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On the Shapes of Geodesics in the Fubinian Space

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§ 0. **Introduction.** The Fubinian space is a typical example of the Kählerian space of constant holomorphic sectional curvature, and locally coincides with such a Kählerian space. In this paper, we shall study what forms the geodesics of the Fubinian space take with respect to the Euclidean metric, and shall show a certain representation which has some analogy to Klein's and Poincaré's models of the non-Euclidean geometry. T. Ōtsuki and Y. Tashiro showed that the geodesics of the Fubinian space are the Möbius circles in the complex projective space ([1]), but do not give shape to them in the Euclidean space. Here we shall investigate them in the Euclidean sense more precisely.

The author should like to express her hearty gratitude to Prof. Shunichi Tachibana for his advice.

§ 1. **The Fubinian space.** In the n-dimensional complex number space $\boldsymbol{C}^n$ with coordinates $z^k=x^k+iy^k(x^k, y^k\in\boldsymbol{R})$, $k=1, \cdots, n$, the relations

$$\frac{\partial}{\partial z^k}=\frac{1}{2}\left(\frac{\partial}{\partial x^k}-i\frac{\partial}{\partial y^k}\right), \quad \frac{\partial}{\partial \bar{z}^k}=\frac{1}{2}\left(\frac{\partial}{\partial x^k}+i\frac{\partial}{\partial y^k}\right) \tag{1.1}$$

hold between the real and the complex natural basis.

A vector $\boldsymbol{a}$ in $\boldsymbol{C}^n$ is written as

$$\boldsymbol{a}=\sum_{k=1}^{n}\left(A^k\frac{\partial}{\partial z^k}+A^{k^*}\frac{\partial}{\partial \bar{z}^k}\right), \qquad A^k, A^{k^*}\in\boldsymbol{C}, \tag{1.2}$$

$$k^*=n+k, \qquad (k=1, \cdots, n).$$

$\boldsymbol{a}$ is called a real vector if it satisfies

$$A^{k^*}=\bar{A}^k \qquad \text{(complex conjugate).} \tag{1.3}$$

Then, putting

$$A^k=a^k+ia^{k^*}, \qquad a^k, a^{k^*}\in\boldsymbol{R},$$

it holds that

$$A^{k^*}=a^k-ia^{k^*},$$

and

$$\boldsymbol{a}=\sum_{k=1}^{n}\left(a^k\frac{\partial}{\partial x^k}+a^{k^*}\frac{\partial}{\partial y^k}\right).$$

Hence a real vector is a vector in $\boldsymbol{R}^{2n}$ identified with $\boldsymbol{C}^n$ naturally. In the following of this paper, a vector means a real vector, and for a real vector $\boldsymbol{a}$, we call (A^k, A^{k^*}) satisfying (1.3) its complex component, and (a^k, a^{k^*}) its real component.

We define the linear transformation J of $\boldsymbol{C}^n$ as an $2n$-dimensional real vector space by

$$J: \frac{\partial}{\partial x^k}\longmapsto\frac{\partial}{\partial y^k}, \qquad \frac{\partial}{\partial y^k}\longmapsto -\frac{\partial}{\partial x^k}.$$

Then the complexification of J (denoted by J again) is the transformation

$$\frac{\partial}{\partial z^k}\longmapsto i\frac{\partial}{\partial z^k}, \qquad \frac{\partial}{\partial \bar{z}^k}\longmapsto -i\frac{\partial}{\partial \bar{z}^k},$$

and for a vector $\boldsymbol{a}$, we have

$$(1.4)\qquad J\boldsymbol{a}=\sum_{k=1}^{n}\left(iA^k\frac{\partial}{\partial z^k}-iA^{k^*}\frac{\partial}{\partial \bar{z}^k}\right)=\sum_{k=1}^{n}\left(-a^{k^*}\frac{\partial}{\partial x^k}+a^k\frac{\partial}{\partial y^k}\right),$$

i. e., the real and complex component of $J\boldsymbol{a}$ are $(iA^k, -iA^{k^*})$ and $(-a^{k^*}, a^k)$ respectively.

We call a plane spanned by a vector $\boldsymbol{a}$ and $J\boldsymbol{a}$ a holomorphic plane of $\boldsymbol{a}$.

Let $\langle\,,\rangle$ be the Euclidean inner product, *i. e.*, for two vectors $\boldsymbol{a}=(a^k, a^{k^*})=(A^k, A^{k^*})$ and $\boldsymbol{b}=(b^k, b^{k^*})=(B^k, B^{k^*})$

$$(1.5)\qquad \langle\boldsymbol{a}, \boldsymbol{b}\rangle=\sum_{k-1}^{n}(a^kb^k+a^{k^*}b^{k^*})=\frac{1}{2}\sum_{k=1}^{n}(A^kB^{k^*}+A^{k^*}B^k)$$

Now we define a function $S(z, \bar{z})$ by

$$(1.6)\qquad S(z, \bar{z})=1+\frac{k}{2}\sum_{k=1}^{n}z^k\bar{z}^k,$$

where k is a real constant. We consider the domain $F^n: S(z, \bar{z})>0$, and introduce a Kählerian metric g by

$$(1.7)\qquad \begin{aligned} g_{ij^*}&=\frac{2}{k}\frac{\partial^2\log S(z, \bar{z})}{\partial z^i\partial\bar{z}^j} \\ g_{i^*j}&=\overline{g_{ij^*}}, \qquad \text{otherwise } 0. \end{aligned}$$

$\{F^n, g\}$ is called an n-dimensional Fubinian space, and the surface $S(z, \bar{z})=0$ its absolute.

It is the purpose of this paper to study what forms the geodesics of the Fubinian space take with respect to the Euclidean metric, provided that $k/2=-l^2<0$.

The covariant and contravariant components of the metric tensor g and the Christoffel symbols of the n-dimensional Fubinian space are as follows:

$$g_{ij^*}=\frac{\delta_{ij}}{S}+l^2\frac{\bar{z}^i z^j}{S^2} \qquad g^{ij^*}=S(\delta_{ij}-l^2 z^i \bar{z}^j)\,, \tag{1.8}$$

$$\Gamma^i_{jk}=\frac{l^2}{S}(\delta^i_j \bar{z}^k+\delta^i_k \bar{z}^j)\,. \tag{1.9}$$

Let $c: z^k=z^k(t)$ be a geodesic, where t is a parameter. We draw the perpendicular with respect to the Euclidean metric from the origin O to c, and denote its foot by P. We may assume that $t=0$ at the point P. Let $\boldsymbol{a}$ be the unit tangent vector at P in the Euclidean sense, and $\boldsymbol{b}$ the position vector of P. Denoting the complex component of the vector $\boldsymbol{a}$ and $\boldsymbol{b}$ by (A^k, A^{k^*}) and (B^k, B^{k^*}) respectively, we have the relations between $\boldsymbol{a}$ and $\boldsymbol{b}$

$$\langle \boldsymbol{a}, \boldsymbol{b}\rangle=\frac{1}{2}\sum_{k=1}^{n}(A^k B^{k^*}+A^{k^*}B^k)=0\,, \tag{1.10}$$

$$\langle \boldsymbol{a}, \boldsymbol{a}\rangle=\frac{1}{2}\sum_{k=1}^{n}(A^k A^{k^*}+A^{k^*}A^k)=\sum_{k=1}^{n}A^k A^{k^*}=1\,. \tag{1.11}$$

We shall investigate the shapes of the geodesics of the Fubinian space in the two cases, in which $\boldsymbol{b}$ is perpendicular or not in the Euclidean sense to the holomorphic plane of $\boldsymbol{a}$.

§ 2. The geodesics in the case of $\langle J\boldsymbol{a}, \boldsymbol{b}\rangle=0$.

In this case, we have $\sum A^k B^{k^*}=\sum A^{k^*}B^k=0$ from $\langle J\boldsymbol{a}, \boldsymbol{b}\rangle=0$ and $\langle \boldsymbol{a}, \boldsymbol{b}\rangle=0$.

The differential equation of geodesics

$$z''^i+\sum_{j,k}\Gamma^i_{jk}z'^j z'^k=0 \tag{2.1}$$

can be written in our Fubinian space as follows:

$$z''^i=-\frac{2l^2}{S}\Big(\sum_k z'^k \bar{z}^k\Big)z'^i\,, \tag{2.2}$$

where $'$ is the differentiation with respect to the arc length s measured from the point P.

Let us consider the line

$$z^i=A^i t+B^i \tag{2.3}$$

where t is a function of s satisfying $t'>0$. We shall show that the line (2.3) satisfies (2.1) with a suitable function t of s.

We differentiate (2.3) by s:

(2.4) $$z'^i = A^i t'$$

(2.5) $$z''^i = A^i t''\,.$$

Using (1.10) and (1.11), and putting $b^2 = \sum B^k B^{k^*}$, we have

(2.6) $$S(z(t), \bar{z}(t)) = 1 - l^2(t^2 + b^2)$$

on the line (2.3). As $S(z, \bar{z}) > 0$ holds on this line, it follows that

(2.7) $$1 - l^2(t^2 + b^2) > 0\,.$$

Hence, the range of t is

(2.8) $$-m_0 < t < m_0\,,$$

where $m_0 = \sqrt{1 - l^2 b^2}/l$. Especially at the point $t = 0$, it holds that

(2.9) $$1 - l^2 b^2 > 0$$

Substituting (2.3), (2.4), (2.5) and (2.6) into (2.2), it follows that

$$A^i t'' = -\frac{2l^2}{1 - l^2(t^2 + b^2)} A^i t t'^2$$

and

$$t''/t' = -2l^2 t t'/\{1 - l^2(t^2 + b^2)\}\,.$$

Integrating it and taking account of $t' > 0$ and (2.7), we get

$$\log t' = \log\{1 - l^2(t^2 + b^2)\} + C$$

where C is the integral constant. Hence

$$t'/\{1 - l^2(t^2 + b^2)\} = C\,, \qquad C > 0: \text{ const.}\,.$$

Rewriting it as

$$t'\left(\frac{1}{t - m_0} - \frac{1}{t + m_0}\right) = C\,, \qquad C < 0: \text{ const.}\,,$$

we integrate and have

$$\log\left(-\frac{t - m_0}{t + m_0}\right) = Cs + D$$

$$C < 0\,, \qquad D: \text{ const.}$$

on taking account of (2.8). As $D = 0$ follows from $t(0) = 0$,

$$-\frac{t - m_0}{t + m_0} = e^{Cs}$$

holds, and we obtain

(2.10) $$t = m_0 \frac{1 - e^{Cs}}{1 + e^{Cs}}\,, \qquad C < 0\,.$$

In the next, we shall determine C, using the relation

(2.11) $$2 \sum_{i,j} g_{ij^*} \frac{dz^i}{ds} \frac{d\bar{z}^j}{ds} = 1 \qquad \text{at } P\,.$$

Differentiating (2.11) with respect to s, it follows that

$$\frac{dt}{ds}=-m_0\frac{2Ce^{Cs}}{(1+e^{Cs})^2}\,.$$

On the geodesic, it holds that

$$\frac{dz^i}{ds}=A^i\frac{dt}{ds}=-m_0A^i\frac{2Ce^{Cs}}{(1+e^{Cs})^2}$$

and

$$\frac{dz^i}{ds}(0)=-\frac{C}{2}m_0A^i \qquad \text{at } P\,.$$

On the other hand we have

$$g_{ij^*}=\frac{\delta_{ij}}{1-l^2b^2}+l^2\frac{B^{i^*}B^j}{(1-l^2b^2)^2} \qquad \text{at } P\,.$$

Substituting them into (2.11), it follows that

$$2\sum_{i,j}\left\{\frac{\delta_{ij}}{1-l^2b^2}+l^2\frac{B^i B^j}{(1-l^2b^2)^2}\right\}A^iA^{j^*}\frac{C^2}{4}\frac{1-l^2b^2}{l^2}=1\,,$$

from which we obtain

$$C=-\sqrt{2}\,l\,.$$

Thus we see that the line (2.3) with

$$t=m_0\frac{1-e^{-\sqrt{2}ls}}{1+e^{-\sqrt{2}ls}} \tag{2.12}$$

satisfies the equation of geodesics. Since a geodesic is determined uniquely for a point and a direction, we conclude that the geodesic through the point P with the direction $\boldsymbol{a}$ is the line.

Next, we shall investigate how the distance between two points on this line is expressed. Let X, Y be the points of intersection of the line and the absolute, and t_+, t_- be the values of t at X, Y, i.e. $t_+=m_0$, $t_-=-m_0$. It is easy to see $s=\pm\infty$ if $t=t_\pm$. We take an arbitrary two points Q and R on this line, and denote the values of t and s at the points by t_1, t_2, s_1, s_2. The length of QR along the geodesic is s_2-s_1, and it holds that

$$t_i=m_0\frac{1-e^{-\sqrt{2}ls_i}}{1+e^{-\sqrt{2}ls_i}}, \qquad i=1,2\,.$$

Hence the non-harmonic ratio (QR, XY) with respect to the parameter t

$$(QR,XY)=\frac{QX}{QY}\Big/\frac{RX}{RY}=\frac{t_+-t_1}{t_--t_1}\frac{t_--t_2}{t_+-t_2}$$

becomes

$$(QR, XY)=e^{\sqrt{-k}\,(s_2-s_1)}$$

in terms of s_1 and s_2, and the distance $d(Q, R)$ between Q and R can be represented as

$$d(Q, R)=s_2-s_1=(1/\sqrt{-k})\log(QR, XY)\,.$$

§ 3. The geodesics of the case of $\langle J\boldsymbol{a}, \boldsymbol{b}\rangle\neq 0$. At the foot P of the perpendicular from the origin O to a geodesic, we consider the holomorphic plane spanned by $\boldsymbol{a}$ and $J\boldsymbol{a}$. The circle formed as the intersection of this holomorphic plane and the absolute is denoted by O'. We denote its center by the same letter O'. We consider the circle A on this plane with a center A on the line $O'P$ and perpendicular to the circle O'. $\boldsymbol{h}=(H^k, H^{k*})$ denotes the vector $\overrightarrow{OO'}$. Such circle A can be written as

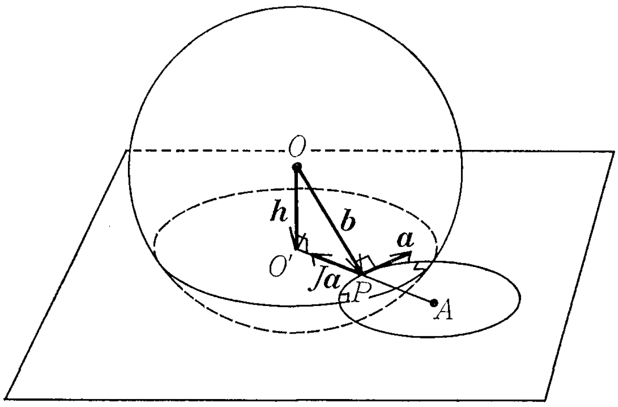


Fig. 1

(3.1) $$z(t)=\overrightarrow{OA}+\boldsymbol{e}\cos t+\boldsymbol{f}\sin t\,,$$

where

(3.2) $$\boldsymbol{f}=r\boldsymbol{a}=(F^k, F^{k*})$$

(3.3) $$\boldsymbol{e}=J\boldsymbol{f}=rJ\boldsymbol{a}=(E^k, E^{k*})=(iF^k, -F^{k*})\,.$$

Then

(3.4) $$\langle \boldsymbol{f}, \boldsymbol{e}\rangle=\frac{1}{2}\sum_k (F^kE^{k*}+F^{k*}E^k)\,,$$

(3.5) $$\langle \boldsymbol{f}, \boldsymbol{f}\rangle=\sum F^kF^{k*}=r^2\,,$$

(3.6) $$\langle \boldsymbol{e}, \boldsymbol{e}\rangle=\sum E^kE^{k*}=r^2\,,$$

(3.7) $$\langle \boldsymbol{f}, \boldsymbol{h}\rangle=\frac{1}{2}\sum (F^kH^{k*}+F^{k*}H^k)\,.$$

(3.8) $$\langle \boldsymbol{e}, \boldsymbol{h}\rangle=\frac{1}{2}\sum (E^kH^{k*}+E^{k*}H^k)$$

hold. From (3.3), (3.7) and (3.8), we have

(3.9) $$\sum F^k H^{k*} = \sum E^{k*} H^k = 0$$

and

(3.10) $$\sum F^k H^{k*} = \sum F^{k*} H^k = 0\,,$$

and from (3.3), we know

(3.11) $$\sum E^{k*} F^k = -i \sum F^{k*} F^k = -ir^2\,.$$

Since $\overrightarrow{O'A} = c\boldsymbol{e}$ (c is a real constant), we have

$$\overrightarrow{OA} = \overrightarrow{OO'} + \overrightarrow{O'A} = \boldsymbol{h} + c\boldsymbol{e}\,,$$

and the equation of the circle A becomes

(3.12) $$z(t) = \boldsymbol{h} + c\boldsymbol{e} + \boldsymbol{e} \cos t + \boldsymbol{f} \sin t\,.$$

The condition that the circle O' and the circle A are perpendicular each other is expressed as

$$1/l^2 - h^2 - r^2 = \langle \overrightarrow{O'A}, \overrightarrow{O'A} \rangle = c^2 r^2$$

i. e.

(3.13) $$1/l^2 - h^2 = (c^2 - 1) r^2\,,$$

since the radius of the absolute is $1/l^2$, where

(3.14) $$c^2 - 1 > 0$$

and h is the length of $\boldsymbol{h}$.

We shall show that the circle (3.12) satisfies (2.1) with a suitable function t of s.

Substituting the equation of the circle A into (2.2) and taking account of (3.5), (3.6), (3.7), (3.8) and (3.13), we have

(3.15) $$S(z(t), \bar{z}(t)) = -2l^2 r^2 (1 + c \cos t)\,.$$

From this formula, it follows that the circle A intersects with the absolute at the point satisfying

(3.16) $$\cos t = -1/c\,,$$

and at the inside of the absolute

(3.17) $$1 + c \cos t < 0\,,$$

since $S > 0$. Especially at the point $t = 0$, it holds that

(3.18) $$1 + c < 0\,.$$

Differentiating (3.12) by the arc length s, we have

$$z'^i = (-E^i \sin t + F^i \cos t) t'$$

$$z''^i = (-t'' \sin t - t'^2 \cos t) E^i + (t'' \cos t - t'^2 \sin t) F^i\,.$$

Substituting them into (2.2), it follows that

$$-2l^2\sum z'^k\bar{z}^k z'^i = 2l^2r^2(c\sin t + ic\cos t + i)(-E^i\sin t + F^i\cos t)t'^2$$

from (3.5), (3.6), (3.9), (3.10) and (3.10) and (3.11). Therefore the equation (3.2) becomes as follows:

$$(3.19)\qquad (-t''\sin t - t'^2\cos t)E^i + (t''\cos t - t'^2\sin t)F^i$$
$$= -\frac{c\cos t + i(c\cos t+1)}{1+c\cos t}(-E^i\sin t + F^i\cos t)t'^2 .$$

Substituting $E^i = iF^i$ into (3.19), we obtain two differential equations

$$\left(t'' + \frac{c\sin t}{1+c\cos t}t'^2\right)\cos t = 0 ,$$
$$\left(t'' + \frac{c\sin t}{1+c\cos t}c'^2\right)\sin t = 0 .$$

These two equations reduce to the single equation:

$$(3.20)\qquad t'' + \frac{c\sin t}{1+c\cos t}t'^2 = 0 .$$

Integrating it at the interval where (3.7) is satisfied and assuming $t'>0$, we get

$$\log t' - \log\{-(1+c\cos t)\} = C ,$$

where C denotes the integral constant. Hence

$$(3.21)\qquad -t'/(1+c\cos t) = C , \qquad C>0 : \text{ const..}$$

In order to integrate it, we put

$$(3.22)\qquad u = \tan(t/2) .$$

Then

$$\cos t = \frac{1-u^2}{1+u^2} , \qquad dt = \frac{2du}{1+u^2} ,$$

and

$$(3.23)\qquad -m_1 < u < m_1 , \qquad m_1 = \sqrt{\frac{c+1}{c-1}} ,$$

from (3.14), (3.17), and (3.18). Substituting them into (3.21) it follows that

$$-\frac{2du}{(1+u^2)\left(1+c\dfrac{1-u^2}{1+u^2}\right)} = Cds , \qquad C>0$$

i. e.

$$-\frac{2du}{(1-c)(u^2-m_1^2)} = Cds , \qquad C>0 .$$

Rewriting it as

$$-\left(\frac{1}{u-m_1}-\frac{1}{u+m_1}\right)\frac{du}{\sqrt{c^2-1}}=Cds\,, \qquad C>0\,,$$

and integrating it at the interval (3.23), we obtain

$$-\frac{1}{\sqrt{c^2-1}}\log\left(-\frac{u-m_1}{u+m_1}\right)=\sqrt{c^2-1}\,Cs\,.$$

Hence, putting $\sqrt{c^2-1}=\alpha$, we have

$$-\frac{u-m_1}{u+m_1}=e^{-\alpha Cs}\,, \qquad C>0\,,$$

$$\alpha=\sqrt{c^2-1}\,.$$

Solving it for u, we obtain

(3.24) $$u=m_1\frac{1-e^{-\alpha Cs}}{1+e^{-\alpha Cs}}\,, \qquad C>0\,.$$

Therefore it follows that

(3.25) $$t=2\tan^{-1}\left(m_1\frac{1-e^{-\alpha Cs}}{1+e^{-\alpha Cs}}\right), \qquad C>0:\ \text{const.}.$$

In the next, we shall determine the constant C, using the relation (2.11). It holds that

$$S(z(0),\bar{z}(0))=-2l^2r^2(1+c)\,,$$

$$g_{ij^*}(z(0),\bar{z}(0))=-\frac{\delta_{ij}}{2l^2r^2(1+c)}+\frac{\{H^{i^*}+(c+1)E^{i^*}\}\{H^j+(c+1)E^j\}}{4l^2r^4(1+c)^2}\,,$$

$$\frac{dz^i}{ds}(0)=F^i t'(0)$$

and

$$\frac{dt}{ds}(0)=-(c+1)C\,;$$

Substituting them into (2.11), and using (3.10) and (3.11). it follows that

$$1=\sum_{i,j}2g_{ij^*}(z(0),\bar{z}(0))\frac{dz^i}{ds}(0)\frac{d\bar{z}^j}{ds}(0)=\frac{\alpha^2}{2l^2}C^2\,,$$

from which we conclude

$$C=\sqrt{2}\,l/\alpha\,.$$

Thus we see that the circle (3.12) with

(3.26) $$t=2\tan^{-1}\left(m_1\frac{1-e^{-\sqrt{2}ls}}{1+e^{-\sqrt{2}ls}}\right)$$

satisfies the equation of geodesics. Since a geodesic is determined uniquely for a point and a direction, we conclude that the geodesic through the point P with the direction $\boldsymbol{a}$ is the circle which is perpendicular to the circle formed as the intersection of the absolute with the holomorphic plane of $\boldsymbol{a}$.

Next, we shall investigate how the distance between two points Q, R on the geodesic is expressed. We take u as a parameter. Denote the values of u and s at Q and R by u_1, u_2, s_1 and s_2. Let X and Y be the points of intersection of the geodesic and the absolute, and denote the values of u at X and Y by u_+ and u_- respectively, *i.e.* $u_+=m_1$, $u_-=-m_1$. The length of QR along the geodesic is s_2-s_1. It holds that

$$u_+-u_i=m_1\left(1-\frac{1-e^{-\sqrt{2}ls_i}}{1+e^{-\sqrt{2}ls_i}}\right),$$

$$u_--u_i=m_1\left(-1-\frac{1-e^{-\sqrt{2}ls_i}}{1+e^{-\sqrt{2}ls_i}}\right), \qquad i=1, 2,$$

from (3.24). Hence the non-harmonic ratio (QR, XY) with respect to the parameter u

$$(QR, XY)=\frac{u_+-u_1}{u_--u_1}\frac{u_--u_2}{u_+-u_2}$$

becomes

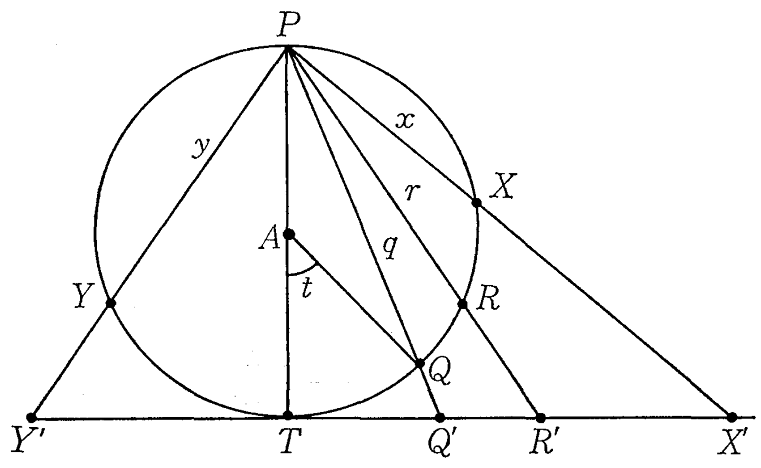
$$(QR, XY)=e^{\sqrt{-k}(s_2-s_1)}$$

in terms of s_1 and s_2, and the distance $d(Q, R)$ can be represented as

$$d(Q, R)=s_2-s_1=(1/\sqrt{-k})\log(QR, XY).$$

REMARK. For four points Q, R, X, Y on the circle with the center A, we take the points Q', R', X', Y', T and the lines q, r, x, y as in the figure. Then it holds that

$$(QR, XY)=(qr, xy)=(Q'R', X'Y')$$



by the property of the non-harmonic ratio. Taking $\angle TAQ=t$ as a parameter for a point Q on the circle, we have

$$TQ'=2k\tan(t/2)\,, \qquad k:\ \text{the diameter.}$$

Hence it is natural to take $2ku$ as a parameter in calculating $(Q'R', X'Y')$ *i.e.* (QR, XY).

§ 4. The representation of the Fubinian space. The results we have obtained in the proceeding sections are summarized as follows:

Consider the Fubinian space with $1+(k/2)\sum z^k\bar{z}^k=0$ $(k<0)$ *as the absolute. Denote the foot of the perpendicular from the origin* O *to a geodesic by* P, *the unit tangent vector in the Eucliden sense at* P *by* $\boldsymbol{a}$, *and the position vector* $\overrightarrow{OP}$ *by* $\boldsymbol{b}$. *Then it holds that* $\langle \boldsymbol{a}, \boldsymbol{b}\rangle=0$ *and*

(i) *the case of* $\langle J\boldsymbol{a}, \boldsymbol{b}\rangle=0$. *The geodesic is the line*

$$\boldsymbol{z}=\boldsymbol{a}t+\boldsymbol{b}$$

and the distance between arbitrary two points A *and* B *on it can be expressed as*

$$d(A, B)=(1/\sqrt{-k})\log(AB, XY)\,,$$

where X, Y, *denote the points of intersection of the geodesic and the absolute* (see fig. 2).

(ii) *the case of* $\langle J\boldsymbol{a}, \boldsymbol{b}\rangle\neq 0$. *The geodesic is the circle*

$$\boldsymbol{z}=\boldsymbol{b}-rJ\boldsymbol{a}+r(J\boldsymbol{a}\cos t+\boldsymbol{a}\sin t)$$

which is perpendicular to the circle that is the intersection of the absolute and the holomorphic plane of $\boldsymbol{a}$, *and the distance between arbitrary two points* A *and* B *on it can be expressed as*

$$d(A, B)=(1/\sqrt{-k})\log(AB, XY)\,,$$

where X, Y *denote the points of intersection of the geodesic and the absolute* (see fig. 1).

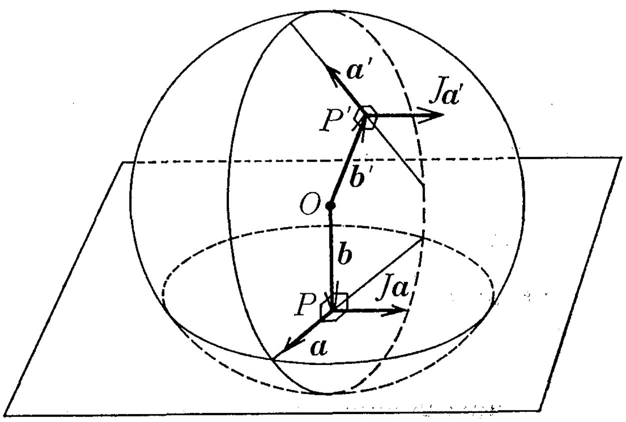


Fig. 2

On the other hand, the following lemma is well known.

LEMMA 1. *For a vector* $\boldsymbol{a}$ *on the holomorphic plane of a vector* $\boldsymbol{a}'$, *the holomorphic plane of* $\boldsymbol{a}$ *coincides with that of* $\boldsymbol{a}'$.

In general, a plane is called an anti-holomorphic plane if $J\boldsymbol{a}$ is always perpendicular to the plane for any vector $\boldsymbol{a}$ on it. We consider the plane spanned by two vectors $\boldsymbol{a}$, $\boldsymbol{b}$ satisfying $\langle J\boldsymbol{a}, \boldsymbol{b}\rangle = \langle \boldsymbol{a}, \boldsymbol{b}\rangle = 0$. Since any vector $\boldsymbol{a}'$ in this plane is represented as a linear combination of $\boldsymbol{a}$ and $\boldsymbol{b}$, $J\boldsymbol{a}'$ is represented as that of $J\boldsymbol{a}$ and $J\boldsymbol{b}$. As $\langle \boldsymbol{b}, J\boldsymbol{a}\rangle = \langle \boldsymbol{b}, J\boldsymbol{b}\rangle = 0$, it holds that $\langle \boldsymbol{b}, J\boldsymbol{a}'\rangle = 0$ i. e. $J\boldsymbol{a}'$ is perpendicular to this plane. Taking account of $g(\ ,\ ) = 2\langle\ ,\ \rangle$ at the origin, the following lemma holds.

LEMMA 2. *In the Fubinian space, the plane through the origin spanned by arbitrary two vector* $\boldsymbol{a}, \boldsymbol{b}$ *satisfying* $\langle J\boldsymbol{a}, \boldsymbol{b}\rangle = \langle \boldsymbol{a}, \boldsymbol{b}\rangle = 0$ *is an anti-holomorphic plane (with respect to both the Fubinian and Euclidean metric).*

In the case of (i), the plane through the origin spanned by $\boldsymbol{a}$ and $\boldsymbol{b}$ is an anti-holomorphic plane from the view of Lemma 2. On this plane, we draw thhe perpendicular from the origin to an arbitrary line, and denote its foot by P', the unit tangent vector in the Euclidean sense by $\boldsymbol{a}'$, and the position vector $\overrightarrow{OP'}$ by $\boldsymbol{b}'$. As $J\boldsymbol{a}'$ is perpendicular to this plane, $\langle J\boldsymbol{a}', \boldsymbol{b}'\rangle = 0$ holds. Therefore all the lines on this plane are geodesics. Furthermore, on this plane, we have seen that the distance between two points A and B is expressed as

$$d(A, B) = (1/\sqrt{-k}) \log (AB, XY),$$

where X, Y denote the points of intersection of the line through A and B and the absolute.

On an arbitrary anti-holomorphic plane through the origin, we have the same statement. Thus the following theorem holds.

THEOREM 1. *In the Fubinian space with* $S(z, \bar{z}) = 1 + (k/2)\sum z^k \bar{z}^k = 0$ *as its absolute, on any anti-holomorphic plane through the origin, the geodesics are lines, and the distance* $d(A, B)$ *between two points* A *and* B *is represented as*

$$d(A, B) = (1/\sqrt{-k}) \log (AB, XY),$$

where X, Y *denote the points of intersection of the line through* A *and* B *and the absolute.*

Therefore, for any anti-holomorphic plane through the origin, it holds so-called the Klein's representation of the non-Euclidean geometry.

Next, we consider the case of (ii). On the holomorphic plane of the vector $\boldsymbol{a}$, consider the geodesic through an arbitrary point Q and with a direction $\boldsymbol{c}$ on the plane. Then they are lines or circles on the holomorphic plane of $\boldsymbol{c}$, which is that of $\boldsymbol{a}$ from Lemma 1.

Drawing the perpendicular from the origin to a geodesic on the holomorphic plane of $\boldsymbol{a}$, we denote its foot by P'. Let $\boldsymbol{a}'$ be the unit tangent vector in the Euclidean sense, and $\boldsymbol{b}'$ the position vector $\overrightarrow{OP'}$. If the geodesic passes through the points O', $\boldsymbol{b}'$ coincides with $\boldsymbol{h}'$ and is perpendicular to this holomorphic plane. And from (i), the geodesic is a line. If the geodesic does not pass through O', since $\boldsymbol{b}'$ cannot be perpendicular to the holomorphic plane then, the geodesic is a circle from (ii). Furthermore, on the holomorphic plane, we have seen that the distance between two points A and B is expressed as

$$d(A, B)=(1/\sqrt{-k}) \log (AB, XY),$$

where X, Y denote the points of intersection of the geodesic and the absolute.

On an arbitrary holomorphic plane, we have the same statement. Thus the following theorem holds.

THEOREM 2. *In the Fubinian space with* $S(z, z)=1+(k/2)\sum z^k\bar{z}^k=0$ *as its absolute, let* H *be any holomorphic plane, and* O' *the foot of the perpendicular from the origin. Then, on* H *the geodesics through* O' *are lines and the geodesics not through* O' *are circles which are perpendicular to the circle of intersection of* H *and the absolute. The distance between two points* A *and* B *is represented as*

$$d(A, B)=(1/\sqrt{-k}) \log (AB, XY),$$

where X, Y *denote the points of intersection of the geodesic and the absolute.*

Therefore, for any holomorphic plane, it holds so-called the Poincaré's representation of the non-Euclidean geometry.

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