

## On a Simplex Homomorphism, II

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### § 1. Introduction.

In [8], we defined a simplex homomorphism  $T$  of a simplex space  $E$ , whose second adjoint operator is a lattice homomorphism, and we investigated some properties of a simplex homomorphism  $T$ . It can be expressed with a mapping  $k: K \rightarrow K$  and a function  $\gamma$  on  $K$  as  $Tf(x) = \gamma(x)f \circ k(x)$  for any  $f \in E$ , where  $K$  is a subset of the state space of  $E$ . In this paper, by assuming some conditions, we shall investigate the behavior of the mapping  $k$  (Lemma 1~4) and we shall show that there is a  $T$ -invariant ideal  $I$  to which the restriction  $T_I$  of  $T$  is uniformly ergodic and  $P_\sigma(T) \cap \Gamma = P_\sigma(T_I) \cap \Gamma$ . We also show that there is a  $T$ -invariant simplex subspace  $A$  to which the restriction  $T_A$  of  $T$  is a simplex isomorphism with  $T_A^n = I_A$  for some  $n \in \mathbb{N}$  and  $P_\sigma(T) \cap \Gamma = P_\sigma(T_A) \cap \Gamma$  (Theorem 1). By using this result, we show that  $P_\sigma(T) \cap \Gamma$  is cyclic if  $T$  is a simplex homomorphism and satisfies some conditions (Theorem 2). In case of a lattice homomorphism  $T$ ,  $P_\sigma(T) \cap \Gamma$  is always cyclic [5, V.4.2 Corollary 2]. As for a simplex homomorphism, we shall give a counter example which shows that the peripheral point spectrum of a simplex homomorphism is not necessarily cyclic.

### § 2. Simplex homomorphism.

An ordered Banach space  $E$  is said to be a *simplex space* if its dual space is an *AL-space* [3]. Due to E. B. Davies [2, Theorem 4.4], a simplex space is a regular ordered Banach space with the Riesz separation property of type  $M$ . An ordered Banach space  $E$  is said to be *regular* if it has the properties

- (i) if  $f, g \in E$  and  $-f \leq g \leq f$ , then  $\|g\| \leq \|f\|$
- (ii) if  $f \in E$  and  $\varepsilon > 0$ , then there is some  $g \in E$  with  $g \geq f$ ,  $-f$  and  $\|g\| \leq \|f\| + \varepsilon$ .

$E$  is said to have the *Riesz separation property* if  $a, b, c, d \in E$  and  $a, b \leq c, d$  imply the existence of  $f \in E$  with

$$a, b \leq f \leq c, d.$$

$E$  is said to be of *type M* if for any non-negative elements  $f, g$  of  $E$  and any  $\varepsilon > 0$ , there exists  $h \in E$  such that

$$h \geq f, g \quad \text{and} \quad \|h\| \leq \max\{\|f\|, \|g\|\} + \varepsilon.$$

Let  $X$  be the set  $\{x \in E'; x \geq 0, \|x\| \leq 1\}$  endowed with the weak\*-topology. Then  $X$  is a simplex and  $E$  may be identified with  $A_0(X)$ , the space of all continuous affine functions on  $X$  vanishing at 0. For each  $x \in X$ , there is a unique maximal representing measure  $\mu_x$  on  $X$  supported by  $\overline{\partial_e X}$  (the weak\*-closure of the set  $\partial_e X$  of all extreme points of  $X$ ). By using this measure, we may further identify  $E$  [4, Theorem 3.3] with the space  $A_0(\overline{\partial_e X})$  ( $= \{f \in C(\overline{\partial_e X}); f(x) = \int f d\mu_x$  for all  $x \in \overline{\partial_e X}$  and  $f(0) = 0\}$ ).

We call  $T \in \mathfrak{S}(E)$  a *simplex homomorphism* if for any  $f, g \in E$  and any  $x \in \partial_e X$ , there exists  $h \in E$  such that  $h \geq f, g$  and  $Th(x) = \max\{Tf(x), Tg(x)\}$ . Then by [8, Theorem 2], there are a function  $\gamma(x)$  on  $\overline{\partial_e X}$  with  $0 \leq \gamma(x) \leq \|T\|$  and a mapping  $k: \overline{\partial_e X} \rightarrow \overline{\partial_e X}$  with  $k(\partial_e X) \subset \partial_e X$  satisfying

$$Tf(x) = \gamma(x) \cdot f \circ k(x) \quad \text{for any } f \in E$$

and

$$k(x_\alpha) = k(x'_\alpha) \quad \text{if } x_\alpha = c_\alpha x'_\alpha \text{ for some } c_\alpha > 0.$$

Put  $\gamma(n, x) = \prod_{j=0}^{n-1} \gamma(k^j(x))$  for any  $x \in \overline{\partial_e X}$ . Then we have

$$T^n f(x) = \gamma(n, x) \cdot f \circ k^n(x) \quad \text{for any } f \in E.$$

Let  $E_1$  be the smallest Banach sublattice of  $E''$  containing  $E$  and let  $F$  be the space  $\{f \in C(\overline{\partial_e X}); f(x_\alpha) = c_\alpha f(x'_\alpha) \text{ for all } \alpha \in \mathcal{A}\}$ , where  $\{(x_\alpha, x'_\alpha, c_\alpha)\}_{\alpha \in \mathcal{A}}$  is a subset of  $\overline{\partial_e X} \times \overline{\partial_e X} \times [0, 1]$  consisting of all the triple  $(x_\alpha, x'_\alpha, c_\alpha)$  such that  $f(x_\alpha) = c_\alpha f(x'_\alpha)$  holds for any  $f \in E$ . Then there is a lattice isomorphism  $\phi$  of  $E_1$  onto  $F$  [6, Theorem 1] and we have

LEMMA 1. *Let  $T$  be a simplex homomorphism of  $E$ . Then for any  $g \in F$ ,  $\gamma \cdot g \circ k$  belongs to  $F$  and  $\tilde{T} := \phi T'' \phi^{-1}$  is a lattice homomorphism of  $F$ .*

PROOF. By [6, Theorem 2], a simplex homomorphism keeps  $E_1$  invariant. So  $g \in F$  implies  $T'' \phi^{-1} g \in E_1$  and  $\phi T'' \phi^{-1} g \in F$ . Since  $T''$  is a lattice homomorphism of  $E''$  [8, Theorem 1] and  $\phi$  is a lattice isomorphism,  $\tilde{T} := \phi T'' \phi^{-1}$  is a lattice homomorphism. //

### § 3. Peripheral point spectrum.

Hereafter let  $E$  be a simplex space satisfying the condition

$$(C1) \quad \inf\{\|x\|; x \in \overline{\partial_e X} \setminus \{0\}\} = \alpha > 0$$

and  $T$  be a simplex homomorphism of  $E$  such that

$$(C2) \quad \sup_n \|T^n\| = b < \infty$$

and

$$(C3) \quad \sigma(T) \cap \Gamma \neq \Gamma,$$

where  $\sigma(T)$  is the spectrum of  $T$  and  $\Gamma = \{\lambda \in \mathbb{C}; |\lambda| = 1\}$ .

First we show some lemmas.

LEMMA 2. Let  $x_0 \in \overline{\partial_e X}$  satisfy  $k^n(x_0) = x_0$  for some  $n \in \mathbb{N}$  and  $k^j(x_0) \neq k^m(x_0)$  for  $0 \leq j < m \leq n-1$ . Put  $c = (\gamma(n, x_0))^{1/n}$ . If  $c \neq 0$ , then  $c \cdot \exp((2\pi i/n)j)$  belongs to the point spectrum of  $T'$  for any  $j \in \mathbb{N}$ .

PROOF. Put

$$\mu = x_0 + \sum_{m=1}^{n-1} \gamma(m, x_0) \left( \frac{1}{c} \cdot \exp\left(-\frac{2\pi i}{n}j\right) \right)^m \cdot k^m(x_0).$$

Then  $\mu \in E'$ ,  $\mu \neq 0$  and we get

$$T'\mu = \left( c \cdot \exp\left(\frac{2\pi i}{n}j\right) \right) \mu$$

by using the relation  $T'(k^m(x_0)) = \gamma(k^m(x_0)) \cdot k^{m+1}(x_0)$ . //

Put

$$N_\infty = \{x \in \overline{\partial_e X}; k^j(x) \neq k^m(x) \quad \text{if } j \neq m\}.$$

For  $x \in \overline{\partial_e X} \setminus N_\infty$ , put

$$r(x) = \min \{r \in \mathbb{N}; k^r(x) = k^s(x) \quad \text{for some } s \in \mathbb{N} \cup \{0\} \text{ with } r > s\}$$

$$s(x) = \min \{s \in \mathbb{N} \cup \{0\}; k^r(x) = k^s(x) \quad \text{for some } r \in \mathbb{N} \text{ with } r > s\}$$

$$n(x) = r(x) - s(x)$$

and  $p(x) = \min \{p \cdot n(x); p \cdot n(x) \geq s(x), p \in \mathbb{N}\}$ .

Define the mapping  $P: \overline{\partial_e X} \setminus N_\infty \rightarrow \overline{\partial_e X} \setminus N_\infty$  by  $Px = k^{p(x)}(x)$ .

Then we have  $k^{n(x)}(Px) = Px$ . The next lemma shows that  $\|T'^n x\|$  is uniformly decreasing to 0 on  $N_\infty$ .

LEMMA 3. For any  $n \in \mathbb{N}$ , there exists  $m_0 \in \mathbb{N}$  such that

$$\sup \{\|T'^m x\|; x \in N_\infty, m \geq m_0\} \leq \frac{1}{n}.$$

PROOF. Suppose there exists  $n_0 \in \mathbb{N}$  such that for any  $m' \in \mathbb{N}$  there exist  $m''(> 2m' + 1)$  and  $x_{m'} \in N_\infty$  satisfying  $\|T'^{m''} x_{m'}\| > 1/n_0$ . By the relation

$$\|T'^{m''} x_{m'}\| > \frac{\|T'^{m''+1} x_{m'}\|}{\|T'\|} \geq \frac{\|T'^{m''+1} x_{m'}\|}{b},$$

we can choose  $m(> m')$  and  $x_m (= x_{m'})$  satisfying

$$\|T'^{2m}x_m\| \geq \frac{1}{n_0b}.$$

Then we have

$$\gamma(2m, x_m) = \frac{\|T'^{2m}x_m\|}{\|k^{2m}(x_m)\|} \geq \frac{1}{n_0b}. \quad (*)$$

For any  $\theta \in \mathbf{R}$ , put  $\lambda = \exp(i\theta)$ .

We shall define a function  $f_m$  on  $\overline{\partial_e X}$  by

$$f_m(x) = \begin{cases} \{\gamma(2m-j, x)/\gamma(2m, x_m)\} \cdot \frac{j}{m} \cdot \lambda^j & \text{for } x \in k^{-2m+j}(k^{2m}(x_m)) \text{ with } j=1, 2, \dots, m-1 \\ \{\gamma(2m-j, x)/\gamma(2m, x_m)\} \cdot \left(2 - \frac{j}{m}\right) \cdot \lambda^j & \text{for } x \in k^{-2m+j}(k^{2m}(x_m)) \text{ with } j=m, m+1, \dots, 2m-1 \\ 0 & \text{for other } x \text{ on } \overline{\partial_e X}. \end{cases}$$

Then we have  $f_m \in F''$  and

$$\begin{aligned} \|f_m\| &\geq |f_m(k^m(x_m))| = \left| \frac{\gamma(m, k^m(x_m))}{\gamma(2m, x_m)} \lambda^m \right| \\ &= \frac{1}{|\gamma(m, x_m)|} = \frac{\|k^m(x_m)\|}{\|T'^m x_m\|} \geq \frac{a}{b} \end{aligned}$$

by using (C1) and (C2).

By Lemma 1,  $\tilde{T} : \phi T'' \phi^{-1}$  is a lattice homomorphism of  $F$ . By using the relation (\*), we get

$$\begin{aligned} \|\tilde{T}'' f_m - \lambda f_m\| &= \sup \{ |\gamma(x) f_m(k(x)) - \lambda f_m(x)| ; x \in \bigcup_{j=0}^{2m} k^{-2m+j}(k^{2m}(x_m)) \} \\ &= \max_{0 \leq j \leq 2m} \frac{\gamma(2m-j, x)}{\gamma(2m, x_m)} \cdot \frac{1}{m} \leq \frac{n_0 b^2}{a} \cdot \frac{1}{m}. \end{aligned}$$

So by letting  $m \rightarrow \infty$ , we obtain that  $\lambda$  belongs to the approximate point spectrum of  $\tilde{T}''$ . Since  $\sigma(\tilde{T}'') \cap \Gamma = \sigma(\tilde{T}) \cap \Gamma = \sigma(T) \cap \Gamma$ , this leads to a contradiction to (C3). Therefore for any  $n_0 \in \mathbf{N}$ , there exists  $m_0 \in \mathbf{N}$  such that  $\sup \{\|T'^m x\| ; m \geq m_0, x \in N_\infty\} \leq 1/n_0$ . //

COROLLARY. For  $n_1, n_2 \in \mathbf{N}$ , put

$$B_{n_1, n_2} = \left\{ x \in \overline{\partial_e X} ; \sup_{m \geq n_2} \|T'^m x\| > \frac{1}{n_1} \right\}.$$

Then for any  $n \in \mathbf{N}$ , there exists  $m_0 \in \mathbf{N}$  such that

$$\sup \{r(x) ; x \in B_{n, m_0}\} < \infty.$$

PROOF. Suppose that there exists  $n_0 \in \mathbf{N}$  such that for any  $m \in \mathbf{N}$ ,

there exists  $x_m \in B_{n_0, 2m+1}$  such that  $r(x_m) > 2m+1$ . This implies  $\|T'^{2m}x_m\| \geq 1/n_0b$ . So in the same way as the proof of lemma 3, this leads to a contradiction to (C3). Therefore for any  $n \in N$ , there exists  $m_0 \in N$  such that  $\sup \{r(x); x \in B_{n, m_0}\} < \infty$ . //

Put

$$N_0 = \{x \in \overline{\partial_e X}; \lim_n \|T'^n x\| = 0\}.$$

Then lemma 3 implies that  $N_\infty \subset N_0$ . Furthermore the next lemma shows that the set  $\{n(x); x \in \overline{\partial_e X} \setminus N_0\}$  is bounded.

LEMMA 4. i) For any  $x \in \overline{\partial_e X} \setminus N_0$ , we have

$$\gamma(n(x), Px) = 1.$$

ii) There exists  $M_0$  such that

$$\sup \{n(x); x \in \overline{\partial_e X} \setminus N_0\} \leq M_0.$$

PROOF. i) For  $x \in \overline{\partial_e X} \setminus N_0$ ,  $n(x)$  and  $r(x)$  are finite numbers and we have a relation:

$$\begin{aligned} T'^{(s \cdot n(x) + p(x))} x &= \gamma(p(x), x) T'^{s \cdot n(x)} Px \\ &= \gamma(p(x), x) \cdot (\gamma(n(x), Px))^s Px \quad \text{for any } s \in N. \end{aligned}$$

So the condition (C2) implies  $\gamma(n(x), Px) = 1$ .

ii) Since  $\sigma(T)$  is closed, (C3) implies that there exist  $c_1$  and  $c_2$  such that  $0 < c_1 < c_2 < 1$  and

$$\{\exp(2\pi i\theta); c_1 < \theta < c_2\} \subset \rho(T) \quad (= \text{the resolvent set of } T). \quad (**)$$

Put  $M_0 = 1/(c_2 - c_1)$ . Since for any  $j = 0, 1, \dots, n(x) - 1$ ,  $\exp((2\pi i/n(x))j)$  belongs to  $P_e(T')$  by i) and lemma 2, the relation (\*\*) implies that  $n(x)$  is less than  $M_0$  for any  $x \in \overline{\partial_e X} \setminus N_0$ . //

Let  $M$  be the least common multiple of the set  $\{n(x); x \in \overline{\partial_e X} \setminus N_0\}$ . Then  $M$  is a finite number by lemma 4 and the following is easily obtained.

LEMMA 5. For any  $x \in \overline{\partial_e X} \setminus N_0$ , we have

- i)  $k^M(Px) = Px$  and  $\gamma(M, Px) = 1$ ,
- ii)  $P(k^j(x)) = k^j(Px)$  for any  $j \in N$ ,
- iii)  $k^{mM}(x) = Px$  and  $\gamma(p(x), x) = \gamma(mM, x)$  for sufficiently large  $m \in N$ .

LEMMA 6. Let  $f$  be an element of  $E$  satisfying  $f(x) = 0$  for all  $x \in N_0$ . Put

and

$$\begin{aligned} g(x) &= \gamma(p(x), x) \cdot f(Px) \quad \text{for } x \in \overline{\partial_e X} \setminus N_0 \\ g(x) &= 0 \quad \text{for } x \in N_0. \end{aligned}$$

Then we have  $g \in E$  and  $g(x) = \lim_{m \rightarrow \infty} T^{mM} f(x)$  for any  $x \in \overline{\partial_e X}$ .

PROOF. For any  $x \in \overline{\partial_e X} \setminus N_0$ , we have

$$\begin{aligned} \lim_{m \rightarrow \infty} T^{mM} f(x) &= \lim_{m \rightarrow \infty} \gamma(mM, x) \cdot f(k^{mM}(x)) \\ &= \gamma(p(x), x) \cdot f(Px) = g(x). \end{aligned}$$

By corollary to lemma 3, for any  $n \in \mathbf{N}$ , there is  $m_0 \in \mathbf{N}$  such that  $c_n := \sup \{r(x); x \in B_{n, m_0}\}$  is finite. Let  $b_n$  be an integer such that  $M \cdot b_n \geq c, m_0$ . Let  $j \geq b_n$ . Then for any  $x \in B_{n, m_0}$ , we have  $k^{jM}(x) = Px$  and  $g(x) = T^{jM} f(x)$ . For any  $x \in \overline{\partial_e X} \setminus B_{n, m_0}$ , we have  $\|T^{jM} x\| \leq 1/n$  and  $|g(x)| \leq \sup_{j \geq b_n} |T^{jM} f(x)| \leq (1/n) \|f\|$ . So we have for  $j \geq b_n$ ,

$$\|T^{jM} f - g\| = \sup \{|T^{jM} f(x) - g(x)|; x \in \overline{\partial_e X} \setminus B_{n, m_0}\} \leq \frac{2}{n} \|f\|.$$

Therefore  $g$  is the limit of a norm convergent sequence  $\{T^{jM} f\}$  of  $E$  and  $g$  belongs to  $E$ . //

We recall that a convex subset  $F$  of  $X$  is said to be a *face* if given  $x, y \in X$  and  $0 < \alpha < 1$  with  $\alpha x + (1 - \alpha)y \in F$ , it follows that  $x, y \in F$ .

LEMMA 7. Denote the smallest face of  $X$  containing  $x \in X$  by  $F(x)$ . Then  $x_0 \in N_0$  implies  $F(x_0) \cap \overline{\partial_e X} \subset N_0$ .

PROOF. It is easily seen that  $F(x_0)$  is the set  $\{y \in X; y \leq cx_0 \text{ for some } c > 0\}$ . For  $y \in F(x_0) \cap \overline{\partial_e X}$  there is some  $c > 0$  such that  $y \leq cx_0$ . Since  $T'$  is positive, we have  $0 \leq T'^n y \leq c \cdot T'^n x_0$ . Therefore  $0 \leq \|T'^n y\| \leq c \|T'^n x_0\|$  for any  $n \in \mathbf{N}$ . Hence  $x_0 \in N_0$  implies  $y \in N_0$ . //

A linear subspace  $I$  of  $E$  is said to be an *ideal* if it has the properties:

(i)  $0 \leq f \leq g \in I$  implies  $f \in I$ .

(ii) If  $f \in I$ , then there is some  $g \in I$  with  $g \geq f, -f$ .

The set  $\{\lambda; |\lambda| = r(T)\} \cap P_\sigma(T)$  is called the *peripheral point spectrum* of  $T$ , where  $r(T)$  is the *spectral radius* of  $T$ . Now we have the main results.

THEOREM 1. Let  $E$  be a simplex space such that  $\inf \{\|x\|; x \in \overline{\partial_e X} \setminus \{0\}\} > 0$  and  $T$  be a simplex homomorphism of  $E$  satisfying  $r(T) = 1$ ,  $\sup \|T^n\| < \infty$  and  $\sigma(T) \cap \Gamma \neq \Gamma$ . Suppose  $\partial_e X \supset \bar{N}_0 \setminus N_0$ , where  $N_0 = \{x \in \overline{\partial_e X}; \lim_{n \rightarrow \infty} \|T^n x\| = 0\}$ . Put  $I = \{f \in E; f(x) = 0 \text{ for all } x \in N_0\}$  and  $A = \{f \in I;$

$f(x) = \gamma(p(x), x) \cdot f(Px)$  for any  $x \in \overline{\partial_e X} \setminus N_0$ , where  $P$  and  $p(x)$  are defined before lemma 3. Then we have

- i)  $I$  is a  $T$ -invariant closed ideal of  $E$ .
- ii)  $A$  is a  $T$ -invariant simplex subspace<sup>2)</sup> of  $E$ , which contains every eigenfunctions pertaining to the peripheral point spectrum of  $T$ .
- iii) The restriction  $T_I$  of  $T$  to  $I$  is uniformly ergodic simplex homomorphism and  $P_e(T_I) \cap \Gamma = P_e(T) \cap \Gamma$ .
- iv) The restriction  $T_A$  of  $T$  to  $A$  is a simplex isomorphism and we have  $T_A^M = I_A$ <sup>3)</sup>, where  $M$  is the number defined before Lemma 5.

PROOF. i) If  $x \in \overline{N_0} \cap (\overline{\partial_e X} \setminus \partial_e X)$ , the assumption  $\partial_e X \supset \overline{N_0} \setminus N_0$  implies  $x \in N_0$ . By lemma 7,  $F(x) \cap \overline{\partial_e X} \subset N_0$ . So the convex closure of  $\overline{N_0}$  is a closed face of  $X$  [1, Problem 28.7] and  $I$  is a closed ideal of  $E$ . Since  $x \in N_0$  implies  $k(x) \in N_0$ ,  $I$  is  $T$ -invariant.

ii) In order to show that  $A$  is a simplex space, it is enough to show that  $A$  is a regular ordered Banach space with the Riesz separation property of type  $M$  as described in § 2. At first, we show that  $A$  has the Riesz separation property. Let  $g_1, g_2, f_1, f_2 \in A$  satisfy  $g_1, g_2 \leq f_1, f_2$ . Since  $E$  has the Riesz separation property, there exists  $h \in E$  such that  $g_1, g_2 \leq h \leq f_1, f_2$ . So  $h \in I$ . Put

$$\tilde{h}(x) = \begin{cases} \gamma(p(x), x) \cdot h(Px) & \text{for } x \in \overline{\partial_e X} \setminus N_0 \\ 0 & \text{for } x \in N_0. \end{cases}$$

Then by Lemma 6,  $\tilde{h} \in E$ . For any  $x \in \overline{\partial_e X} \setminus N_0$ ,  $\tilde{h}(Px) = h(Px)$  holds by Lemma 4. Therefore we have  $\tilde{h} \in A$ . It is easily seen that  $g_1, g_2 \leq \tilde{h} \leq f_1, f_2$ . So  $A$  has the Riesz separation property.

Next we show that  $A$  is regular. For  $f \in A$ , there exists  $h \in I$  such that  $h \geq f, -f$  since  $I$  is an ideal of  $E$  by i). By Lemma 6, we have  $h_0 := \lim_{j \rightarrow \infty} T^{jM} h \in A$  and  $h_0 \geq f, -f$ . Put

$$\phi(x) = \inf \{h(x); h \in E, h \geq f, -f\}$$

and

$$\phi(h) = \inf \{h(x); h \in A, h \geq f, -f\} \quad (***)$$

for all  $x \in X$ . Then  $\phi$  is a bounded function on  $X$  and  $\phi(x) = 0$  for any  $x \in \overline{N_0}$ . It is clear that  $\phi \geq \phi \geq 0$  and  $\phi$  is upper semi-continuous affine on  $X$  and  $\phi(x) = |f(x)|$  for any  $x \in \partial_e X$ , since  $X$  is a simplex [1, § 28]. Since  $A$  has the Riesz separation property, the set  $\{h \in A; h \geq f, -f\}$  is directed downward and so  $\phi$  is upper semi-continuous affine on  $X$ . If  $\phi \neq \phi$ , there

2) A subspace  $A$  of  $E$  is said to be a simplex subspace if  $A$  itself is a simplex space.

3)  $I_A$  is the identity operator in  $A$ .

exists  $x_0 \in \partial_e X \setminus N_0$  such that  $\phi(x_0) > \phi(x_0)$  by [8, Lemma 1]. Then we get

$$\begin{aligned}\phi(Px_0) &= \{\gamma(p(x), x_0)\}^{-1} \cdot \phi(x_0) > \{\gamma(p(x), x_0)\}^{-1} \cdot \phi(x_0) \\ &= \{\gamma(p(x), x_0)\}^{-1} \cdot |f(x_0)| = |f(Px_0)| = \phi(Px_0)\end{aligned}$$

by using  $Px_0 \in \partial_e X$ . So there exists  $\varepsilon_0 > 0$  such that  $\phi(Px_0) - \varepsilon_0 > \phi(Px_0)$ . Put

$$g(x) = \begin{cases} 0 & x \in \bar{N}_0 \\ \phi(Px_0) - \varepsilon_0 & x = Px_0 \\ \|\phi\| & x \in X \setminus (\bar{N}_0 \cup \{Px_0\}). \end{cases}$$

Then  $g$  is a lower semi-continuous concave function on  $X$  and  $g \geq \phi$ . By [1, 28.6 (vii)], there exists  $h \in E$  such that  $g \geq h \geq \phi$ . Then  $h \in I$ . So by lemma 6, we have  $h_0 := \lim_{j \rightarrow \infty} T^{jM} h \in A$ . Furthermore we have  $h_0 \geq f, -f$  and  $h_0(Px_0) = h(Px_0) \leq g(Px_0) = \phi(Px_0) - \varepsilon_0$ . This is a contradiction to (\*\*). Therefore  $\phi = \phi$ . So for any  $\varepsilon > 0$  and for any  $x \in \bar{\partial_e X}$  we can find  $h_x \in A$  such that  $h_x > \phi$  and  $\phi(x) > h_x(x) - \varepsilon$ . Since  $\bar{\partial_e X}$  is compact and  $A$  has the Riesz separation property, we have  $g_0 \in A$  such that  $g_0 \geq \phi \geq f, -f$  and  $\|g_0\| \leq \|\phi\| + \varepsilon = \|f\| + \varepsilon$ . So  $A$  is a regular ordered Banach space. We can show that  $A$  is of type  $M$  in a similar way. Therefore  $A$  is a simplex space.

For  $f \in A$ , we have  $Tf \in A$  by using the relation:

$$\begin{aligned}Tf(x) &= \gamma(x) \cdot f(k(x)) = \gamma(p(x) + 1, x) \cdot f(Pk(x)) \\ &= \gamma(p(x), x) \cdot Tf(Px) \quad \text{for any } x \in \bar{\partial_e X} \setminus N_0.\end{aligned}$$

So  $A$  is a  $T$ -invariant simplex subspace of  $E$ .

Suppose  $Tf = \alpha f$ ,  $|\alpha| = 1$ . Then  $\sigma(T'') \cap \Gamma = \sigma(T) \cap \Gamma \neq \Gamma$  and the periodicity of  $\sigma(T'')$  [5, V.4.4] imply that there exist  $p$  and  $q \in \mathbf{N}$  such that  $\alpha = \exp((q/p)2\pi i)$ . We have

$$f(T'^{pj}x) = T^{pj}f(x) = f(x) \quad \text{for all } j \in \mathbf{N}.$$

So  $f(x) = 0$  holds for all  $x \in N_0$ . For any  $x \in \bar{\partial_e X} \setminus N_0$ , we have

$$\begin{aligned}f(x) &= T^{p \cdot p(x)}f(x) = \gamma(p \cdot p(x), x) \cdot f(k^{p \cdot p(x)}(x)) \\ &= \gamma(p(x), x) \cdot f(Px) \quad \text{by Lemma 6.}\end{aligned}$$

Therefore  $f \in A$ .

iii) Suppose  $f \in I$ . Then  $g = \lim_{j \rightarrow \infty} T^{jM} f$  exists as an element of  $E$  by Lemma 6. Moreover, in the proof of Lemma 6, we have that for any  $n \in \mathbf{N}$ , there exists  $b_n \in \mathbf{N}$  such that  $\|T^{jM} f - g\| \leq (2/n)\|f\|$  for any  $j \geq b_n$ , which implies that  $T_I^M$  is mixing and  $T_I$  is uniformly ergodic.  $P_\sigma(T) \cap \Gamma = P_\sigma(T_I) \cap \Gamma$  follows from ii) since  $I$  contains  $A$ .

iv) For any  $f \in A$  and any  $x \in \overline{\partial_e X} \setminus N_0$ , we have

$$\begin{aligned} T^M f(x) &= \gamma(M, x) \cdot f(k^M(x)) \\ &= \gamma(M, x) \cdot \gamma(p(x), k^M(x)) \cdot f(Pk^M(x)) \\ &= \gamma(p(x) + M, x) \cdot f(Px) = \gamma(p(x), x) \cdot f(Px) \\ &= f(x). \end{aligned}$$

Therefore  $T_A^M = I_A$  and  $T_A$  is a uniformly ergodic, simplex isomorphism of  $A$ . //

We recall that a subset  $S \subset C$  is said to be *cyclic* if  $\alpha \in S$ ,  $\alpha = |\alpha|\lambda$  implies  $|\alpha|\lambda^j \in S$  for all  $j \in \mathbb{Z}$ .

As for the peripheral point spectrum, we have

**THEOREM 2.** *Let  $E$  be a simplex space such that  $\inf \{\|x\|; x \in \overline{\partial_e X} \setminus \{0\}\} > 0$  and  $T$  be a simplex homomorphism of  $E$  satisfying  $r(T) = 1$ ,  $\sup \|T^n\| < \infty$  and  $\sigma(T) \cap \Gamma \neq \Gamma$ . Suppose  $\partial_e X \supset \overline{N_0} \setminus N_0$ , where  $N_0 = \{x \in \overline{\partial_e X}; \lim_{n \rightarrow \infty} \|T^n x\| = 0\}$ . Then the peripheral point spectrum of  $T$  is cyclic.*

**PROOF.** Suppose  $Tf = \alpha f$ ,  $|\alpha| = 1$  and  $f \in E$ . Then by Theorem 1,  $f$  belongs to  $A$  defined at Theorem 1 and  $T_A^M = I_A$ . By the spectral mapping theorem,  $\lambda \in \sigma(T)$  implies  $\lambda^M = 1$ . With this fact and by [7, Theorem 3 and Theorem 5], it follows that  $\sigma(T_A) \cap \Gamma$  is cyclic and consists of poles of  $R(\lambda, T_A)$ . Hence for any  $j \in \mathbb{N}$ , there exists  $g_j \in A$  such that  $T_A g_j = \alpha^j g_j$ . Therefore  $T g_j = \alpha^j g_j$  and  $g_j \in E$ , which shows that the peripheral point spectrum of  $T$  is cyclic. //

If  $T$  is a Markov operator, the above theorems can be rewritten as follows.

**COROLLARY.** *Let  $E$  be a simplex space with the order unit  $\mathbf{1}$  and  $T$  be a simplex homomorphism of  $E$  such that  $T\mathbf{1} = \mathbf{1}$  and  $\sigma(T) \cap \Gamma \neq \Gamma$ . Then  $T$  is uniformly ergodic and the peripheral point spectrum of  $T$  is cyclic.*

If we omit the condition  $\partial_e X \supset \overline{N_0} \setminus N_0$  in Theorem 2, the peripheral point spectrum is not necessarily cyclic as shown in the following example.

**EXAMPLE.** Let  $K$  be the subset

$$\{1, 2, 3, 4\} \cup \bigcup_{j=1}^{\infty} [3+2j, 4+2j]$$

of  $\mathbb{R}$  and  $E$  be the space

$$\left\{ f \in C(K); f(3+2j) = \frac{1}{2} \{f(j) + f(j+2)\}, j=1, 2 \right\}.$$

Let  $T$  be defined by

$$Tf(x) = \begin{cases} f(x+1) & x=1, 2, 3 \\ f(1) & x=4 \\ (6-x)f(x+2) & 5 \leq x \leq 6 \\ (8-x)f(x-2) & 7 \leq x \leq 8. \end{cases}$$

Then  $E$  is a simplex space,  $T$  is a simplex homomorphism of  $E$ ,  $\partial_e X \simeq (K \setminus \{5, 7\})$  and  $N_0 \simeq \bigcup_{j=1}^2 (3+2j, 4+2j]$ . Therefore  $\bar{N}_0 \simeq \bigcup_{j=1}^2 [3+2j, 4+2j]$  and  $\partial_e X \not\supset \bar{N}_0 \setminus N_0$ . Let

$$f_0(x) = \begin{cases} i^x & x=1, 2, 3, 4 \\ 0 & x \in [5, 8] \cap K, \end{cases}$$

where  $i$  is the imaginary unit.

Then  $f_0 \in E$  and  $Tf_0 = i \cdot f_0$ . So  $\alpha = i$  belongs to the point spectrum of  $T$ . But  $\alpha^2 = i^2 = -1$  does not belong to the point spectrum of  $T$ .

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