty$. From this we can deduce that

$$\lim_{p\to 1}\sum_{H\subset A}\varphi(H)\square^{p}=\sum_{H\subset A}\varphi(H)\,.$$

In fact, this is obvious if A contains only a finite number of the intervals H. In the opposite case, we arrange all the $H\subset A$ in an infinite sequence $\langle H_1, H_2, \cdots\rangle$. Given any positive number $\varepsilon<1$, let us take a positive integer N so as to fulfil the inequality

$$\sum_{n>N}|\varphi(H_n)|<\varepsilon, \qquad \text{so that } \Big|\sum_{n>N}\varphi(H_n)\square^{p}\Big|\leqq\sum_{n>N}|\varphi(H_n)|^{p}<\varepsilon.$$

This being so, we choose a $\delta>0$ such that

$$\Big|\sum_{n=1}^{N}\varphi(H_n)\square^{p}-\sum_{n=1}^{N}\varphi(H_n)\Big|<\varepsilon \qquad \text{if } 1<p<1+\delta\,.$$

It follows that, for the same values of p,

$$\Big|\sum_{H\subset A}\varphi(H)\square^{p}-\sum_{H\subset A}\varphi(H)\Big|\leqq\Big|\sum_{n=1}^{N}\varphi(H_n)\square^{p}-\sum_{n=1}^{N}\varphi(H_n)\Big|$$
$$+\Big|\sum_{n>N}\varphi(H_n)\square^{p}\Big|+\Big|\sum_{n>N}\varphi(H_n)\Big|<3\varepsilon\,,$$

which establishes the stated result.

Since $\varphi(x)$ fulfils the Dirichlet condition on Q, we have on the other hand $\lim_{p\to 1}\sum_{H\subset A}\varphi(H)\square^{p}=\varphi(A)$. This, together with what we have proved above, shows that $\sum_{H\subset A}\varphi(H)=\varphi(A)$. The function $\varphi(x)$ thus fulfils the condition (B) on the set Q.

re (ii): Suppose that a function $\varphi(x)$ fulfils the condition (B) on Q. Then $\sum|\varphi(H)|<+\infty$, where H has the same meaning as above. Hence we can take the number $\rho(H)$ of Theorem 2 to be $\varphi(H)$, and it follows directly from that theorem and Lemma 3 that the function $\varphi(x)$ is AC on Q as well as AD to zero at almost all points of Q.

We go on to show that $\varphi(x)$ fulfils the Dirichlet condition on Q. Since $\sum|\varphi(H)|<+\infty$, we have $|\varphi(H)|<1$ for all the H except perhaps a finite number of them. Accordingly $\sum|\varphi(H)|^p<+\infty$ for every $p>1$.

To show further that $(p-1)\sum|\varphi(H)|^p\to 0$ as $p\to 1$, we may plainly assume that there are an infinity of the intervals H. Arranging all the H in a sequence $\langle H_1, H_2, \cdots\rangle$ as above, we take a positive integer N such that $|\varphi(H_{N+1})|+|\varphi(H_{N+2})|+\cdots<1$. Then

$$\lim_{p\to 1}(p-1)\sum_{n=1}^{N}|\varphi(H_n)|^p=0 \quad \text{and} \quad \sum_{n>N}|\varphi(H_n)|^p<1\,,$$

whence

$$\lim_{p\to 1}(p-1)\sum_{n=1}^{\infty}|\varphi(H_n)|^p=0\,.$$

We already saw in the above that the condition $\sum|\varphi(H)|<+\infty$ implies the relation $\lim\limits_{p\to 1}\sum\limits_{H\subset A}\varphi(H)\square^p=\sum\limits_{H\subset A}\varphi(H)$, where the right-hand side coincides with $\varphi(A)$, since $\varphi(x)$ fulfils the condition (B) on Q. Thus the function $\varphi(x)$ fulfils the Dirichlet condition on Q.

Let R be any compact nonconnected set contained in Q. we proceed to show that the function $\varphi(x)$ fulfils the condition (B) on R. Writing G for a generic open interval contiguous to R, we have

$$|\varphi(G)|=|\sum_{H\subset G}\varphi(H)|\leqq\sum_{H\subset G}|\varphi(H)|\,,$$

where a possible void sum means zero. It follows that

$$\sum_{G}|\varphi(G)|\leqq\sum_{G}\sum_{H\subset G}|\varphi(H)|\leqq\sum_{H}|\varphi(H)|<+\infty\,.$$

If, further, C is any closed interval such that the set R contains the end points of C without containing C itself, then we have $H\subset C$ when and only when there is a G such that $H\subset G\subset C$. Consequently

$$\varphi(C)=\sum_{H\subset C}\varphi(H)=\sum_{G\subset C}\sum_{H\subset G}\varphi(H)=\sum_{G\subset C}\varphi(G)\,.$$

The function $\varphi(x)$, which thus fulfils the condition (B) on R, fulfils the Dirichlet condition on R by what was already proved in the above. We conclude that $\varphi(x)$ is Dirichlet continuous on Q, and the proof is complete.

THEOREM 4. *Given a compact nonconnected set Q and given a function $\varphi(x)$ which is* AC *on this set, let $\lambda(x)$ be the linear modification of $\varphi(x)$ with respect to Q and let H be a generic open interval contiguous to Q. Then*

(i) *the function $\lambda(x)$ is* AC *on the closed interval spanned by Q;*

(ii) *we have $\sum|\varphi(H)|<+\infty$;*

(iii) *the function $\varphi(x)$ fulfils the condition* (B) *on Q, provided that it is* AD *to zero at almost every point of Q.*

PROOF. *re* (i) and (ii): The absolute continuity of the function $\lambda(x)$ is the assertion of Theorem 15 of [4]; the absolute convergence of $\sum\varphi(H)$ is incidentally shown in the proof of the mentioned theorem.

re (iii): Let I be the closed interval spanned by the set Q. The function $\lambda(x)$, which is AC on I, is derivable almost everywhere on I. For definiteness, let us set $\lambda'(x)=0$ for each point $x\in\boldsymbol{R}$ at which $\lambda(x)$ is not derivable. Since $\lambda(x)=\varphi(x)$ on Q and since $\varphi(x)$ is AD to 0 at almost all points of Q, Lemma 3 shows that $\lambda(x)$ is derivable to 0 at almost all points of Q. It follows that

$$\varphi(I)=\lambda(I)=\int_I\lambda'(x)\,dx=\sum_H\int_H\lambda'(x)\,dx+\int_Q\lambda'(x)\,dx=\sum_H\lambda(H)=\sum_H\varphi(H).$$

This being so, let A be any closed interval such that the set Q contains the end points of A without containing the whole interval A. Writing $R=A\cap Q$, we find at once that R is a compact nonconnected set and that the open intervals contiguous to R are precisely those intervals H which are contained in the interval A. On the other hand, the function $\varphi(x)$ is AC on Q and AD to zero at almost all points of Q, and we may replace here the set Q by its subset R. By what was already established, it follows that $\sum_{H\subset A}\varphi(H)=\varphi(A)$, which completes the proof.

LEMMA 4. *A function which is* GAC *on a set necessarily fulfils the condition* (N) *of Luzin on this set* (see Saks [5], p. 225).

LEMMA 5. *Let $F(x)$ be a function which is continuous on a closed interval I and let $\xi(y)$ denote for each $y\in\boldsymbol{R}$ the number* (*finite or infinite*) *of the interior points of I at which the function $F(x)$ assumes the value y. Then the function $\xi(y)$ is Borel measurable on $\boldsymbol{R}$ and, denoting by* $\mathrm{W}(F;I)$ *the absolute variation of $F(x)$ over I, we have the relation* (see Saks [5], p. 280)

$$\int_{\boldsymbol{R}}\xi(y)\,dy=\mathrm{W}(F;I).$$

THEOREM 5. *Given a compact nonconnected set Q, the following four conditions on a function $\varphi(x)$ are equivalent to one another:*

(i) *The function fulfils the condition* (B) *on the set Q.*

(ii) *The function is both* AC *and Dirichlet continuous, on Q.*

(iii) *The function is* AC *on Q and* AD *to zero at almost all points of Q.*

(iv) *The function is* AC *on Q and the image $\varphi[Q]$ is a null set.*

PROOF. It is obvious by Theorem 3 and Theorem 4 that the conditions (i), (ii), and (iii) are equivalent. We shall show in what follows that the same is true of (iii) and (iv).

(a) Condition (iii) implies condition (iv). To prove this, let us consider the set M of all the points of Q at which the function $\lambda(x)$ of Theorem 4 is derivable to 0, so that $|Q\setminus M|=0$ as in the proof of that theorem. It is known that *if a function $F(x)$ is derivable at every point of a measurable set S, then*

$$|F[S]|\leqq\int_S|F'(x)|\,dx \qquad \text{(see Saks [5], p. 227).}$$

We therefore have $|\lambda[M]|=0$. On the other hand, the function $\lambda(x)$, which is AC on Q, fulfils the condition (N) on Q by Lemma 4. But $|Q\setminus M|=0$ as mentioned above, and it follows that $|\lambda[Q\setminus M]|=0$. Since we have $\varphi[Q]=\lambda[Q]=\lambda[M]\cup\lambda[Q\setminus M]$, we conclude that $|\varphi[Q]|=0$.

(b) Condition (iv) implies condition (iii). Let I be the closed interval spanned by the set Q and let $\lambda(x)$ be the same function as considered above. If D is an open interval contained in I, we shall denote for each $y\in\boldsymbol{R}$ by $N(y;D)$ the number (finite or infinite) of the points of D at which the function $\lambda(x)$ assumes the value y. We find by Lemma 5 that $N(y;D)$, as function of y, is Borel measurable on $\boldsymbol{R}$ and that

$$\int_{\boldsymbol{R}}N(y;D)\,dy=\mathrm{W}(\lambda;\overline{D}),$$

where $\overline{D}$ is the closure of D. But $\lambda(x)$ is AC on I, and hence

$$\mathrm{W}(\lambda;\overline{D})=\int_D|\lambda'(x)|\,dx\,,$$

where we set $\lambda'(x)=0$ for each $x\in\boldsymbol{R}$ at which $\lambda(x)$ is not derivable.

When D is especially the interior I° of I, we shall write simply $N(y)$ for $N(y;I^\circ)$.

Given an open set $G\subset\boldsymbol{R}$, let $\lambda^{-1}[G]$ be the inverse image of G under the mapping $\lambda(x)$. If we write for short $S=\lambda^{-1}[G]\cap I^\circ$, then S is an open set contained in I. Suppose that S is nonvoid and let us arrange all the component open intervals of S in a sequence $\langle D_1, D_2, \cdots\rangle$. We then plainly have

$$N(y)=\sum_n N(y;D_n) \qquad \text{for } y\in G .$$

Noting that $N(y;D_n)=0$ unless $y\in G$, we find that

$$\int_G N(y)\,dy=\sum_n \int_G N(y;D_n)\,dy=\sum_n \int_R N(y;D_n)\,dy$$
$$=\sum_n \int_{D_n} |\lambda'(x)|\,dx=\int_S |\lambda'(x)|\,dx .$$

This being so, we go on to show that $\lambda'(x)=0$ at almost all points of the set Q. We may evidently assume that Q is an infinite set. Let G be any open set containing the null set $\lambda[Q]=\varphi[Q]$ and let us write $S=\lambda^{-1}[G]\cap I^\circ$ as above. Then S is a nonvoid open set containing the intersection $T=Q\cap I^\circ$ and we have

$$\int_Q |\lambda'(x)|\,dx=\int_T |\lambda'(x)|\,dx\leqq\int_S |\lambda'(x)|\,dx=\int_G N(y)\,dy .$$

But the function $N(y)$ is summable over $\boldsymbol{R}$ on account of $\mathrm{W}(\lambda; I)<+\infty$. Consequently, given any $\varepsilon>0$ there is a $\delta>0$ such that

$$\int_G N(y)\,dy<\varepsilon \qquad \text{whenever } |G|<\delta ,$$

where G has the same meaning as above. Since $|\lambda[Q]|=0$, we can choose G so as to satisfy $|G|<\delta$. We thus have

$$\int_Q |\lambda'(x)|\,dx<\varepsilon \qquad \text{for every } \varepsilon>0 .$$

It follows that this last integral vanishes and that, therefore, $\lambda'(x)=0$ at almost all points of Q. By Lemma 3, the function $\varphi(x)$, which coincides with $\lambda(x)$ on Q, must then be AD to zero at almost all points of Q. This completes the proof.

REMARK. We do not know whether the following assertion is true: *If a function $\varphi(x)$ is Dirichlet continuous on a compact nonconnected set Q, then $|\varphi[Q]|=0$ and the function is* AD *to* 0 *at almost all points of Q.*

THEOREM 6. *If a function $\varphi(x)$ which is Dirichlet continuous on a compact nonconnected set Q, is* GBV *on this set, then the function is* GAC *on Q.*

PROOF. By hypothesis the set Q is expressible as the union of a sequence of closed subsets on each of which the function $\varphi(x)$ is BV. Let E be any one of these subsets. It is enough to verify that the function is AC on this set.

We may clearly suppose E to be an infinite set. If E if an interval, then $\varphi(x)$ is a constant on E, and hence it is AC on E. Otherwise E is a compact nonconnected set, and by hypothesis the function fulfils the Dirichlet condition on E. Then the function must be AC on E in virtue of Theorem 3.

REMARK. The contents of this section will be used subsequently only in the case in which the underlying set Q is null. However, we thought it better not to restrict the basic notions too narrowly from the beginning.

§ 2. Powerwise continuous functions.

A fnnction will be said to fulfil the *condition* (P) on a linear set E, if either the function is AC on E, or else if there exists a CT null set which contains E and on which the function is Dirichlet continuous. When this is the case, the function is necessarily continuous on E. Further, *the multiple, by any constant, of a function which fulfils the condition* (P) *on a set, itself does so on this set.*

LEMMA 6. *A function which is continuous on a set M and which is* AC *on a subset of M everywhere dense in M, is necessarily* AC *on the whole set M* (see Saks [5], p. 224).

THEOREM 7. *A function which fulfils the condition* (P) *on a set E, does so also on every subset of E. If, in particular, this function is continuous on the closure of E, then the function fulfils the condition* (P) *on the whole of this closure.*

PROOF. The first half of the assertion is evident, while the second half is an immediate consequence of the preceding lemma.

THEOREM 8. *The sum, or more generally, any linear combination with constant coefficients, of two function $\varphi(x)$ and $\psi(x)$ which fulfil the condition* (P) *on a closed set E, itself does so on this set.*

PROOF. It suffices to deal with the case of the sum $\xi(x)=\varphi(x)+\psi(x)$. Let us suppose first that one at least, say $\varphi(x)$, of the two functions $\varphi(x)$ and $\psi(x)$ is AC on E. We distinguish two cases according as $\psi(x)$ is also AC on E, or not. In the former case, the sum $\xi(x)$ is plainly AC on E, and hence fulfils the condition (P) on E. In the latter case, the set E is infinite and there is a CT null set M which contains E and on which the function $\psi(x)$ is Dirichlet continuous. It follows that E, which is closed by hypothesis, is also a CT null set and that $\psi(x)$ is Dirichlet continuous on E. But the function $\varphi(x)$, being AC on E, must be Dirichlet continuous on E on account of Theorem 5, parts (ii) and (iii). Then $\xi(x)$ fulfils the condition (P) on E, since it is Dirichlet continuous on E together with $\varphi(x)$ and $\psi(x)$.

It remains to treat the case in which none of $\varphi(x)$ and $\psi(x)$ is AC on E. By what was shown in the above, the function $\psi(x)$ is Dirichlet continuous on E which must now be a CT null set. The same is of course true of $\varphi(x)$, and hence of $\xi(x)$ also. Thus $\xi(x)$ fulfils the condition (P) on E. This completes the proof.

A function will be called *powerwise continuous* on a linear set E, if it is continuous on E and if this set is expressible as the union of a sequence (finite or enumerable) of sets E_n on each of which the function fulfils the condition (P). Plainly, *a function which is* GAC *on a set is necessarily powerwise continuous on this set.* Again, *a function which is powerwise continuous on a set, is so also on every subset of this set,* as we see at once from Theorem 7.

THEOREM 9. *Every function which is powerwise continuous on a measurable set E is* AD *at almost all points of this set.*

PROOF. By definition of the powerwise continuity, the set E contains a subset on which the function is GAC and which consists of almost all points of E. But *any function which is* GAC *on a measurable set is* AD *at almost all points of this set* (see Saks [5], p. 223). Hence the result.

THEOREM 10. *Any linear combination, with constant coefficients, of two functions $\varphi(x)$ and $\psi(x)$ which are powerwise continuous on a closed set E, is itself powerwise continuous on this set.*

PROOF. By definition of the powerwise continuity, the set E is the union of a sequence of sets A_i on each of which the function $\varphi(x)$ fulfils the condition (P). Since E is a closed set, these sets A_i may be supposed closed, on account of Theorem 7. Similarly we can express E as the union of a sequence of closed sets B_j on each of which the function $\psi(x)$ fulfils the condition (P). Then, by Theorem 7 and Theorem 8, each linear combination of $\varphi(x)$ and $\psi(x)$ fulfils the condition (P) on all the intersections $A_i \cap B_j$, which are closed sets. But E is evidently the union of these intersections, and this completes the proof.

THEOREM 11. *In order that a function which is continuous on a nonvoid closed set E, be powerwise continuous on E, it is necessary and sufficient that each nonvoid closed subset of E contain a portion on which the function fulfils the condition* (P).

REMARK. This may be established in the same way as for Theorem (9.1) of Saks [5], p. 233. We shall, however, give a full account of the proof, since this theorem is important to our theory.

PROOF. (i) Necessity. Let $\varphi(x)$ be a function which is powerwise continuous on E. The set E is the union of a sequence of sets E_n on each of which $\varphi(x)$ fulfils the condition (P). By the continuity of $\varphi(x)$ on E and by Thorem 7, these sets E_n may be supposed closed. Then, by Baire's Theorem, every nonvoid closed subset of E has a portion contained wholly in one of the sets E_n. The function $\varphi(x)$, which fulfils the condition (P) on each of these sets, certainly does so also on this portion.

(ii) Sufficiency. Suppose that a function $\varphi(x)$ which is continuous on a nonvoid closed set E, fulfils the condition of the theorem. Let $\langle I_1, I_2, \cdots\rangle$ be the sequence of all the open intervals I with rational end points and such that $\varphi(x)$ is powerwise continuous on the intersection $E \cap I$. By hypothesis, there certainly exist such intervals I. Let S be the union of all the sets $E \cap I_n$ and write $K = E \setminus S$. The function $\varphi(x)$ is plainly powerwise continuous on S, and we need only show that the set K is void.

Suppose, if possible, that K is nonvoid. Since K is clearly closed, there exists, by hypothesis, an open interval J such that $K \cap J$ is nonvoid and that the function $\varphi(x)$ fulfils the condition (P) on $K \cap J$. We may evidently assume that the end points of the interval J are rational. Thus $\varphi(x)$, which is powerwise continuous on the set S, is so also on the set $E \cap J \subset (K \cap J) \cup S$. This requires J to be one of the intervals I_n and we have a contradiction, since the set K, by definition, is disjoint with each of the intervals I_n. This completes the proof.

LEMMA 7. *If a function which is both continuous and* BV *on a compact set, fulfils the condition* (N) *on this set, then the function is necessarily* AC *on the same set* (see Saks [5], p. 227).

THEOREM 12. *If two functions are powerwise continuous on a closed interval I and if they are* AED *at almost all points of I, then the functions differ on I only by an additive constant.*

This follows directly from the following theorem.

THEOREM 13. *Given two functions $\varphi(x)$ and $\psi(x)$ powerwise continuous on a closed interval I, if $\varphi'_{\mathrm{ap}}(x) \leqq \psi'_{\mathrm{ap}}(x)$ at almost every point x of I at which both the functions are* AD, *then the difference $\psi(x)-\varphi(x)$ is both* AC *and nondecreasing, on the interval I.*

PROOF. The difierence $\xi(x)=\psi(x)-\varphi(x)$ is powerwise continuous on I by Theorem 10. Since $-\infty<\varphi'_{\mathrm{ap}}(x) \leqq \psi'_{\mathrm{ap}}(x)<+\infty$ almost everywhere on I, the function $\xi(x)$ has a finite nonnegative approximate derivative at almost every point of I. From these two facts it will be deduced, in what follows, that $\xi(x)$ is AC and nondecreasing over I.

Let us define a subset S of I as follows: a point of I belongs to S if and only if there exists no open interval G containing this point and such that the function $\xi(x)$ is GAC on the interval $I \cap G$. We find at once that the set S is closed.

We shall begin by showing that if a finite interval D (closed, open, or half open) contained in I is disjoint with the set S, then $\xi(x)$ is GAC and nondecreasing on D. For this purpose, let K be any closed interval contained in the interior of D. It is enough to verify that $\xi(x)$ is GAC and nondecreasing on K. Let us enclose, as we plainly can, each point of K in an open interval contained in D and on which $\xi(x)$ is GAC. Then K is covered by a finite number of such intervals, and it follows at once that $\xi(x)$ is GAC on the whole interval K. But we know that *if the approximate derivative of a function which is* GAC *on an interval is nonnegative almost everywhere on this interval, then the function is nondecreasing over the same interval* (see Saks [5], p. 225). Hence $\xi(x)$ is not only GAC, but also nondecreasing, on the interval K, as required.

This result implies that the closed set S is perfect. To see this, let us assume, to the contrary, that S contains an isolated point c. If c is an interior point of I, there are two points a, b of I such that $a<c<b$ and that the open intervals (a, c) and (c, b) are disjoint with S. Then $\xi(x)$ is GAC on each of these two intervals, and it follows, in view of the continuity of $\xi(x)$ on I, that $\xi(x)$ is GAC on the whole interval (a, b). But this contradicts the fact that c is a point of S. If, on the other hand, c is an end point of I, we likewise arrive at a contradiction by an argument similar to the above.

If the set S is void, the function $\xi(x)$ is GAC and nondecreasing on the whole interval I, by what was already proved. It then follows from Lemma 4 and Lemma 7 that $\xi(x)$ is AC on I. The proof therefore reduces to obtaining a contradiction from the assumption that S is nonvoid.

Suppose therefore, if possible, that S is nonvoid. There exists, by Theorem 11, an open interval H such that the intersection $M=S\cap H$ is nonvoid and that $\xi(x)$ fulfils the condition (P) on the set M. Then, either $\xi(x)$ is AC on M, or else there is a CT null set Q which contains M and on which $\xi(x)$ is Dirichlet continuous. We shall deal with these alternatives separately. We observe in passing that M is an infinite set; for otherwise each point of M would be an isolated point of S.

Supposing that $\xi(x)$ is AC on M, let U be the union of all the open intervals that are contiguous to the set M and let us write $T=U\cup M$. We find without difficulty that T is a finite interval contained in I and that $\xi(x)$ is GAC on T. This contradicts the definition of the set S, since M is an infinite subset of T and hence intersects the interior of T.

We pass on to the remaining case. Let Q be a CT null set which contains M and on which $\xi(x)$ is Dirichlet continuous. Then the closure $\overline{M}$ is a CT null set contained in Q and hence $\xi(x)$ is Dirichlet continuous on $\overline{M}$. Since $M=S\cap H$, where H is an open interval, $\overline{M}$ is contained in S and each open interval G contiguous to $\overline{M}$ is at the same time contiguous to S. Hence $\xi(x)$ is GAC and nondecreasing on each G. This, together with the continuity of $\xi(x)$ on I, shows that $\xi(x)$ is further nondecreasing on $\overline{G}$ and that, in particular, $\xi(G)\geqq 0$. Consequently $\xi(x)$, which we saw to be Dirichlet continuous on $\overline{M}$, is nondecreasing on $\overline{M}$. We then find by Theorem 3 that $\xi(x)$ is AC on $\overline{M}$. It thus follows that the function $\xi(x)$ is GAC on the closed interval spanned by $\overline{M}$. This contradicts the definition of the set S, since $\overline{M}$ is an infinite subset of S.

THEOREM 14. *If a function $\varphi(x)$ which is powerwise continuous on a closed set E, is* GBV *on this set, then the function is necessarily* GAC *on E. If, in particular, the set E is compact and the function is* BV *on E, then it is* AC *on E.*

PROOF. To prove the first half of the theorem, we express the set E as the union of a sequence of sets M on each of which the function $\varphi(x)$ fulfils the condition (P). In view of Theorem 7, we may assume the sets M closed. It suffices to verify that the function is GAC on each M.

Suppose, if possible, that $\varphi(x)$ is non-GAC on some M. Then M must be an infinite set. Moreover, by definition of the condition (P), M is contained in a CT null set on which $\varphi(x)$ is Dirichlet continuous. It follows that M, which is originally closed, is itself a CT null set and that $\varphi(x)$ is Dirichlet continuous on M. But $\varphi(x)$ is GBV on M, since $M\subset E$. Theorem 6 then shows that $\varphi(x)$ is GAC on M, which is a contradiction.

The second half of the theorem follows directly from Lemma 4, Lemma 7, and the first half.

REMARK. We do not know whether the following assertion is true: *A function which is Dirichlet continuous on a compact nonconnected set Q, is necessarily powerwise continuous on this set.* This is obvious if the set Q is of measure zero.

§ 3. Powerwise integration.

We shall define the powerwise integration in two ways, descriptive and constructive. Let us begin with the descriptive definition.

A function $f(x)$ will be termed *powerwise integrable* on a closed interval I, if there exists a function $\varphi(x)$ which is powerwise continuous on I and which has $f(x)$ for its approximate derivative at almost all points of I. The function $\varphi(x)$ is then called *indefinite powerwise integral* of $f(x)$ on I. By the *definite powerwise integral* of $f(x)$ over I we shall mean the increment $\varphi(I)$ of its indefinite integral over the interval I. This definite integral will be denoted by $\mathrm{P}(f; I)$.

It is known that *the extreme approximate derivates of any finite measurable function of one real variable are themselves measurable functions* (see Saks [5], p. 299). Consequently, *any function $f(x)$ which is powerwise integrable on a closed interval I is necessarily measurable on I.* Furthermore, *the powerwise integral over I of such a function $f(x)$ is uniquely determined,* since its indefinite integral is determined on I except for an additive constant on account of Theorem 12. More generally, *if two functions coincide almost everywhere on a closed interval I and if the one is powerwise integrable on I, then so is the other and the two functions have the same definite integral over I.*

The following three theorems are easily derived from the properties, established in the preceding section, of powerwise continuous functions.

THEOREM 15. *Every function $f(x)$ which is Denjoy integrable on a closed interval I is powerwise integrable on I and the powerwise integral $\mathrm{P}(f;I)$ coincides with the Denjoy integral $\mathrm{D}(f;I)$.*

THEOREM 16. *Any function $f(x)$ which is powerwise integrable on a closed interval I is necessarily so also on every closed subinterval J of I, and its definite integral $\mathrm{P}(f; J)$, regarded as function of J, is an additive interval function on I.*

THEOREM 17. *If two functions $f(x)$ and $g(x)$ are powerwise integrable on a closed interval I, the same is true of any linear combination*

$af(x)+bg(x)$ *of these functions, and we have*

$$\mathrm{P}(af+bg;I)=a\mathrm{P}(f;I)+b\mathrm{P}(g;I);$$

in brief, the definite powerwise integral is a linear functional of the integrand function.

THEOREM 18. *If a function $f(x)$ which is powerwise integrable on a closed interval I is almost everywhere nonnegative on I, then the function is necessarily summable over I.*

PROOF. Let $\varphi(x)$ be an indefinite integral of $f(x)$ on I, so that $\varphi(x)$ is powerwise continuous on I and has $f(x)$ for its approximate derivative almost everywhere on I. It follows from Theorem 13 that the function $\varphi(x)$ is AC on I. Consequently, $\varphi(x)$ is derivable almost everywhere on I and its derivative $\varphi'(x)$ is summable over I, it being agreed that we set $\varphi'(x)=0$ for each $x\in\boldsymbol{R}$ at which $\varphi(x)$ is not derivable. But $f(x)$ equals $\varphi'(x)$ almost everywhere on I, and hence $f(x)$ is summable over I.

THEOREM 19. *If a function $f(x)$ is powerwise integrable on a closed interval I, then every nonvoid closed subset of I contains a portion for which we have the following alternatives:*

either (i) *the function $f(x)$ is summable on the closure Q of this portion and the series of the definite powerwise integrals of $f(x)$ over the closed intervals (if existent) contiguous to Q is absolutely convergent;*

or else (ii) *the closure Q is a* CT *null set and each indefinite powerwise integral of $f(x)$ on the interval I is Dirichlet continuous on Q.*

PROOF. Let $\varphi(x)$ be an indefinite powerwise integral of $f(x)$ on the interval I, i. e. let $\varphi(x)$ be a function which is powerwise continuous on I and which has $f(x)$ for its approximate derivative almost everywhere on I. By Theorem 11, each nonvoid closed subset of I contains a portion on which the function $\varphi(x)$ fulfils the condition (P). It follows from Theorem 7 that $\varphi(x)$ fulfils the same condition on the closure Q of this portion.

Suppose first that $\varphi(x)$ is AC on Q. If Q is a closed interval or a singletonic set, then $\varphi(x)$ must have $f(x)$ for its derivative almost everywhere on Q, and it follows that the function $f(x)$ is summable over Q. If, on the other hand, the closure Q is a compact nonconnected set, then writing K for a generic closed interval contiguous to Q, we find by part (ii) of Theorem 4 that $\sum|\mathrm{P}(f;K)|=\sum|\varphi(K)|<+\infty$. Moreover, if we denote by $\lambda(x)$ the linear modification of $\varphi(x)$ with respect to Q, then part (i) of Theorem 4 shows that $\lambda(x)$ is AC on the interval spanned by Q. On the other hand, since $\varphi(x)$ and $\lambda(x)$ coincide on Q, we find by Lemma

3 that these two functions are AED at almost all points of Q. It thus follows that $f(x)$ is the derivative of $\lambda(x)$ at almost all points of Q. We conclude that $f(x)$ is summable over Q.

It remains to consider the case in which the function $\varphi(x)$ is not AC on Q, so that Q is an infinite set. There exists a CT null set which contains Q and on which $\varphi(x)$ is Dirichlet continuous. Then $\varphi(x)$ must be so also on Q which is itself a CT null set. This completes the proof, since each indefinite powerwise integral of $f(x)$ on the interval I differs on I from $\varphi(x)$ only by an additive constant.

We shall now proceed to the constructive definition of the powerwise integration. Our treatment of this subject will be modelled after that of Saks [5], pp. 254–259.

The general notion of integration stated in Saks [5] reads essentially as follows, where an *interval*, by itself, means a finite closed one, the void set and the singletonic sets not being counted among intervals.

Let $\mathfrak{T}$ be a functional operation by which there corresponds to each interval $I=[a, b]$ a nonvoid class of functions, and to each function $f(x)$ of this class a finite real number depending on both the function and the interval I. This class of functions will be written $\Delta(\mathfrak{T}; I)$ and called $\mathfrak{T}$-*domain on the interval* I, while the number associated with $f(x)$ will be denoted by $\mathfrak{T}(f; I)$. As it stands to reason, we require that given two functions $f(x)$ and $g(x)$ coinciding on an interval I, if one of them belongs to $\Delta(\mathfrak{T}; I)$, then so does the other also and we have $\mathfrak{T}(f; I)=\mathfrak{T}(g; I)$.

An operation $\mathfrak{T}$ of the above kind will be termed an *integration*, if the following three conditions are fulfilled:

(i) If a function $f(x)$ belongs to the $\mathfrak{T}$-domain on an interval I_0, the function belongs also to the $\mathfrak{T}$-domain on any interval $I\subset I_0$, and $\mathfrak{T}(f; I)$ is a continuous additive function of the interval $I\subset I_0$.

(ii) If a function belongs to the $\mathfrak{T}$-domain on each of two abutting intervals I_1 and I_2, the function belongs also to $\Delta(\mathfrak{T}; I_1\cup I_2)$.

(iii) A function $f(x)$ which vanishes identically on an interval I belongs to $\Delta(\mathfrak{T}; I)$, and we have $\mathfrak{T}(f; I)=0$.

Saks then introduces the notions of $\mathfrak{T}$-integrability and $\mathfrak{T}$-integral on an interval, as well as the corresponding notions on a set, as follows:

(a) If $\mathfrak{T}$ is an integration, a function $f(x)$ which belongs to the $\mathfrak{T}$-domain on an interval I will be called $\mathfrak{T}$-*integrable* on I and the number $\mathfrak{T}(f; I)$ will be termed *definite* $\mathfrak{T}$-*integral* of the function $f(x)$ on I.

(b) We shall say that a function $f(x)$ is $\mathfrak{T}$-*integrable* on a bounded set E, if the function $g(x)$ which coincides with $f(x)$ on E and vanishes outside E, is $\mathfrak{T}$-integrable on every interval $I \supset E$. The number $\mathfrak{T}(g; I)$ is then independent of the choice of the interval $I \supset E$; we shall call this number *definite $\mathfrak{T}$-integral* of the function $f(x)$ on the set E and we shall denote it by $\mathfrak{T}(f; E)$.

The two definitions (a) and (b) ought, needless to say, to be compatible in the special case in which the set E is an interval. Seeing that Saks mentions nothing as to this compatibility, it seems that he has no doubts about it. The fact of the matter is that the two definitions are inconsistent, as will be manifested by the following simple example.

EXAMPLE. Let us define a functional operation $\mathfrak{T}$ as follows. The $\mathfrak{T}$-domain on an interval I consists of all those functions each of which is a constant over I. If a function $f(x)$ belongs to $\Delta(\mathfrak{T}; I)$, we mean by $\mathfrak{T}(f; I)$ the product $c|I|$, where c is the value of $f(x)$ on I.

It is readily seen that this operation $\mathfrak{T}$ is an integration in the Saks sense. However, the function which is identically equal to 1 on the real line, is $\mathfrak{T}$-integrable on each interval according to definition (a), without being so also according to (b).

This defect of the Saks notion of general integration can be saved if we replace the above condition (iii) with the following one which is somewhat stronger, and we agree to adopt this replacement.

(iv) A function $f(x)$ which vanishes identically on the interior of an interval I belongs to $\Delta(\mathfrak{T}; I)$ and we have $\mathfrak{T}(f; I) = 0$.

As to general integration and related definitions, we shall conform to the Saks treatment, excepting the one amendment put forward just now. In order to prevent any misunderstanding, we observe here again that *an interval, by itself, will always mean a closed interval,* so far as we are concerned with the constructive definition of the powerwise integration. This last agreement will not apply to § 4, neither to § 5.

We find easily that the powerwise integration fulfils the conditions (i), (ii), (iv) and hence is a general integration.

Given an integration $\mathfrak{T}$, a function $f(x)$, and an interval I, we shall say after Saks that a point c of I is a *$\mathfrak{T}$-singular point* of $f(x)$ in I, if there exist in I arbitrarily small intervals containing c and on each of which the function is not $\mathfrak{T}$-integrable. We find at once that the set S of all such points c is closed and that the function $f(x)$ is $\mathfrak{T}$-integrable on every interval contained in the set $I \setminus S$.

With each integration $\mathfrak{T}$ we now associate three "generalized" integrations $\mathfrak{T}^{C}$, $\mathfrak{T}^{H}$, and $\mathfrak{T}^{P}$. The respective definition of $\mathfrak{T}^{C}$ and of $\mathfrak{T}^{H}$ is the same as stated on p. 255 of Saks [5], and $\mathfrak{T}^{P}$ is defined as follows.

Given any interval I, the $\mathfrak{T}^{P}$-domain on I is the class of all the functions $f(x)$ which fulfil the following two conditions:

(p^1) The set S of all $\mathfrak{T}$-singular points of $f(x)$ in I is of measure zero, and the function $f(x)$ is $\mathfrak{T}$-integrable on each of the intervals I_k contiguous to the compact set Q consisting of the points of S and of the end points of I.

(p^2) There is a function $\varphi(x)$ which is continuous over I, Dirichlet continuous on the set Q, and further such that $\varphi(K)=\mathfrak{T}(f; K)$ whenever K is a subinterval of any one of the above intervals I_k. (It is evident that the function $\varphi(x)$ is uniquely determined over the interval I to within an additive constant.)

For any such function $f(x)$, we define $\mathfrak{T}^{P}(f;I)$ to mean $\varphi(I)$.

As we verify without especial difficulty, the operations $\mathfrak{T}^{C}$, $\mathfrak{T}^{H}$, and $\mathfrak{T}^{P}$ all fulfil the conditions (i), (ii), (iv) for an operation to be an integration, though the verification is somewhat toilful. These operations are therefore integrations, and each of them plainly includes the integration $\mathfrak{T}$.

The powerwise integration, when regarded as a general integration, will be denoted by $\mathfrak{P}$. To denote the definite powerwise integral, however, we shall write $\mathrm{P}(f;I)$ as before, instead of writing $\mathfrak{P}(f;I)$. We see at once that for each integration $\mathfrak{T}\subset\mathfrak{P}$, we have also $\mathfrak{T}^{C}\subset\mathfrak{P}$ and $\mathfrak{T}^{P}\subset\mathfrak{P}$. It is not quite so obvious that the relation $\mathfrak{T}\subset\mathfrak{P}$ implies $\mathfrak{T}^{H}\subset\mathfrak{P}$. This last assertion is a consequence of the following theorem.

THEOREM 20. *Given a compact nonconnected set Q, write $\langle I_k\rangle$ for the sequence of the intervals contiguous to the set Q; and suppose that $f(x)$ is a function powerwise integrable on Q as well as on each of the intervals I_k, and that (in the case in which the sequence $\langle I_k\rangle$ is infinite)*

$$\sum_k |\mathrm{P}(f;I_k)|<+\infty \quad \textit{and} \quad \lim_k \mathrm{O}(\mathfrak{P};f;I_k)=0\,.$$

Then the function $f(x)$ is powerwise integrable on the interval I spanned by the set Q and we have

$$\mathrm{P}(f;I)=\mathrm{P}(f;Q)+\sum_k \mathrm{P}(f;I_k)\,.$$

PROOF. By Theorem 2 there exists a function $\theta(x)$ which is AC on Q, derivable to zero at almost all points of Q, subject to the condition (B) on Q, and such that $\theta(I_k)=\mathrm{P}(f;I_k)$ for every interval I_k.

Altering the values of the function $\theta(x)$ suitably on the set $D=I\setminus Q$, we now construct a function $\varphi(x)$ coinciding with $\theta(x)$ on the set Q and such that $\varphi(K)=\mathrm{P}(f;K)$ whenever K is a subinterval of any one of the intervals I_k. This function $\varphi(x)$ is AC on Q and powerwise continuous on every interval I_k. The hypothesis $\lim \mathrm{O}(\mathfrak{P};f;I_k)=0$ ensures further the continuity of $\varphi(x)$ on I. We thus find that $\varphi(x)$ is powerwise continuous on the whole interval I.

Now Lemma 3 shows that the function $\varphi(x)$ is AD to zero at almost all points of Q; on the other hand, we have $\varphi'_{\mathrm{ap}}(x)=f(x)$ at almost all points of the set $I\setminus Q$. Hence, noting the powerwise continuity of $\varphi(x)$ on I, we see that the function equal to $f(x)$ on $\boldsymbol{R}\setminus Q$ and to 0 on Q has $\varphi(x)$ for an indefinite powerwise integral on I. Again, the function equal to $f(x)$ on Q and to 0 on $\boldsymbol{R}\setminus Q$ is, by hypothesis, powerwise integrable on I. It ensues that the function $f(x)$ is itself powerwise integrable on I and that $\mathrm{P}(f;I)=\varphi(I)+\mathrm{P}(f;Q)$, where we have

$$\varphi(I)=\theta(I)=\sum_k \theta(I_k)=\sum_k \mathrm{P}(f;I_k)\,,$$

since the function $\theta(x)$ fulfils the condition (B) on Q. This completes the proof.

We are now in a position to state the constructive definition of the powerwise integration. Let $\langle\mathfrak{L}^\xi\rangle$ be a transfinite sequence of integrations which is defined, by an induction starting with the Lebesgue integration $\mathfrak{L}$, as follows:

$$\mathfrak{L}^0=\mathfrak{L}\,, \qquad \mathfrak{L}^\rho=\Big(\bigcup_{\xi<\rho}\mathfrak{L}^\xi\Big)^{\mathrm{CHP}} \qquad \text{for } \rho>0\,.$$

Writing Ω for the smallest ordinal of the third class, we shall show that

$$\mathfrak{P}=\mathfrak{L}^\Omega=\mathfrak{M}\,, \qquad \text{where } \mathfrak{M}=\bigcup_{\xi<\Omega}\mathfrak{L}^\xi\,.$$

Since $\mathfrak{L}\subset\mathfrak{P}$, we find at once by induction that $\mathfrak{L}^\xi\subset\mathfrak{P}$ for every ordinal number ξ and that especially $\mathfrak{L}^\Omega\subset\mathfrak{P}$. Since $\mathfrak{M}\subset\mathfrak{L}^\Omega$, the assertion will follow if we show that $\mathfrak{P}\subset\mathfrak{M}$, or in other words, that each function $f(x)$ which is powerwise integrable on an interval I is $\mathfrak{L}^\xi$-integrable on I for some index $\xi<\Omega$. Using Theorem 19, we can prove this in almost the same way as on pp. 258–259 of Saks [5]. The details may be omitted.

§ 4. Existence of functions which are powerwise integrable without being Denjoy integrable.

Let us begin by recalling the notion of *elementary figure,* or simply *figure,* defined on p. 58 of Saks [5]. We shall only consider linear figures in what follows. By a *component interval,* or briefly *component,* of a nonvoid elementary figure E, we mean any maximal closed interval contained in E. Plainly, E is the union of a finite number of components which are mutually disjoint.

We shall denote by $\boldsymbol{N}$ and $\boldsymbol{M}$ the set of the positive integers and that of the nonnegative integers, respectively.

Given a closed interval $I=[a, b]$ and given an integer $k \in \boldsymbol{M}$, we write

$$\lambda=\frac{b-a}{2k+1} \quad \text{and} \quad c_i=a+i\lambda\,, \qquad \text{where } i=0, 1, \cdots\cdot, 2k+1\,.$$

We denote by $I(k+1)$ the elementary figure whose components are the $k+1$ closed intervals $[c_{2j}, c_{2j+1}]$, where $j=0, 1, \cdots\cdot, k$. We shall call *ramification of size* $k+1$ the operation that makes correspond to I the figure $I(k+1)$. The components of $I(k+1)$ will be termed *successors* to I under this ramification. It may be observed that $I(1)$ coincides with I itself.

Now let E be a nonvoid figure with the components $A_1, \cdots\cdot, A_r$. We can, for each $n \in \boldsymbol{N}$, associate with E another figure, written $E(n)$ and defined by $E(n)=A_1(n)\cup\cdots\cdot\cup A_r(n)$. In other words, $E(n)$ is obtained from E by ramifying each component of E to size n. We agree that also this operation on E be called *ramification of size* n. When $m \in \boldsymbol{N}$ and $n \in \boldsymbol{N}$, the figure which is the result of ramifying the figure $E(m)$ to size n will be denoted by $E(m; n)$.

Given any closed interval I, we attach to I an infinite sequence of figures $\langle E_0, E_1, \cdots\cdot\rangle$ defined inductively as follows:

$$E_0=I\,, \qquad E_{k+1}=E_k(3\,; 3^{2k}) \qquad \text{for } k \in \boldsymbol{M}\,.$$

The sequence thus constructed, is clearly monotone descending. If we denote by N_k the number of the components of E_k, then we have $N_0=1$ and $N_{k+1}=3^{2k+1}N_k$. It follows that $N_k=3^{k^2}$ for each $k \in \boldsymbol{M}$. Furthermore, we find at once that $|E_{k+1}| \leqq (3/5)|E_k|$, whence we derive

$$|E_k| \leqq \left(\frac{3}{5}\right)^k |I| \qquad \text{for } k \in \boldsymbol{M}\,.$$

From the above sequence $\langle E_0, E_1, \cdots\cdot\rangle$ we obtain two sets Γ and Δ which we define as follows:

$$\Gamma=\Gamma_I=E_0 \cap E_1 \cap \cdots\cdot \quad \text{and} \quad \Delta=\Delta_I=I \setminus \Gamma\,.$$

It is obvious that Γ is a CT null set spanning the interval I and that Δ is an open set with measure equal to $|I|$. Since $E_0 \supset E_1 \supset \cdots\cdot$, we can express Δ also in the form $\Delta=\cup(E_{n-1}\setminus E_n)$, where $n \in \boldsymbol{N}$. From this expression we find easily that the closed intervals contiguous to Γ are mutually disjoint and all contained in the interior of I. Accordingly Γ is a closed set with no isolated points; in other wards, Γ is a perfect set.

Besides Γ and Δ, we consider the set of all the end points of the closed intervals contiguous to Γ and we denote this countable set by Θ or Θ_I. We see at once that Γ is the closure of Θ.

This being so, let J be an arbitrary closed interval and let us write $J=[s, s+5l]$. We define a function $\Phi(x)=\Phi(x;J;h)$, where $h>0$, by the following three conditions.

(i) $\Phi(x)=0$ for $x\leqq s+l$ and for $x\geqq s+4l$,

(ii) $\Phi(x)=h$ for $x\in[s+2l, s+3l]$,

(iii) $\Phi(x)$ is linear on $[s+l, s+2l]$ and on $[s+3l, s+4l]$.

Thus defined, $\Phi(x)$ is a continuous function on $\boldsymbol{R}$ with values belonging to the interval $[0, h]$.

Let us return to the interval I and the sequence $\langle E_0, E_1, \cdots\cdot\rangle$. Given a positive number $\delta<1$, we define for each $m\in\boldsymbol{M}$ a function

$$\Phi_m(x)=\Phi_m(x;I;\delta) \quad \text{by} \quad \Phi_m(x)=\sum_K \Phi(x;K;\delta^{m^2}),$$

where K ranges over all the components of the figure E_m. This function is clearly continuous and we have $0\leqq\Phi_m(x)\leqq\delta^{m^2}\leqq\delta^m$ for all $x\in\boldsymbol{R}$. Consequently, writing

$$\Psi(x)=\Psi(x;I;\delta)=\sum_{m=0}^{\infty}\Phi_m(x;I;\delta),$$

we find that this series is uniformly convergent over $\boldsymbol{R}$ and that $\Psi(x)$ is therefore a continuous function. We have also $0\leqq\Psi(x)\leqq(1-\delta)^{-1}$, and $\Psi(x)$ is evidently linear on each closed interval contiguous to the set Γ.

Let $\langle a_0, a_1, \cdots\cdot\rangle$ be any infinite sequence such that $a_m\in\{0, 1\}$ for every $m\in\boldsymbol{M}$, where $\{0, 1\}$ is the set of the two numbers 0 and 1. We write $\mathfrak{A}$ for the class of all such sequences. For brevity we shall signify by an $\mathfrak{A}$-*sequence* any member of the class $\mathfrak{A}$.

LEMMA 8. *Given a closed interval I, let c be a fixed point of the set Γ_I. If we write $\psi(\delta)=\Psi(c;I;\delta)$, where $0<\delta<1$, then $\psi(\delta)$ is a continuous function of δ.*

PROOF. Since $\Gamma=E_0\cap E_1\cap\cdots\cdot$ by definition, the point c belongs, for each $m\in\boldsymbol{M}$, to a component interval, say I_m, of the fiigure E_m. The figure $I_m(3)$, which is obtained from I_m by ramification of size 3, is the union of three successors to I_m. Since $c\in E_{m+1}=E_m(3;3^{2m})$, we find that c belongs to one, say S_m, of these successors. We now define a_m to be 1 or 0 according as S_m is the middle successor or not, respectively. We have thus attached to c an $\mathfrak{A}$-sequence $\sigma=\sigma(c)=\langle a_m; m\in \boldsymbol{M}\rangle$.

This being so, let us recall the definition of the function $\Phi_m(x;I;\delta)$:

$$\Phi_m(x;I;\delta)=\sum_K \Phi(x;K;\delta^{m^2}) \qquad \text{for } x\in \boldsymbol{R},$$

where K ranges over the components of E_m. Now the function $\Phi(x;K;h)$, where $h>0$, vanishes by definition unless $x\in K$. Consequently, in view of the above choice of a_m, we have

$$\Phi_m(c;I;\delta)=\Phi(c;I_m;\delta^{m^2})=a_m\delta^{m^2}.$$

This, combined with the definition of $\Psi(x;I;\delta)$, namely

$$\Psi(x;I;\delta)=\sum_{m=0}^{\infty}\Phi_m(x;I;\delta),$$

leads to the expression $\psi(\delta)=\sum_{m=0}^{\infty}a_m\delta^{m^2}$, which clearly implies the asserted continuity of $\psi(\delta)$.

REMARK. The lemma holds true as well for the points c of the set $\Delta=I\setminus\Gamma$. We omit the proof, since we shall not require this fact later on.

Supposing as hitherto that $0<\delta<1$, we now define

$$W(\sigma)=W(\sigma;\delta)=\sum_{m=0}^{\infty}a_m\delta^m \qquad \text{for } \sigma=\langle a_0, a_1, \cdots\cdots\rangle\in\mathfrak{A},$$

and we denote by $S(\delta)$ the set of all the values $W(\sigma)$.

LEMMA 9. *If especially $0<\delta<2^{-1}$, the set $S(\delta)$ defined just now is of measure zero.*

PROOF. For each $m\in\boldsymbol{M}$ let us denote by $\mathfrak{A}_m$ the class of the $\mathfrak{A}$-sequences $\langle a_0, a_1, \cdots\cdots\rangle$ such that $a_i=0$ for all $i>m$. Each sequence σ of the class $\mathfrak{A}_m$ is of the form $\sigma=\langle a_0, \cdots\cdots, a_m, 0, 0, \cdots\cdots\rangle$, and hence $\mathfrak{A}_m$ consists of exactly 2^{m+1} sequences of $\mathfrak{A}$. If we write $\sigma'=\langle a_0, \cdots\cdots, a_m, 1, 1, \cdots\cdots\rangle$, then we have

$$W(\sigma')-W(\sigma)=\sum_{i>m}\delta^i=\lambda\delta^{m+1}, \qquad \text{where } \lambda=\frac{1}{1-\delta}.$$

Now let L_m be the figure which is the union of all the closed intervals $I_m(\sigma)=[W(\sigma), W(\sigma')]$, where σ ranges over the class $\mathfrak{A}_m$. Since $|I_m(\sigma)|=\lambda\delta^{m+1}$, we have

$$|L_m|\leqq\sum_\sigma|I_m(\sigma)|=\sum_\sigma\lambda\delta^{m+1}=\lambda(2\delta)^{m+1}.$$

But $2\delta<1$ by hypothesis, and hence $|L_m|\to 0$ as $m\to+\infty$. Consequently, writing $L(\delta)=L_0\cap L_1\cap\cdots\cdots$, we find that $L(\delta)$ is a null set. It therefore suffices to show that $S(\delta)\subset L(\delta)$.

Given any $\mathfrak{A}$-sequence $\tau=\langle a_i\,; i\in \boldsymbol{M}\rangle$, consider the sequences

$$\sigma_m=\langle a_0, \cdots, a_m, 0, 0, \cdots\rangle \quad\text{and}\quad \sigma'_m=\langle a_0, \cdots, a_m, 1, 1, \cdots\rangle\,,$$

where $m\in \boldsymbol{M}$. Then $\sigma_m\in\mathfrak{A}_m$ and we have

$$W(\sigma_m)\leqq W(\tau)\leqq W(\sigma'_m)\,, \qquad \text{whence } W(\tau)\in I_m(\sigma_m)\subset L_m\,.$$

It follows that $W(\tau)\in L(\delta)=L_0\cap L_1\cap\cdots$. Since the $\mathfrak{A}$-sequence τ is arbitrary, we conclude that $S(\delta)\subset L(\delta)$, which completes the proof.

REMARKS. (i) We can strengthen the inclusion $S(\delta)\subset L(\delta)$ to the equality $S(\delta)=L(\delta)$. The proof is not difficult. (ii) When $\delta=2^{-1}$, the lemma ceases to be valid, since $S(2^{-1})=[0, 2]$ as is readily seen. In the special case in which $\delta=3^{-1}$, we have

$$S(3^{-1})=\left\{\frac{3}{2}t\,; t\in C_0\right\}, \quad \text{or briefly } S(3^{-1})=\frac{3}{2}C_0\,,$$

where C_0 stands for the Cantor perfect set. This follows directly from

$$C_0=\{\sum_{m=0}^{\infty} 2a_m 3^{-m-1}\,; \langle a_0, a_1, \cdots\rangle\in\mathfrak{A}\}\,,$$

which is the well-known ternary scale expression of C_0.

LEMMA 10. *If* $0<\delta<2^{-1}$, *the function* $\Psi(x;I;\delta)$ *fulfils the condition* (N) *on the interval* I.

PROOF. As we saw in the proof of Lemma 8, there corresponds to each point c of the set $\Gamma=\Gamma_I$ an $\mathfrak{A}$-sequence $\sigma(c)=\langle a_0, a_1, \cdots\rangle$ such that

$$\Psi(c;I;\delta)=\sum_{m=0}^{\infty} a_m\delta^{m^2}\,.$$

From this sequence we derive another $\mathfrak{A}$-sequence $\tau(c)=\langle b_0, b_1, \cdots\rangle$, by determining b_k for each $k\in\boldsymbol{M}$ as follows:

(a) if k is a square number, then we set $b_k=a_m$, where $m=\sqrt{k}$;
(b) otherwise we set simply $b_k=0$.

We then have $\Psi(c;I;\delta)=\sum b_k\delta^k\in S(\delta)$. Since this is true for every $c\in\Gamma$, we find that $\Psi[\Gamma]\subset S(\delta)$. But $|S(\delta)|=0$ by Lemma 9, and hence $|\Psi[\Gamma]|=0$.

On the other hand, the function $\Psi(x;I;\delta)$, which is linear on each closed interval contiguous to Γ, fulfils the condition (N) on the set $\Delta=I\setminus\Gamma$. This, together with $|\Psi[\Gamma]|=0$, completes the proof.

Given a closed interval I and given a positive number $\delta<1$, we write

$$\Psi_m(x;I;\delta)=\sum_{k=0}^{m}\Phi_k(x;I;\delta) \qquad \text{for } m\in\boldsymbol{M}\,,$$

so that

$$\Psi(x;I;\delta)=\sum_{k=0}^{\infty}\Phi_k(x;I;\delta)=\lim_{m\to\infty}\Psi_m(x;I;\delta)\,.$$

We shall consider the absolute variation of the function $\Psi_m(x)=\Psi_m(x;I;\delta)$ over the interval I and the weak variation of the function $\Psi(x)=\Psi(x;I;\delta)$ over the set Γ_I. With the notation of Saks [5], p. 221, these quantities will be written $\mathrm{V}(\Psi_m;I)$ and $\mathrm{V}(\Psi;\Gamma_I)$, respectively, in what follows.

LEMMA 11. *Suppose that* $0<\delta\leqq 3^{-1}$ *and that* $m\in\boldsymbol{M}$. *If* $\delta<3^{-1}$, *we have the relations*

$$\mathrm{V}(\Psi_m;I)=2\sum_{k=0}^{m}(3\delta)^{k^2}\quad\textit{and}\quad \mathrm{V}(\Psi;\Gamma_I)=2\sum_{k=0}^{\infty}(3\delta)^{k^2}\,;$$

while if $\delta=3^{-1}$, *the corresponding relations are*

$$\mathrm{V}(\Psi_m;I)=2(m+1)\quad\textit{and}\quad \mathrm{V}(\Psi;\Gamma_I)=+\infty\,.$$

Moreover, if $\delta=3^{-1}$ *and if* K *is any component of the figure* E_m, *we have* $\mathrm{V}(\Psi;\Gamma_I\cap K)=+\infty$.

PROOF. Let $0<\delta\leqq 3^{-1}$ and write for short

$$\mathcal{V}_m=\mathrm{V}(\Psi_m;I),\quad \mathcal{V}=\mathrm{V}(\Psi;\Gamma_I),\quad \mathcal{V}^*=2\sum_{k=0}^{\infty}(3\delta)^{k^2}.$$

The assertion concerning $\mathcal{V}_m$ and $\mathcal{V}$ then amounts to

$$\mathcal{V}_m=2\sum_{k=0}^{m}(3\delta)^{k^2}\quad\text{and}\quad \mathcal{V}=\mathcal{V}^*\,,$$

inclusive of the case $\delta=3^{-1}$. But this expression for $\mathcal{V}_m$ is readily obtained by an inspection of the graph of the function $\Psi_m(x)$.

We proceed to the proof of $\mathcal{V}=\mathcal{V}^*$. Since the function $\Psi(x)$ is continuous and since the set Γ_I is the closure of the set Θ_I (see p. 23), we have $\mathcal{V}=\mathrm{V}(\Psi;\Theta_I)$. Consequently

$$\mathcal{V}=\sup_{\gamma}\sum_{j=1}^{n}|\Psi(x_j)-\Psi(x_{j-1})|\qquad(n\in\boldsymbol{N})\,,$$

where $\langle x_0,x_1,\cdots,x_n\rangle$ is any increasing sequence of $n+1$ points of Θ_I and where γ is short for this sequence.

Let us keep γ fixed for a while. We find at once the existence of an integer $m\geqq 0$ such that for each $j=0,1,\cdots,n$ the point x_j is an end point of some component of the figure E_m. It follows that

$$\Psi(x_j)=\sum_{k=0}^{\infty}\Phi_k(x_j;I;\delta)=\sum_{k=0}^{m}\Phi_k(x_j;I;\delta)=\Psi_m(x_j)\,;$$

in fact, $\Phi_k(x;I;\delta)$ vanishes for $k>m$. We thus have

$$\sum_{j=1}^{n}|\Psi(x_j)-\Psi(x_{j-1})|=|\sum_{j=1}^{n}\Psi_m(x_j)-\Psi_m(x_{j-1})|\leqq \mathcal{V}_m \leqq \mathcal{V}^* .$$

We now make vary the sequence $\gamma=\langle x_0, x_1, \cdots, x_n\rangle$ kept fixed hitherto, and we obtain

$$\mathcal{V}=\sup_{\gamma}\sum_{j=1}^{n}|\Psi(x_j)-\Psi(x_{j-1})|\leqq \mathcal{V}^* .$$

The equality $\mathcal{V}^*=\mathcal{V}$ will therefore follow if we verify further that $\mathcal{V}_k \leqq \mathcal{V}$ for every $k\in \boldsymbol{M}$. For this purpose, it is enough to verify the inequality

$$\sum_{j=1}^{n}|\Psi_k(t_j)-\Psi_k(t_{j-1})|\leqq \mathcal{V} \qquad (n\in \boldsymbol{N})$$

for every increasing sequence $\langle t_0, t_1, \cdots, t_n\rangle$ of points of the interval I. Examining the graph of the function $\Psi_k(x)$, however, we see without difficulty that each of these points may be assumed to be an end point of some component of the figure $E_k(3)$ obtained from E_k by 3-sized ramification. We then have $\Psi_k(t_j)=\Psi(t_j)$ for $j=0, 1, \cdots, n$. But this implies that

$$\sum_{j=1}^{n}|\Psi_k(t_j)-\Psi_k(t_{j-1})|\leqq \mathrm{V}(\Psi\,;\Theta_I)=\mathcal{V}, \qquad \text{Q. E. D.}$$

Finally, in the case in which $\delta=3^{-1}$, the relation $\mathrm{V}(\Psi\,;\Gamma_I \cap K)=+\infty$ is an immediate consequence of $\mathcal{V}=+\infty$ just established. In fact, the figure E_k has precisely 3^{k^2} components K of the same length, and the weak variation $\mathrm{V}(\Psi\,;\Gamma_I \cap K)$ is evidently independent of K. Accordingly, denoting by v this variation and by G generically any of the $3^{k^2}-1$ open intervals contiguous to E_k, we find easily that if $v<+\infty$, then

$$\mathcal{V}=\sum_{K} v+\sum_{G}|\Psi(G)|=3^{k^2}v+\sum_{G}|\Psi(G)|<+\infty .$$

From this contradiction we deduce that $v=+\infty$. This completes the proof of the lemma.

LEMMA 12. *Let us write* $\Psi(x)=\Psi(x;I;\delta)$ *as hitherto.*

(i) *If* $0<\delta<3^{-1}$, *the function* $\Psi(x)$ *is* AC *on* I *and Dirichlet continuous on the set* Γ_I ;

(ii) *if* $\delta=3^{-1}$, *then* $\Psi(x)$ *is not* GBV *on the set* Γ_I. *More minutely, for each* $k\in \boldsymbol{M}$, *this function is not* GBV *on the intersection* $\Gamma_I \cap K$, *where* K *is any component of the figure* E_k.

PROOF. *re* (i): In this case, $\Psi(x)$ is BV on Γ_I by the foregoing lemma. It then follows from Lemma 10 and Lemma 7 that $\Psi(x)$ is AC on this set. But $\Psi(x)$ is linear on each closed interval contiguous to Γ_I. Hence $\Psi(x)$ is AC on the whole interval I, on account of part (i) of Theorem 4. We find further by Theorem 5, parts (ii) and (iii), that $\Psi(x)$ is Dirichlet continuous on Γ_I.

re (ii): Supposing that $\delta=3^{-1}$, let c be a point of $\Gamma_I \cap K$ and let us take any open interval G containing c. Since $\Gamma_I=E_0 \cap E_1 \cap \cdots$, the point c belongs, for each $m \in \boldsymbol{M}$, to the figure E_m and hence to a component, say I_m, of this figure. Since $|I_m| \leqq |E_m| \leqq (3/5)^m |I|$, we can choose an integer $m>k$ such that $I_m \subset G$. Noting the evident inclusion $I_m \subset K \cap G$, we then have $\mathrm{V}(\Psi; \Gamma_I \cap K \cap G)=+\infty$, since a stronger relation $\mathrm{V}(\Psi; \Gamma_I \cap I_m)=+\infty$ is valid by the foregoing lemma. Thus $\Psi(x)$ is not BV on any portion of $\Gamma_I \cap K$.

On the other hand, it is well-known that *if a function which is continuous on a nonvoid closed set, is* GBV *on this set, then the same set contains a portion on which the function is* BV (see Saks [5], p. 233). We thus conclude that $\Psi(x)$ is not GBV on $\Gamma_I \cap K$.

By a lattice point in the plane $\boldsymbol{R}^2$ we mean, as usual, any point $\langle r, s \rangle$ with integer coordinates r, s. We make correspond to this point a plane interval $[r, r+1) \times [s, s+1)$ open on the right, which will be denoted by $I_{r,s}$ and called *lattice square* with *leading vertex* $\langle r, s \rangle$. This correspondene is biunique, for to different leading vertices there evidently correspond disjoint lattice sequares. Again, each point of the plane belongs to one and only one of the lattice squares. In other words, the family of the lattice sqares is disjoint and covers the whole plane.

LEMMA 13. *Given a real number $a \geqq 0$, let us denote by $\nu(a)$ the number of the lattice points contained in the following set:*

$$D(a)=\{\langle x, y \rangle ;\ x^2+y^2 \leqq a^2,\ x \geqq 0,\ y \geqq 0\},$$

which is a closed quarter disc of radius a. We then have

$$0 \leqq \nu(a)-4^{-1}\pi a^2 < 3a+2.$$

PROOF. Writing $M(a)$ for the union of all the lattice squares $I_{r,s}$ such that $\langle r, s \rangle \in D(a)$, let us prove first the relation

$$D(a) \subset M(a) \subset D(a+\sqrt{2}),$$

from which the assertion can easily be derived, as we shall show below.

Let $\langle x, y \rangle$ be any point of the set $D(a)$, and let $I_{r,s}$ be that lattice square to which this point belongs. Since $0 \leqq r \leqq x$ and $0 \leqq s \leqq y$, we have $r^2+s^2 \leqq x^2+y^2 \leqq a^2$. Consequently $\langle r, s \rangle \in D(a)$, and hence $I_{r,s} \subset M(a)$, which implies that $\langle x, y \rangle \in M(a)$. We have thus proved the inclusion $D(a) \subset M(a)$.

Suppose next that $\langle x, y \rangle$ is any point of $M(a)$, and let $I_{r,s}$ be that lattice square to which this point belongs, so that $\langle r, s \rangle \in D(a)$. Then

$$r^2+s^2 \leqq a^2, \quad 0 \leqq r \leqq x < r+1, \quad 0 \leqq s \leqq y < s+1 .$$

Accordingly, by the triangle inequality,

$$\sqrt{x^2+y^2} \leqq \sqrt{r^2+s^2} + \sqrt{(x-r)^2+(y-s)^2} < a + \sqrt{2},$$

and hence we get $\langle x, y \rangle \in D(a+\sqrt{2})$. This establishes the second inclusion $M(a) \subset D(a+\sqrt{2})$.

Taking plane measure, we deduce from $D(a) \subset M(a) \subset D(a+\sqrt{2})$ proved already the inequalities

$$|D(a)| \leqq |M(a)| \leqq |D(a+\sqrt{2})| ,$$

where we plainly have

$$|D(a)| = 4^{-1}\pi a^2, \quad |M(a)| = \nu(a), \quad |D(a+\sqrt{2})| = 4^{-1}\pi(a+\sqrt{2})^2 .$$

The required inequalities follow now at once.

Using the lemma proved just now, we proced to appraise the magnitude of the following function:

$$T(q) = \sum_{m=0}^{\infty} q^{m^2} , \qquad \text{where } 0 < q < 1 .$$

LEMMA 14. *If* $p>1$ *and* $q=3^{1-p}$, *then* $T(q) < \dfrac{2}{\sqrt{1-q}} < \dfrac{2^p}{\sqrt{p-1}}$.

PROOF. Let us denote by $\mu(m)$, where $m \in \boldsymbol{M}$, the number of the non-negative integral solutions $\langle x, y \rangle$ of the equation $x^2+y^2=m$. We then find successively that

$$\mu(n) = \nu(\sqrt{n}) - \nu(\sqrt{n-1}) \qquad \text{for } n \in \boldsymbol{N},$$

$$T^2(q) = \Big(\sum_{r=0}^{\infty} q^{r^2}\Big)\Big(\sum_{s=0}^{\infty} q^{s^2}\Big) = \sum_{m=0}^{\infty} \mu(m) q^m = 1 + \sum_{n=1}^{\infty} \{\nu(\sqrt{n}) - \nu(\sqrt{n-1})\} q^n$$

$$= \sum_{m=0}^{\infty} \nu(\sqrt{m}) q^m - \sum_{m=0}^{\infty} \nu(\sqrt{m}) q^{m+1} = (1-q) \sum_{m=0}^{\infty} \nu(\sqrt{m}) q^m ,$$

where the series $\sum \nu(\sqrt{m}) q^m$ $(m \in \boldsymbol{M})$, which we shall denote by $U(q)$, is convergent since, by the foregoing lemma, we have $\nu(\sqrt{n}) = O(n)$ for $n \in \boldsymbol{N}$.

Writing $\rho(m) = \nu(\sqrt{m}) - 4^{-1}\pi m$, so that $0 \leqq \rho(m) < 3\sqrt{m}+2$ by the same lemma, we go on as follows:

$$U(q) = \sum_{m=0}^{\infty} \left\{ \frac{\pi m}{4} + \rho(m) \right\} q^m = \frac{\pi q}{4} \sum_{n=1}^{\infty} n q^{n-1} + \sum_{m=0}^{\infty} \rho(m) q^m ,$$

$$\sum_{m=0}^{\infty} \rho(m) q^m < 3 \sum_{m=0}^{\infty} \sqrt{m}\, q^m + 2 \sum_{m=0}^{\infty} q^m \leqq 3q \sum_{n=1}^{\infty} n q^{n-1} + \frac{2}{1-q}.$$

We therefore have successively

$$U(q)<4q\sum_{n=1}^{\infty} nq^{n-1}+\frac{2}{1-q}=\frac{4q}{(1-q)^2}+\frac{2}{1-q}<\frac{4}{(1-q)^2},$$

$$T^2(q)=(1-q)U(q)<\frac{4}{1-q},\quad T(q)<\frac{2}{\sqrt{1-q}}<2\sqrt{\frac{3^{p-1}}{p-1}},$$

the last inequality being a consequence of $1-q=1-3^{1-p}>3^{1-p}(p-1)$. Since $2\sqrt{3^{p-1}}<2^p$, the proof is complete.

LEMMA 15. *Given a closed interval I, the function $\Xi(x)=\Psi(x;I;3^{-1})$ fulfils the Dirichlet condition on the set Γ_I.*

REMARK. This result is imperfect and may be skipped over. It will be superseded later on by Theorem 21 which is decisive. We have inserted this lemma in order to show, by way of contrast, how much we can say about the function $\Xi(x)$ at the present stage.

PROOF. Let us denote, as before, by H a generic open interval contiguous to Γ_I and let us keep H fixed for the moment, writing $H=(u, v)$. The function $\Psi(x)=\Psi(x;I;\delta)$, where $0<\delta<1$, was defined as follows:

$$\Psi(x;I;\delta)=\sum_{m=0}^{\infty}\Phi_m(x;I;\delta)=\sum_{m=0}^{\infty}\sum_{K}\Phi(x;K;\delta^{m^2}),$$

where K ranges over all the components of the figure E_m. As it follows easily from this definition, we have the alternatives:

either (i) the increment $\Psi(H)=\Psi(v;I;\delta)-\Psi(u;I;\delta)$ vanishes for all δ (of the interval $0<\delta<1$, needless to say);

or else (ii) there exists an integer $m\in\boldsymbol{M}$, uniquely determined by H and such that either $\Psi(H)=\delta^{m^2}$ for all δ, or $\Psi(H)=-\delta^{m^2}$ for all δ.

From now on, let us suppose that $\delta=3^{-p}$, where $p>1$. Making vary arbitrarily the interval H kept fixed hitherto, we find at once, from what was said in the above, that $\sum|\Xi(H)|^p=\sum|\Psi(H)|$. Here the series on the right is convergent, since the function $\Psi(x)=\Psi(x;I;\delta)$ is BV on Γ_I by Lemma 11 on account of $\delta=3^{-p}<3^{-1}$.

The function $\Psi(x)$, which vanishes outside I and which is linear on each interval H, is derivable at all points of the set $\boldsymbol{R}\setminus\Gamma_I$. For definiteness, we shall set $\Psi'(x)=0$ for every point x at which the function is not derivable. Since $\Psi(x)$ is AC on I by Lemma 12, its absolute variation $\mathrm{V}(\Psi;I)$ is expressed as follows:

$$\mathrm{V}(\Psi;I)=\int_I|\Psi'(x)|\,dx=\int_\Delta|\Psi'(x)|\,dx,$$

where $\Delta=\Delta_I=I\setminus\Gamma_I$. It follows, in view of the linearity of $\Psi(x)$ on the closure of each H, that

$$\mathrm{V}(\Psi;I)=\sum_H\int_H|\Psi'(x)|\,dx=\sum_H|\Psi(H)|\,.$$

On the other hand, the same linearity implies that $\mathrm{V}(\Psi;I)$ coincides with the weak variation $\mathrm{V}(\Psi;\Gamma_I)$. On account of Lemma 11, we conclude that

$$\sum_H|\Xi(H)|^p=\sum_H|\Psi(H)|=2\sum_{m=0}^{\infty}(3\delta)^{m^2}.$$

Writing $q=3\delta=3^{1-p}$ and using Lemma 14, we thus obtain the estimation:

$$\sum_H|\Xi(H)|^p=2T(q)=O\Big(\frac{1}{\sqrt{p-1}}\Big)=o\Big(\frac{1}{p-1}\Big)\,,\qquad \text{as } p\to 1\,.$$

Suppose now that the end points of a closed interval A belong to the set Γ_I. Then A itself cannot be contained in Γ_I, since $|\Gamma_I|=0$. By the same argument as used above to deduce $\sum|\Xi(H)|^p=\sum|\Psi(H)|$, we get

$$\sum_{H\subset A}\Xi(H)\square^p=\sum_{H\subset A}\Psi(H)\,.$$

But $\Psi(x)$, which is AC on I as stated already, fulfils the condition (B) on the null set Γ_I in virtue of part (iii) of Theorem 4. It follows that

$$\sum_{H\subset A}\Xi(H)\square^p=\Psi(A)=\Psi(\beta;I;3^{-p})-\Psi(\alpha;I;3^{-p})\,,$$

where we write $A=[\alpha,\beta]$. Making $p\to 1$ here and using Lemma 8, we obtain

$$\lim_{p\to 1}\sum_{H\subset A}\Xi(H)\square^p=\Psi(\beta;I;3^{-1})-\Psi(\alpha;I;3^{-1})=\Xi(A)\,,$$

which completes the proof.

Let Q be a compact nonconnected set and let G denote a generic open interval contiguous to Q. Given any function $\varphi(x)$, we write by definition:

$$\Lambda(\varphi;Q;p)=\sum_G|\varphi(G)|^p\qquad\text{for } p>1\,.$$

In the case in which $\Lambda(\varphi;Q;p)<+\infty$, we define further

$$\Upsilon(\varphi;Q;p)=\sum_G\varphi(G)\square^p\qquad\text{for } p>1\,,$$

the series on the right being absolutely convergent.

For each $m\in\boldsymbol{M}$ we shall henceforth denote by B_m the boundary of the figure E_m, i.e. the set of the end points of the component intervals of E_m.

LEMMA 16. *Given a closed interval I, a positive number $\delta\leqq 3^{-1}$, a number $p>1$, and a* CT *set Q contained in Γ_I, suppose that $\Lambda(\Psi;Q;p)$ is finite, where $\Psi(x)=\Psi(x;I;\delta)$.*

If $B_m\subset Q$ for an integer $m\geqq 0$ and if we write $Q^*=Q\cup B_{m+1}$, then

$$|\Upsilon(\Psi;Q;p)-\Upsilon(\Psi;Q^*;p)|<2^{5p}(3\delta^p)^{m^2}(p-1)\,.$$

If, further, a closed set $R\subset\Gamma_I$ *fulfils the condition* $R\cup B_{m+1}=Q^*$, *we have*

$$|\Upsilon(\Psi;Q^*;p)-\sum_U\Psi(U)\Box^p|<3^p(3\delta^p)^{m^2}\,,$$

where U *ranges over the open intervals contiguous to* R *and contained in the figure* E_{m+1}, *and where a possible void sum means zero.*

REMARK. The fact of the matter is that the hypothesis $\Lambda(\Psi;Q;p)<+\infty$ of the lemma is always fulfilled. If $0<\delta<3^{-1}$, this is immediate from part (i) of Lemma 12. The case $\delta=3^{-1}$ is delicate and will be treated only later on. Again, since B_{m+1} is a finite set, the hypothesis $\Lambda(\Psi;Q;p)<+\infty$ implies that $\Lambda(\Psi;Q^*;p)<+\infty$; consequently the series $\Upsilon(\Psi;Q^*;p)$ is absolutely convergent.

PROOF. Let us consider any component open interval, say $D=(\alpha,\beta)$, of the open set $E_m\setminus E_{m+1}$. Then D is contained in a component K of the figure E_m. Writing $Q'=Q\cup\{\alpha,\beta\}$, we proceed to appraise the difference

$$d=\Upsilon(\Psi;Q;p)-\Upsilon(\Psi;Q';p).$$

For this purpose, we express d in the form

$$d=\Upsilon(\Psi;Q\cap K;p)-\Upsilon(\Psi;Q'\cap K;p)\,,$$

which follows directly from the fact that the end points of the interval K belong to Q on account of $B_m\subset Q$. We denote by α' the rightmost point of the set $Q\cap(-\infty,\alpha]$ and by β' the leftmost point of $Q\cap[\beta,+\infty)$, so that we have $\alpha'\leqq\alpha<\beta\leqq\beta'$, $\alpha'\in K$, $\beta'\in K$. By means of α' and β', the difference d can now be written explicitly as follows:

$$d=\Psi([\alpha',\beta'])\Box^p-\Psi([\alpha',\alpha])\Box^p-\Psi([\alpha,\beta])\Box^p-\Psi([\beta,\beta'])\Box^p\,,$$

where $[\alpha',\alpha]$ is singletonic if $\alpha'=\alpha$, and similarly for $[\beta,\beta']$. It follows from Lemma 2 that

$$|d|\leqq3^p(p-1)\max\{|\Psi([\alpha',\beta'])|^p,\ |\Psi([\alpha',\alpha])|^p,\ |\Psi([\alpha,\beta])|^p,\ |\Psi([\beta,\beta'])|^p\}\,.$$

This being so, let us ramify to size 3 the interval K under consideration and let the three successors to K, arranged in their natural ordering, be K_1, K_2, K_3. We now distinguish two cases for the interval D considered in the above: D will be called, for the nonce, *of the first* or *second kind* according as it is, respectively, contiguous to or contained in the figure $K(3)=K_1\cup K_2\cup K_3$.

As is obvious from the construction of the function $\Psi(x)$, this function is expressible on each of the intervals K_i in the following form:

$$\Psi(x)=c(K)+c_i\delta^{m^2}+\sum_{k>m}\Phi_k(x;I;\delta) \qquad \text{for } x\in K_i\,,$$

where $c_1=c_3=0$, $c_2=1$ and where $c(K)$ is a constant which depends on the interval K. We find at once that

$$0\leqq\sum_{k>m}\Phi_k(x;I;\delta)\leqq\sum_{k=0}^{\infty}\delta^{(m+1+k)^2}\leqq\sum_{k=0}^{\infty}\delta^{(m+1)^2+k}\leqq\frac{3}{2}\delta^{(m+1)^2}\leqq\frac{1}{2}\delta^{m^2},$$

where we made use of the hypothesis $0<\delta\leqq 3^{-1}$.

We now return to the difference d and we shall deal first with the case in which the interval $D=(\alpha,\beta)$ is of the first kind. To fix the ideas, let us suppose that D is contiguous to the figure $K_1\cup K_2$. Then α is the right-hand end point of K_1 and β is the left-hand end point of K_2, so that $\Psi(\alpha)=c(K)$ and $\Psi(\beta)=c(K)+\delta^{m^2}$. Furthermore, we have $\alpha'\in K_1$ and $\beta'\in K_2\cup K_3$. We thus get

$$0\leqq\Psi(\alpha')-\Psi(\alpha)\leqq\frac{1}{2}\delta^{m^2},\quad -\delta^{m^2}\leqq\Psi(\beta')-\Psi(\beta)\leqq\frac{1}{2}\delta^{m^2},$$
$$-\frac{1}{2}\delta^{m^2}\leqq\Psi(\beta')-\Psi(\alpha')\leqq\frac{3}{2}\delta^{m^2}.$$

Consequently the above inequality for $|d|$ gives

$$|d|\leqq 3^p(p-1)\max\left\{\left(\frac{3}{2}\delta^{m^2}\right)^p,\left(\frac{1}{2}\delta^{m^2}\right)^p,(\delta^{m^2})^p\right\}=\left(\frac{9}{2}\right)^p\delta^{pm^2}(p-1)\,.$$

Of course this appraisal is valid also when D is contiguous to $K_2\cup K_3$.

We pass on to the case in which the interval $D=(\alpha,\beta)$ is of the second kind. Then D must be contained in one of the intervals K_1, K_2, K_3 and we have $\Psi(\alpha)=\Psi(\beta)=c(K)+c_i\delta^{m^2}$ if $D\subset K_i$. But it will be shown below that we may suppose the set Q to contain all the end points of the three intervals K_i. We see at once that, under this supposition, the points α' and β' both belong to the interval K_i which contains D. We therefore appraise $|d|$ by Lemma 1 as follows:

$$|d|\leqq 2^{p-1}(p-1)\cdot 2\left\{\frac{3}{2}\delta^{(m+1)^2}\right\}^p=3^p\delta^{p(m+1)^2}(p-1).$$

The figure E_m has 3^{m^2} components K and for each K the intersection $K\cap E_{m+1}$ is composed of 3^{2m+1} components, so that the open set $K\setminus E_{m+1}$ has precisely $3^{2m+1}-1$ components. It follows that the number, say N, of the components $D=(\alpha,\beta)$ of the open set $E_m\setminus E_{m+1}$ is expressed by

$$N=3^{m^2}(3^{2m+1}-1)=3^{(m+1)^2}-3^{m^2}.$$

Each interval K contains exactly two intervals of the first kind and hence there are $2\cdot 3^{m^2}$ intervals of the first kind, in all. Let us write $M=2\cdot 3^{m^2}$.

We now arrange all the intervals D in a sequence $\langle D_1,\cdots,D_N\rangle$, in such a manner that the first M intervals $D_1,\cdots,D_M$ are of the first kind and the remaining $N-M$ intervals $D_{M+1}, \cdots, D_N$ are of the second kind. We then define a sequence of $N+1$ sets $\langle Q_0, \cdots, Q_N\rangle$ inductively as follows, where we write $D_n=(\alpha_n, \beta_n)$:

$$Q_0=Q, \quad Q_n=Q_{n-1}\cup\{\alpha_n, \beta_n\} \qquad \text{for } n=1, \cdots, N.$$

By what was already proved we have

$$|\Upsilon(\Psi;Q_{n-1};p)-\Upsilon(\Psi;Q_n;p)|\leqq 2^{3p}\delta^{pm^2}(p-1)$$

for $n=1, \cdots, M$ and it follows at once that

$$|\Upsilon(\Psi;Q;p)-\Upsilon(\Psi;Q_M;p)|\leqq 2^{4p}(3\delta^p)^{m^2}(p-1).$$

On the other hand, for each component K of the figure E_m, the set Q_M contains all the end points of the three successors to K. We thus have

$$|\Upsilon(\Psi;Q_M;p)-\Upsilon(\Psi;Q_N;p)|\leqq 3^p(3\delta^p)^{(m+1)^2}(p-1)\leqq 3^p(3\delta^p)^{m^2}(p-1).$$

Noting that the set Q_N coincides with the set $Q^*=Q\cup B_{m+1}$, we obtain

$$|\Upsilon(\Psi;Q;p)-\Upsilon(\Psi;Q^*;p)|<2^{5p}(3\delta^p)^{m^2}(p-1).$$

This establishes the first of the asserted inequalies.

To prove the second inequality, let L denote a generic component of the figure E_{m+1}. By the symmetry of the graph of $\Psi(x)$ and by the inclusion $B_{m+1}\subset Q^*$, we have

$$\Upsilon(\Psi;Q^*;p)=\sum_L \Upsilon(\Psi;Q^*\cap L;p).$$

Fixing an L and writing $L=[u, v]$, so that

$$Q^*\cap L=(R\cup B_{m+1})\cap L=(R\cap L)\cup\{u, v\},$$

we find that if the set $R\cap L$ is CT, then the interval spanned by $R\cap L$ is a subinterval, say $[u', v']$, of the interval L. In this case we have

$$\Upsilon(\Psi;Q^*\cap L;p)=\sum_{U\subset L}\Psi(U)\square^p+\Psi([u, u'])\square^p+\Psi([v, v'])\square^p,$$

where U means a generic open interval contiguous to R and contained in E_{m+1}. But L is contained in some component K of E_m and, if as above we denote by K_1, K_2, K_3 the successors to K under 3-sized ramification of K, then necessarily one of these contains the whole of L. Accordingly we have the inequality $\mathrm{O}(\Psi;L)\leqq 2^{-1}\cdot 3\delta^{(m+1)^2}$, which implies that

$$|\Upsilon(\Psi;Q^*\cap L;p)-\sum_{U\subset L}\Psi(U)\square^p|\leqq 3^p\delta^{p(m+1)^2}.$$

Again, if the set $R\cap L$ is not CT, so that L contains none of the intervals U, we have similarly

$$|\Upsilon(\Psi;Q^*\cap L;p)|\leqq 3^p\delta^{p(m+1)^2}.$$

Since E_{m+1} has exactly $3^{(m+1)^2}$ components L, we conclude that

$$|\Upsilon(\Psi;Q^*;p)-\sum_U \Psi(U)|^p| \leqq 3^p(3\delta^p)^{(m+1)^2} < 3^p(3\delta^p)^{m^2},$$

on account of $3\delta^p \leqq 3^{1-p} < 1$. This completes the proof.

LEMMA 17. *Given a closed interval I, a positive number $\delta \leqq 3^{-1}$, and a number $p>1$, suppose that a closed set $R \subset \Gamma_I$ contains the end points of I and that $\Lambda(\Psi;R;p)<+\infty$, where $\Psi(x)=\Psi(x;I;\delta)$. If we write $\delta=3^{-s}$ (so that $s \geqq 1$), then*

$$|\Upsilon(\Psi;R;p)| < 2^{6ps}\cdot\sqrt{p-1}$$

PROOF. Let us write $R_m=R\cup B_m$ for each $m\in\boldsymbol{M}$, where B_m denotes as before the boundary of E_m. Since B_m is a finite set, the hypothesis $\Lambda(\Psi;R;p)<+\infty$ clearly implies that $\Lambda(\Psi;R_m;p)<+\infty$. We may therefore specialize the set Q of the preceding lemma to R_m. Noting the relation $Q^*=Q\cup B_{m+1}=R_{m+1}$, we have

$$|\Upsilon(\Psi;R_m;p)-\Upsilon(\Psi;R_{m+1};p)| < 2^{5p}(3\delta^p)^{m^2}(p-1)$$

for every $m\in\boldsymbol{M}$. But $3\delta^p=3(3^{-s})^p=3^{1-ps}<1$ on account of $ps \geqq p>1$, and it follows at once that

$$|\Upsilon(\Psi;R;p)-\Upsilon(\Psi;R_{m+1};p)| < 2^{5p}(p-1)T(3^{1-ps}),$$

where $T(q)=\sum_{k=0}^{\infty} q^{k^2}$ for $0<q<1$ as before. Since $R\cup B_{m+1}=R_m\cup B_{m+1}$, Lemma 16 gives further

$$|\Upsilon(\Psi;R_{m+1};p)| < 3^p(3^{1-ps})^{m^2}+\sum_U |\Psi(U)|^p,$$

where U ranges over the open intervals contiguous to R and contained in E_{m+1}. The last two inequalities together lead to

$$|\Upsilon(\Psi;R;p)| < 2^{5p}(p-1)T(3^{1-ps})+3^p(3^{1-ps})^{m^2}+\sum_U |\Psi(U)|^p.$$

We now make $m\to+\infty$ here. On account of $3^{1-ps}<1$ and $\Lambda(\Psi;R;p)<+\infty$, we have $\lim_{m\to\infty} 3^p(3^{1-ps})^{m^2}=0$ and $\lim_{m\to\infty}\sum_U |\Psi(U)|^p=0$; consequently

$$|\Upsilon(\Psi;R;p)| \leqq 2^{5p}(p-1)T(3^{1-ps}).$$

Inserting here the appraisal of Lemma 14, we obtain

$$|\Upsilon(\Psi;R;p)| < 2^{5p}(p-1)\frac{2^{ps}}{\sqrt{ps-1}} \leqq 2^{6ps}\cdot\sqrt{p-1}.$$

LEMMA 18. *Suppose given a closed interval I, a positive number $\delta<3^{-1}$, a number $p>1$, and a* CT *set Q contained in Γ_I.*

If $B_m\subset Q$ for an integer $m\geqq 0$ and if we write $Q^*=Q\cup B_{m+1}$, then

$$\Lambda(\Psi;Q;p)-\Lambda(\Psi;Q^*;p) < 2^{2p}(3\delta^p)^{m^2}(p-1),$$

where $\Psi(x)=\Psi(x;I;\delta)$.

If, further, a closed set $R\subset\Gamma_I$ *fulfils the condition* $R\cup B_{m+1}=Q^*$, *we have the inequality*

$$\Lambda(\Psi;Q^*;p)<2\sum_{k=0}^{m}(3\delta^p)^{k^2}+3^p(3\delta^p)^{m^2}+\sum_U|\Psi(U)|^p,$$

where U *ranges over the open intervals contiguous to* R *and contained in* E_{m+1}, *and where a possible void sum means zero.*

REMARK. Both $\Lambda(\Psi;Q;p)$ and $\Lambda(\Psi;Q^*;p)$ are finite, since $\Psi(x)$ is Dirichlet continuous on Γ_I by part (i) of Lemma 12.

PROOF. This can be established by the same means as used for Lemma 16 and we need only give an outline of the proof.

Let D be any component of the open set $E_m\setminus E_{m+1}$. Then D is contained in some component K of E_m. Writing $D=(\alpha,\beta)$ and $Q'=Q\cup\{\alpha,\beta\}$, we shall appraise the difference

$$d=\Lambda(\Psi;Q;p)-\Lambda(\Psi;Q';p)=\Lambda(\Psi;Q\cap K;p)-\Lambda(\Psi;Q'\cap K;p).$$

For this purpose, we consider the minimal closed interval $[\alpha',\beta']$ such that $[\alpha,\beta]\subset[\alpha',\beta']\subset K$, $\alpha'\in Q$, $\beta'\in Q$. Then

$$d=|\Psi([\alpha',\beta'])|^p-|\Psi([\alpha',\alpha])|^p-|\Psi([\alpha,\beta])|^p-|\Psi([\beta,\beta'])|^p.$$

We begin with the case in which the interval D is of the first kind. We may suppose that D is contiguous to $K_1\cup K_2$, where K_1, K_2 K_3 are the three successors to K under 3-sized ramification of K, arranged in their natural ordering. Then

$$\Psi([\alpha,\beta])=\delta^{m^2},\quad \Psi(\alpha)\leqq\Psi(\alpha')\leqq\Psi(\beta),\quad \Psi(\alpha)\leqq\Psi(\beta').$$

Consequently, if $\Psi(\beta')\leqq\Psi(\beta)$, we have $|\Psi([\alpha',\beta'])|\leqq\Psi([\alpha,\beta])$, whence $d\leqq 0$. On the other hand, if $\Psi(\beta)<\Psi(\beta')$, then $0\leqq\Psi([\alpha',\beta'])\leqq\Psi([\alpha,\beta'])$, and we find that

$$d\leqq\Psi([\alpha,\beta'])^p-\Psi([\alpha,\beta])^p-\Psi([\beta,\beta'])^p,$$

where $\Psi([\beta,\beta'])\leqq 2^{-1}\delta^{m^2}$. It follows from Lemma 1 that

$$d\leqq 2^{p-1}(p-1)(\delta^{pm^2}+2^{-p}\delta^{pm^2})<2^p\delta^{pm^2}(p-1).$$

Suppose next that the interval D is of the second kind. We may suppose that the set Q contains all the end points of the three intervals K_i. Then

$$\Psi(\alpha')\geqq\Psi(\alpha)=\Psi(\beta)\leqq\Psi(\beta'),$$

whence we have either $|\Psi([\alpha',\beta'])|\leqq|\Psi([\alpha',\alpha])|$, or $|\Psi([\alpha',\beta'])|\leqq\Psi([\beta,\beta'])$. It thus follows that $d\leqq 0$.

Using the above results and arguing as in the proof of Lemma 16, we arrive at the first inequlity of the assertion:

$$\Lambda(\Psi;Q;p)-\Lambda(\Psi;Q^*;p)<2^{2p}(3\delta^p)^{m^2}(p-1)\,.$$

The proof of the second inequality is quite similar to that of the corresponding inequality of Lemma 16. We omit the details.

LEMMA 19. *Given a closed interval I, suppose that a closed set $R\subset\Gamma_I$ contains the end points of I. If $s>1$ and $\delta=3^{-s}$, then*

$$\Lambda(\Psi;R;p)<\frac{2^{3ps}p}{\sqrt{p-1}}\qquad \textit{for } p>1\,,$$

where $\Psi(x)=\Psi(x;I;\delta)$.

PROOF. We shall only outline the proof, since it resembles that of Lemma 17. We may specialize the set Q of Lemma 18 to $R_m=R\cup B_m$, where $m\in\boldsymbol{M}$. Noting that $Q^*=Q\cup B_{m+1}=R_{m+1}$, we have

$$\Lambda(\Psi;R_m;p)-\Lambda(\Psi;R_{m+1};p)<2^{2p}(3\delta^p)^{m^2}(p-1)$$

for every $m\in\boldsymbol{M}$. But $3\delta^p=3^{1-ps}<1$, and it follows that

$$\Lambda(\Psi;R;p)-\Lambda(\Psi;R_{m+1};p)<2^{2p}(p-1)T(3^{1-ps})\,.$$

Since $R\cup B_{m+1}=R_m\cup B_{m+1}$, Lemma 18 shows further that

$$\Lambda(\Psi;R_{m+1};p)<2\sum_{k=0}^{m}(3^{1-ps})^{k^2}+3^p(3^{1-ps})^{m^2}+\sum_U|\Psi(U)|^p\,.$$

The last two inequalities together yield

$$\Lambda(\Psi;R;p)<2^{2p}p\cdot T(3^{1-ps})+3^p(3^{1-ps})^{m^2}+\sum_U|\Psi(U)|^p\,.$$

Making $m\to+\infty$ here, we obtain $\Lambda(\Psi;R;p)\leqq 2^{2p}p\cdot T(3^{1-ps})$. Using the appraisal of Lemma 14, we find that

$$\Lambda(\Psi;R;p)<2^{2p}p\frac{2^{ps}}{\sqrt{ps-1}}<\frac{2^{3ps}p}{\sqrt{p-1}},\qquad \text{Q. E. D.}$$

LEMMA 20. *Given a closed interval I, suppose that a closed set $R\subset\Gamma_I$ contains the end points of I. Writing as before $\Xi(x)=\Psi(x;I;3^{-1})$, we have*

$$\Lambda(\Xi;R;p)<\frac{2^{4p}p}{\sqrt{p-1}}\qquad \textit{for } p>1, \textit{ so that } \Lambda(\Xi;R;p)=o\left(\frac{1}{p-1}\right).$$

PROOF. The proof will be based on approximating the function $\Xi(x)$ by the manageable function $\Psi(x)=\Psi(x;I;\delta)$, where $0<\delta<3^{-1}$. In fact, if we fix a point x of Γ_I, then $\Psi(x)$ tends to $\Xi(x)$ as $\delta\to3^{-1}$, in virtue of Lemma 8.

Let $\langle G_1,\cdots,G_n\rangle$ be any finite distinct sequence of open intervals contiguous to the set R and let us write for short

$$g=\langle G_1, \cdots, G_n\rangle, \quad \Lambda_g=\sum_{i=1}^{n} |\Xi(G_i)|^p, \quad \Lambda_g(\delta)=\sum_{i=1}^{n} |\Psi(G_i)|^p,$$

so that $\Lambda_g(\delta)\leqq\Lambda(\Psi;R;p)$.

Writing $\delta=3^{-s}$, where $s>1$, and using the foregoing lemma, we have

$$\Lambda_g(3^{-s})<\frac{2^{3ps}p}{\sqrt{p-1}} \qquad \text{for } p>1\,.$$

Hence, if we keep the sequence g fixed and make $s\to1$, we get

$$\Lambda_g=\lim_{s\to1} \Lambda_g(3^{-s})\leqq\frac{2^{3p}p}{\sqrt{p-1}}\,.$$

We now make the sequence g vary and we have

$$\Lambda(\Xi;R;p)=\sup_g \Lambda_g\leqq\frac{2^{3p}p}{\sqrt{p-1}}<\frac{2^{4p}p}{\sqrt{p-1}}\,,$$

which completes the proof.

LEMMA 21. *Under the same hypothesis as in Lemma 20, we have*

$$|\Upsilon(\Xi;R;p)|<2^{6p}\cdot\sqrt{p-1} \qquad \textit{for } p>1\,,$$

so that $\lim_{p\to1} \Upsilon(\Xi;R;p)=0=\Xi(I)$.

PROOP. Let us take $\delta=3^{-1}$ in Lemma 17, so that $s=1$. Then $\Psi(x)$ coincides with $\Xi(x)$ and we have $\Lambda(\Xi;R;p)<+\infty$ by the preceding lemma. Hence the result.

THEOREM 21. *Given a closed interval I, the function $\Xi(x)=\Psi(x;I;3^{-1})$ is Dirichlet continuous, without being* GAC, *on the set Γ_I. More minutely, for each $m\in\boldsymbol{M}$, the function is not* GAC *on the intersection $\Gamma_I\cap K$, where K is any component of the figure E_m.*

PROOF. The function $\Xi(x)$ is continuous on $\boldsymbol{R}$ and hence its Dirichlet continuity on Γ_I is an immediate consequence of Lemma 20 and Lemma 21.

By Lemma 12, $\Xi(x)$ is not GBV on $\Gamma_I\cap K$. But *any function which is* GAC *on a set, is necessarily* GBV *on the same set* (see Saks [5], p. 223). Hence $\Xi(x)$ cannot be GAC on $\Gamma_I\cap K$. This completes the proof.

The function $\Xi(x)=\Psi(x;I;3^{-1})$ vanishes outside I and is linear on each closed interval contiguous to Γ_I. Accordingly $\Xi(x)$ is derivable at every point of the set $\boldsymbol{R}\setminus\Gamma_I$. For definiteness, we shall write $\Xi'(x)=0$ for each point x at which the function is not derivable.

Since $\Xi(x)$ is moreover Dirichlet continuous on the CT null set Γ_I by the above theorem, the function must be powerwise continuous on the interval I. The function $f(x)=\Xi'(x)$ is therefore powerwise integrable over I. However, $f(x)$ cannot be Denjoy integrable on I. To see this, suppose if possible that the contrary is true. There then exists a function $\varphi(x)$ which is GAC, and hence also powerwise continuous, on I and which has $f(x)$ for its approximate derivative almost everywhere on I. It follows that $\varphi(x)$ and $\Xi(x)$ are AED almost everywhere on I and that they therefore differ on I only by an additive constant, on account of Theorem 12. We thus find that $\Xi(x)$ is GAC on I, a fact which contradicts the above Theorem 21.

It is obvious that if c is any nonzero constant, then the function $cf(x)$ is likewise powerwise integrable, but not Denjoy integrable, on I.

We thus obtain the following theorem:

THEOREM 22. *Given any closed interval, there exist functions which are powerwise integrable, without being Denjoy integrable, on this interval.*

§ 5. The Luzin integration does not include the powerwise integration.

Given a closed interval $I=[a, b]$ and a number $h>0$, we shall denote by $\theta(x;I;h)$ the function which vanishes for $x\leqq a$, equals h for $x\geqq b$, and is linear on I.

If S and T are two nonvoid linear sets and if we have $x<y$ whenever $x\in S$ and $y\in T$, we shall write $S<T$ as on p. 8 of [3].

Given a nonvoid elementary figure E and a number $h>0$, let the component intervals of E be $I_1<I_2<\cdots\cdot<I_n$ and let us define

$$\theta(x;E;h)=\theta(x;I_1;n^{-1}h)+\cdots\cdot+\theta(x;I_n;n^{-1}h) \qquad \text{for } x\in \boldsymbol{R}.$$

The following lemma is obvious and will be used without quotation in the rest of the paper.

LEMMA 22. *Given a nonvoid figure E and a number $h>0$, let the interval spanned by E be $I=[a, b]$. Then*

(i) *the function $\theta(x;E;h)$ vanishes for $x\leqq a$, equals h for $x\geqq b$, and we have $0\leqq\theta(x;E;h)\leqq h$ for $x\in\boldsymbol{R}$;*

(ii) *the function $\theta(x;E;h)$ is continuous and nondecreasing over $\boldsymbol{R}$, as well as linear on each component of the figure E;*

(iii) *if E has exactly n components, where $n\geqq 2$, and if the closed intervals contiguous to E are $J_1<J_2<\cdots\cdot<J_{n-1}$, we have*

$$\theta(x;E;h)=n^{-1}ih \qquad \textit{for } x\in J_i, \textit{ where } i=1, 2, \cdots\cdot, n-1.$$

LEMMA 23. *Given a nonvoid figure E whose components are $I_1<I_2<\cdots\cdot<I_n$ and given for each $k=1, 2, \cdots\cdot, n$ a nonvoid figure Q_k spanning*

the interval I_k, *write* $Q=Q_1\cup\cdots\cup Q_n$ *and suppose that the number of the components of* Q_k *is independent of* k. *We then have the following results, where* h *is any positive number.*

(a) $\theta(x;Q;h)=\sum_{k=1}^{n}\theta(x;Q_k;n^{-1}h)$ *for* $x\in\boldsymbol{R}$;

(b) $\theta(x;Q;h)=\theta(x;E;h)$ *unless* x *is interior to* E;

(c) $|\theta(x;Q;h)-\theta(x;E;h)|\leqq n^{-1}h$ *for* $x\in\boldsymbol{R}$;

(d) $\theta(x;Q;h)=\theta(x;Q_k;n^{-1}h)+n^{-1}h(k-1)$ *for* $x\in I_k$,

where $k=1,2,\cdots,n$, *so that the increment of the function* $\theta(x;Q;h)$ *over the interval* I_k *is equal to* $n^{-1}h$;

(e) *if there exist a positive integer* $r<n$ *and a real number* $\alpha>0$ *such that* $Q_{r+1}=Q_r+\alpha$, *where* $Q_r+\alpha$ *means the translation of the figure* Q_r *by the number* α, *then*

$$\theta(x+\alpha;Q;h)=\theta(x;Q;h)+n^{-1}h \qquad for\ x\in I_r .$$

PROOF. *re* (a): For each $k=1,2,\cdots,n$ let $J_1^{(k)}<J_2^{(k)}<\cdots<J_m^{(k)}$ be the components of the figure Q_k, the number m being the same for all k by hypothesis. Then, for any $x\in\boldsymbol{R}$,

$$\theta(x;Q;h)=\sum_{k=1}^{n}\sum_{i=1}^{m}\theta(x;J_i^{(k)};h/mn)=\sum_{k=1}^{n}\theta(x;Q_k;n^{-1}h).$$

re (b): Suppose that x is not an interior point of E. Then x is not interior to I_k for any $k=1,2,\cdots,n$ either and hence we have

$$\theta(x;Q_k;n^{-1}h)=\theta(x;I_k;n^{-1}h) \qquad \text{for all } k .$$

It therefore follows from the relation (a) proved just now that

$$\theta(x;Q;h)=\sum_{k=1}^{n}\theta(x;I_k;n^{-1}h)=\theta(x;E;h).$$

re (c): On account of part (b), We may assume x to be interior to E, so that x belongs to one, say I_r, of the intervals $I_1,\cdots,I_n$. Then

$$\theta(x;Q_k;n^{-1}h)=\theta(x;I_k;n^{-1}h) \qquad \text{for } k\neq r .$$

This, combined with the relation (a), gives

$$\theta(x;Q;h)-\theta(x;E;h)=\sum_{k=1}^{n}\{\theta(x;Q_k;n^{-1}h)-\theta(x;I_k;n^{-1}h)\}$$

$$=\theta(x;Q_r;n^{-1}h)-\theta(x;I_r;n^{-1}h)$$

for $x\in\boldsymbol{R}$. But this last difference has absolute value $\leqq n^{-1}h$, since

$$0\leqq\theta(x;Q_r;n^{-1}h)\leqq n^{-1}h \quad \text{and} \quad 0\leqq\theta(x;I_r;n^{-1}h)\leqq n^{-1}h .$$

re (d): Keeping k fixed, let $x\in I_k$ and consider positive integers $i\leqq n$.

Then
$$\theta(x;Q_i;n^{-1}h)=\begin{cases}0 & \text{for } i>k, \text{ if } k<n;\\ n^{-1}h & \text{for } i<k, \text{ if } k>1.\end{cases}$$

The required result follows at once from this and the relation (a).

re (e): Let x be any point of I_r. The above relation (d) implies that
$$\theta(x+\alpha;Q;h)=\theta(x+\alpha;Q_{r+1};n^{-1}h)+n^{-1}rh,$$
$$\theta(x;Q;h)=\theta(x;Q_r;n^{-1}h)+n^{-1}h(r-1).$$
But we clearly have $\theta(x+\alpha;Q_{r+1};n^{-1}h)=\theta(x;Q_r;n^{-1}h)$. Hence the result.

Let I be a closed interval. As before, we associate with I a descending sequence of figures $\langle E_0, E_1, \cdots\rangle$ constructed inductively as follows:
$$E_0=I, \qquad E_{m+1}=E_m(3\,;3^{2m}) \qquad \text{for } m\in\boldsymbol{M}.$$
The compact set Γ is then defined as $\Gamma=E_0\cap E_1\cap\cdots$.

We denote by K a generic component of the figure E_m for each $m\in\boldsymbol{M}$, and by $K_1<K_2<K_3$ the successors to K under 3-sized ramification of K. We write further $\lambda=(2/5)|K|$.

We shall consider the functions $F_m(x)$ defined on $\boldsymbol{R}$ by
$$F_m(x)=\theta(x;E_m;1), \qquad \text{where } m\in\boldsymbol{M}.$$

The above notation will be retained throughout the sequel.

The reader is requested to keep in his memory the following facts, where the figure $E_m(3)$ is the result of 3-sized ramification applied to E_m.

(i) The figure E_m has exactly 3^{m^2} components K;
(ii) the figure $E_m(3)$ has exactly 3^{m^2+1} components;
(iii) the intervals K_1, K_2, K_3 are components of $E_m(3)$;
(iv) we have $K_2=K_1+\lambda$ and $K_3=K_2+\lambda$.

LEMMA 24. *Let m and p be two integers such that $0\leqq m<p$. Then, with the same notation as above, we have*
(i) $F_p(K)=3^{-m^2}$ *for each component K of E_m,*
(ii) $F_p(x+\lambda)=F_p(x)+3^{-m^2-1}$ *for $x\in K_i$, where $i=1,2$.*

PROOF. *re* (i): The figure E_m has just 3^{m^2} components K, and the figure E_p is obtained from E_m by replacing each K with the figure $K\cap E_p$ which spans K and which has exactly $3^{p^2-m^2}$ components. Consequently it follows from part (d) of Lemma 23 that the increment of the function $F_p(x)=\theta(x;E_p;1)$ over K is 3^{-m^2}.

re (ii): The figure $E_m(3)$ has just 3^{m^2+1} components, which we shall denote generically by L. The figure E_p is obtained from $E_m(3)$ by replacing each L with the figure $L \frown E_p$ which spans L and which has exactly $3^{p^2-m^2-1}$ components. The relation (ii) then follows immediately from part (e) of Lemma 23, if we take $E=E_m(3)$ and $Q=E_p$ in that lemma.

LEMMA 25. *The sequence of functions* $\langle F_0(x), F_1(x), \cdots\cdot\rangle$ *converges uniformly on* $\boldsymbol{R}$ *and its limiting function, which we denote by* $F(x\,;I)$ *or simply by* $F(x)$, *has the following properties, where we write* $I=[a,b]$.

(i) *The function* $F(x)$ *vanishes for* $x \leqq a$, *equals* 1 *for* $x \geqq b$, *and is continuous as well as nondecreasing, over the whole line* $\boldsymbol{R}$;

(ii) $F(x)$ *is a constant on each closed interval contiguous to the compact set* $\Gamma=E_0 \frown E_1 \frown \cdots\cdot$;

(iii) $F(x+\lambda)=F(x)+3^{-m^2-1}$ *for* $x \in K_i$, *where* $i=1,2$.

PROOF. *re* (i): The figure E_{m+1} is obtained from E_m by replacing each component K of E_m with the figure $K \frown E_{m+1}=K(3\,;3^{2m})$ which spans K and which has exactly 3^{2m+1} components. Consequently it follows from part (c) of Lemma 23 that

$$|F_{m+1}(x)-F_m(x)|=|\theta(x\,;E_{m+1}\,;1)-\theta(x\,;E_m\,;1)| \leqq 3^{-m^2} \leqq 3^{-m}$$

for all $x \in \boldsymbol{R}$. This appraisal implies that the series $\sum\{F_{m+1}(x)-F_m(x)\}$, where m ranges over $\boldsymbol{M}$, is uniformly convergent on $\boldsymbol{R}$; therefore the sequence $\langle F_m(x)\,;\, m \in \boldsymbol{M}\rangle$ converges uniformly on $\boldsymbol{R}$ to its limit $F(x)$.

The assertion (i) is certainly true if we replace there the function $F(x)$ by $F_m(x)$. This fact, together with the uniform convergence of $\langle F_m(x)\,;\, m \in \boldsymbol{M}\rangle$, completes the proof of (i).

re (ii): Let A be any closed interval contiguous to the set Γ. As is readily seen from the definition of the sequence $\langle E_m\,;\, m \in \boldsymbol{M}\rangle$, there exists a least integer $q>0$ such that A is contiguous to the figure E_q. We then find that A is contiguous to E_m for every integer $m \geqq q$. Accordingly, by part (b) of Lemma 23, we have

$$F_{m+1}(x)=\theta(x\,;E_{m+1}\,;1)=\theta(x\,;E_m\,;1)=F_m(x) \qquad \text{for } x \in A\,,$$

provided that $m \geqq q$. It follows that

$$F(x)=\lim_{m\to\infty} F_m(x)=F_q(x) \qquad \text{for } x \in A\,.$$

But $F_q(x)$ is a constant on A by part (iii) of Lemma 22, since A is contiguous to the figure E_q. Hence the result.

re (iii): We need only make $p \to +\infty$ in part (ii) of the foregoing lemma.

We now remind us of a few basic notions concerning the Luzin integration (see [4], p. 26 and p. 31). A function $\varphi(x)$ is said to be *approxi-*

mately derivable (N), or ADN, on a closed interval I, if on this interval the function is continuous, subject to the condition (N), and further AD almost everywhere. Again, we call a function $\varphi(x)$ to be *stable* (N) on I, if $\varphi(x)$ is ADN on I and if every function wbich is ADN on I as well as AED with $\varphi(x)$ almost everywhere on I, differs over I from $\varphi(x)$ only by an additive constant.

THEOREM 23. *The function* $\Xi(x)=\Psi(x;I;3^{-1})$ *of the foregoing* § *is* ADN *on* I *without being stable* (N) *on* I.

PROOF. The function $\Xi(x)$ is continuous on $\boldsymbol{R}$ and it fulfils the condition (N) on I by Lemma 10. Moreover, $\Xi(x)$ is linear on each closed interval A contiguous to the compact null set Γ which spans the interval I. $\Xi(x)$ is therefore ADN on I.

On the other hand, the function $F(x)=\lim F_m(x)$ which is continuous on $\boldsymbol{R}$, is a constant on each interval A considered just now (see Lemma 25). Consequently, if we write

$$\Omega(x)=\Xi(x)+3F(x) \qquad \text{for } x\in\boldsymbol{R}\,,$$

the function $\Omega(x)$ is not only continuous on $\boldsymbol{R}$ and AD almost everywhere on I, but also AED with $\Xi(x)$ almost everywhere on I. Furthermore, the difference $\Omega(x)-\Xi(x)$, which equals $3F(x)$, is nonconstant on I. Accordingly we need only show, in what follows, that $\Omega(x)$ fulfils the condition (N) on I and hence is ADN on I.

Fixing an integer $m\geqq 0$, let us consider any component K of the figure E_m. We shall prove first that the measure of the compact set $\Omega[K\cap\Gamma]$ is independent of the choice of K and hence depends only on m.

We have to verify that $|\Omega[K'\cap\Gamma]|=|\Omega[K\cap\Gamma]|$, where K' is any component of E_m other than K. As we find easily, there exist real numbers α and β such that $K'=K+\alpha$ and that

$$\Xi(x+\alpha)=\Xi(x)+\beta \qquad \text{for } x\in K\,.$$

We can show further the existence of a real number γ such that

$$F(x+\alpha)=F(x)+\gamma \qquad \text{for } x\in K\,,$$

the proof being the same as for part (iii) of Lemma 25. Hence

$$\Omega(x+\alpha)=\Omega(x)+\beta+3\gamma \qquad \text{for } x\in K\,.$$

But we have $K'\cap\Gamma=(K\cap\Gamma)+\alpha$ as an easy consequence of $K'=K+\alpha$, and it follows that

$$\Omega[K'\cap\Gamma]=\Omega[K\cap\Gamma]+\beta+3\gamma\,.$$

This plainly implies the required result $|\Omega[K'\cap\Gamma]|=|\Omega[K\cap\Gamma]|$.

We proceed to establish the inequality

$$|\Omega[K\cap\Gamma]|\leqq 2^m 3^{m(m-1)}|\Omega[L\cap\Gamma]| \qquad \text{for } m\in \boldsymbol{M},$$

where L is any component of E_{m+1} and where the measure $|\Omega[L\cap\Gamma]|$ is independent of L by what we proved just now.

Consider the intervals K_1, K_2, K_3. We have

$$K\cap\Gamma=\bigcup_{i=1}^{3}(K_i\cap\Gamma), \quad \text{so that } \Omega[K\cap\Gamma]=\bigcup_{i=1}^{3}\Omega[K_i\cap\Gamma].$$

But $\Omega[K_2\cap\Gamma]=\Omega[K_3\cap\Gamma]$, as we shall show below. Thus

$$\Omega[K\cap\Gamma]=\bigcup_{i=1}^{2}\Omega[K_i\cap\Gamma], \quad \text{whence } |\Omega[K\cap\Gamma]|\leqq \sum_{i=1}^{2}|\Omega[K_i\cap\Gamma]|.$$

From the definition of the function $\Xi(x)$ we find that

$$\Xi(x+\lambda)=\Xi(x)-3^{-m^2} \qquad \text{for } x\in K_2.$$

On, the other hand, by part (iii) of Lemma 25, we have

$$F(x+\lambda)=F(x)+3^{-m^2-1} \qquad \text{for } x\in K_2.$$

Hence $\Omega(x+\lambda)=\Omega(x)$ for $x\in K_2$. This, together with $K_3\cap\Gamma=(K_2\cap\Gamma)+\lambda$ which is a direct consequence of $K_3=K_2+\lambda$, shows that $\Omega[K_3\cap\Gamma]$ coincides with $\Omega[K_2\cap\Gamma]$ as stated above.

We go on to appraise the measure $|\Omega[K_i\cap\Gamma]|$ for $i=1, 2$. Writing $S=K_1\cap E_{m+1}$ for short, we have $K_1\cap\Gamma=S\cap\Gamma$, so that $\Omega[K_1\cap\Gamma]=\Omega[S\cap\Gamma]$. But the figure S has exactly 3^{2m} components, and these are at the same time components of E_{m+1}. If, therefore, L denotes as above a generic component of E_{m+1}, then

$$\Omega[K_1\cap\Gamma]=\bigcup_{L\subset S}\Omega[L\cap\Gamma], \quad |\Omega[K_1\cap\Gamma]|\leqq 3^{2m}|\Omega[L\cap\Gamma]|.$$

Of course, the inequality $|\Omega[K_2\cap\Gamma]|\leqq 3^{2m}|\Omega[L\cap\Gamma]|$ admits a similar treatment.

The inequality $|\Omega[K\cap\Gamma]|\leqq|\Omega[K_1\cap\Gamma]|+|\Omega[K_2\cap\Gamma]|$ which was already established, combined with what we showed just now, gives

$$|\Omega[K\cap\Gamma]|\leqq 2\cdot 3^{2m}|\Omega[L\cap\Gamma]| \qquad \text{for } m\in\boldsymbol{M}.$$

From this we readily find, by induction, that

$$|\Omega[\Gamma]|\leqq 2^m 3^{m(m-1)}|\Omega[K\cap\Gamma]| \qquad \text{for } m\in\boldsymbol{M},$$

where K is any component of the figure E_m, as hitherto.

By means of this inequality we shall prove that $|\Omega[\Gamma]|=0$. For this purpose, we appraise the measure $|\Omega[K\cap\Gamma]|$ as follows.

By what was shown in the proof of Lemma 16 (see p. 33), there is a constant $c(K)$ depending on K and such that

$$c(K)\leqq\Xi(x)\leqq c(K)+3^{1-m^2} \qquad \text{for } x\in K.$$

On the other hand, if p is any integer $>m$, part (i) of Lemma 24 asserts that $F_p(K)=3^{-m^2}$ for each K. Making $p\to+\infty$ here and passing to the limit, we get $F(K)=3^{-m^2}$. By part (i) of Lemma 25, however, $F(x)$ is nondecreasing on $\boldsymbol{R}$. We thus find the existence of a constant $c'(K)$ depending on K and such that

$$c'(K)\leqq F(x)\leqq c'(K)+3^{-m^2} \quad \text{for } x\in K.$$

The above results together lead at once to the following estimation of the function $\Omega(x)=\Xi(x)+3F(x)$:

$$c''(K)\leqq \Omega(x)\leqq c''(K)+3^{2-m^2} \quad \text{for } x\in K,$$

where $c''(K)$ is a certain constant which depends on K. It follows that

$$|\Omega[K\cap\Gamma]|\leqq \mathrm{d}(\Omega[K\cap\Gamma])\leqq \mathrm{d}(\Omega[K])\leqq 3^{2-m^2}.$$

We thus arrive at

$$|\Omega[\Gamma]|\leqq 2^m 3^{m(m-1)}|\Omega[K\cap\Gamma]|\leqq 9\left(\frac{2}{3}\right)^m.$$

Since m is arbitrary, this implies that $|\Omega[\Gamma]|=0$.

By part (ii) of Lemma 25, the function $\Omega(x)$ is linear on each closed interval A contiguous to the set Γ, and hence $\Omega(x)$ fulfils the condition (N) on each A. Combining this fact with $|\Omega[\Gamma]|=0$ obtained just now, we find that $\Omega(x)$ fulfils the condition (N) on the whole interval I. This completes the proof.

THEOREM 24. *The Luzin integration does not include the powerwise integration.*

PROOF. As we saw on p. 38, if $f(x)$ denotes the function which equals $\Xi'(x)$ at every point x of derivability of $\Xi(x)$ and which vanishes everywhere else, so that $\Xi(x)$ is derivable to $f(x)$ almost everywhere on I, then $f(x)$ is powerwise integrable on I.

We shall show that the function $f(x)$ is not Luzin integrable on I. Suppose, if possible, that the contrary is true. There then exists a function $\psi(x)$ which is stable (N) on I and which has $f(x)$ for its approximate derivative almost everywhere on I (see [4], p. 32). Then the function $\Xi(x)$, which is ADN on I (see Theorem 23) and AED with $\psi(x)$ almost everywhere on I, differs over I from $\psi(x)$ only by an additive constant. Consequently $\Xi(x)$ itself must be stable (N) on I, contradicting Theorem 23. This completes the proof.

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