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Abstract Green Operators and Semigroups

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§ 0. Introduction.

In her preceding paper [3], the author has considered an operator L which is a generalization of the differential operators of the form $\Delta+\sum_{i=1}^{n} a_i(x)\partial/\partial x_i$ and, by means of an abstract method, proved that the existence of nonconstant L-harmonic function implies the existence of a Green operator G associated with L.

In the present paper, we construct an extension $\hat{G}$ of the operator G in a certain function space E, and prove that the inverse $\hat{G}^{-1}$ of $\hat{G}$ generates a semigroup $\{T_t\}$ of operators in E. The result will show the relation between the contents of the author's preceding paper [3] and the characterization of abstract potential operators by K. Yosida [9].

§ 1. Preliminary notions and some results of the preceding paper [3].

The following assumptions and definitions are mentioned in the author's preceding paper [3].

All functions are assumed to be real valued.

Let X be a connected, locally compact and σ-compact Hausdorff space, $C(X)$ be the set of all continuous functions on X, $C_b(X)$ be the set of all bounded functions in $C(X)$ and $C_0(X)$ be the set of all functions in $C(X)$ with compact support. $C(D)$, $C_0(D)$ and $C(\bar{D})$ are defined analogously for any subdomain D of X. The norm $\|f\|$ of any bounded function f on X (or D, $\bar{D}$) is defined by $\|f\|=\sup_x|f(x)|$, and the completion of $C_0(X)$ (resp. $C_0(D)$) with respect to the norm is denoted by $\overline{C_0(X)}$ (resp. $\overline{C_0(D)}$). The dual space of $C(\bar{D})$ is the set $\mathfrak{M}(\bar{D})$ of all signed measures on $\bar{D}$. We denote by $\mathfrak{M}_0(D)$ the set of $\rho\in\mathfrak{M}(\bar{D})$ with compact support in the interior of D.

Let L be a linear operator with the following properties and satisfying the axioms (α), (β), (γ) and (δ) mentioned later. The domain $\mathfrak{D}(L)$ is a linear subspace of $C(D)$ such that $\mathfrak{D}(L)\cap C_0^+(D)$ is dense in $C_0^+(D)$ for any subdomain D of X, and L is a linear operator of $\mathfrak{D}(L)$ into $C(X)$; any constant c belongs to $\mathfrak{D}(L)$ and $Lu=0$. L is assumed to be a local operator in the sence that, if $f\in\mathfrak{D}(L)$ and $f(x)\equiv 0$ in a neighborhood of a point $x_0\in X$, then $(Lf)(x_0)=0$.

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The operator L_D which is obtained by localizing the operator L to any subdomain D of X will be denoted by L briefly (see [3; § 1]). We may consider the dual operator L_D^* of the restriction of L on $C(\bar{D})\cap\mathfrak{D}(L)$.

A subdomain D of X is called a regular domain if the closure $\bar{D}$ is compact and, for any $\varphi\in C(\partial D)$, there exists a solution $u\in\mathfrak{D}(L)\cap C(\bar{D})$ of the boundary value problem: $Lu=0$ in D and $u=\varphi$ on ∂D. We assume that there exist sufficiently many regular domains, that is, for any domains D_1 and D_2 with compact closure and satisfying $\bar{D}_1\subset D_2$, there exists a regular domain D such that $\bar{D}_1\subset D\subset D_2$.

The operator L is assumed to satisfy the following axioms.

(α) If $Lu\geqq 0$ and u is nonconstant in D, then u does not take its maximum in the interior of D (maximum principle).

(β) If $\{u_n\}$ and $\{Lu_n\}$ are uniformly bounded on D, then a subsequence $\{u_{n_\nu}\}$ of $\{u_n\}$ converges uniformly on every compact subset of D (Harnack property).

(γ) For any regular domain D, any $\lambda>0$ and any $f\in\mathfrak{D}(L)\cap C(\bar{D})$, there exists $u\in\mathfrak{D}(L)\cap\overline{C_0(D)}$ satisfying $(\lambda-L)u=f$.

(δ) If $u\in C(D)$ satisfies $\langle u, L^*\rho\rangle=0$ for any $\rho\in\mathfrak{D}(L^*)\cap\mathfrak{M}_0(D)$, then $u\in\mathfrak{D}(L_D)$ and $Lu=0$ (cf. Weyl's lemma).

By definition, a function u on a domain $D\subset X$ is said to be L-harmonic if $u\in\mathfrak{D}(L)$ and satisfies $Lu=0$ in D. A linear operator G of $C_0(X)$ into $C(X)$ is called a Green operator associated with L if, for any $f\in\mathfrak{D}(L)\cap C_0(X)$, $u=Gf$ belongs to $\mathfrak{D}(L)$ and satisfies $Lu=-f$ on X.

Using the assumptions mentioned above, we have shown the following results in [3].

We fix a regular domain D.

LEMMA 1.1. *If $u\in\mathfrak{D}(L_D)$ takes its maximum at $x_0\in D$, then $Lu(x_0)\leqq 0$.*

LEMMA 1.2. *If $(\lambda-L)u\leqq 0$ (resp. $\geqq 0$) in D ($\lambda>0$), then u does not take its positive maximum (resp. negative minimum) in the interior of D.*

LEMMA 1.3. *Suppose $\lambda>0$, $f\in\mathfrak{D}(L)\cap C(\bar{D})$ and $u\in\mathfrak{D}(L)\cap\overline{C_0(D)}$. If $(\lambda-L)u=f$, then $\|u\|\leqq\|f\|/\lambda$. Accordingly the function u in (γ) is uniquely determind by f ([3; § 2]).*

By virtue of Lemma 1.3, we can define the operator $J_\lambda^D=(\lambda-L)^{-1}$ of $\mathfrak{D}(L)\cap\overline{C_0(D)}$ into $\overline{C_0(D)}$, which is uniquely extended to a bounded linear operator in $\overline{C_0(D)}$ with norm $\leqq 1/\lambda$; we denote the extended operator by J_λ^D again. Then $\{J_\lambda^D\}_{\lambda>0}$ satisfies that

(1.1) $$J_\lambda^D-J_\mu^D=(\mu-\lambda)J_\lambda^D J_\mu^D \quad \text{(resolvent equation)},$$

(1.2) $$\text{s-}\lim_{\lambda\uparrow\infty}\lambda J_\lambda^D=I \quad \text{and} \quad \text{s-}\lim_{\lambda\downarrow 0}\lambda J_\lambda^D=0$$

in the Banach space $\overline{C_0(D)}$. Hence, by the result of K. Yosida [9], there exists a Green operator G^D such that

(1.3) $$G^D f = \text{s-}\lim_{\lambda \downarrow 0} J_\lambda^D f \qquad \text{for any} \quad f \in \mathfrak{D}(L) \cap C_0(D);$$

here $J_\lambda^D f$ increases monotonously as $\lambda \downarrow 0$ if $f \in \mathfrak{D}(L) \cap C_0^+(D)$ ([3; § 3]).

If the space X admits a positive nonconstant L-harmonic function, then there exists a Green operator G associated with L, and

(1.4) $$(Gf)(x) = \lim_{D \uparrow X} (G^D f)(x) \qquad \text{(pointwise convergence)}$$

for any $f \in \mathfrak{D}(L) \cap C_0(X)$; in particular, $(G^D f)(x)$ increases monotonously as $D \uparrow X$ if $f \in \mathfrak{D}(L) \cap C_0^+(X)$. Furthermore there exists a family of measures $\{\Phi(x, E) | x \in X\}$ such that

(1.5) $$(Gf)(x) = \int_X \Phi(x, dy) f(y) \qquad \text{for any} \quad f \in C_0(X)$$

([3; § 4 and § 5]).

§ 2. The operators $J_\lambda (\lambda > 0)$ in $C_b(X)$.

In the sequel, we always assume that the space X admits a positive nonconstant L-harmonic function.

We first notice the following facts. Let D be an arbitrary relatively compact domain in X. Then any function $w \in \overline{C_0(D)}$ may be regarded as an element of $C_0(X)$ by putting $w(x) = 0$ for $x \in X - D$. Similarly any function $w \in C_0(D) \cap \mathfrak{D}(L)$ may be regarded as an element of $C_0(X) \cap \mathfrak{D}(L)$ since L is a local operator. We shall use these facts without repeating the above notices.

For every regular domain D in X and every $\lambda > 0$, let J_λ^D be the bounded linear operator in $\overline{C_0(D)}$ mentioned in § 1. We shall define the operator J_λ in $C_b(X)$ which corresponds to J_λ^D in $\overline{C_0(D)}$.

Let $w \in C_0(X) \cap \mathfrak{D}(L)$, and define u_λ^D by

$$u_\lambda^D = J_\lambda^D w$$

for any $\lambda > 0$ and any regular domain $D \supset \operatorname{supp} w$.

PROPOSITION 2.1. *If $w \in C_0^+(X) \cap \mathfrak{D}(L)$, then $u_\lambda^D = J_\lambda^D w$ increases monotonously as D increases.*

PROOF. Suppose that $D_1 \subset D_2$ and we put $u = u_\lambda^{D_2} - u_\lambda^{D_1}$. Then $u \geqq 0$ on ∂D_1 and

(2.1) $$(\lambda - L) u = 0 \quad \text{in} \quad D_1$$

since $(\lambda - L) u_\lambda^{D_1} = (\lambda - L) u_\lambda^{D_2} = w$ in D_1. If u takes negative value at some point in D_1, then $-u$ takes the positive maximum at some point $x_1 \in D_1$, and $L(-u)(x_1) \leqq 0$ from Lemma 1.1 in § 1. Hence $(\lambda - L)(-u)(x_1) > 0$, which contradicts (2.1). Thus we get $u \geqq 0$ in D_1.

Since J_λ^D is a positive operator, we have $0 \leqq J_\lambda^D w \leqq \frac{1}{\lambda} \|w\|$ for any $w \in C_0^+(X) \cap \mathfrak{D}(L)$. Hence the limit function

$$u_\lambda = \lim_{D \uparrow X} u_\lambda^D = \lim_{D \uparrow X} J_\lambda^D w \tag{2.2}$$

exists by Proposition 2.1. Since any $w \in C_0(X) \cap \mathfrak{D}(L)$ is expressible as $w = w_1 - w_2$ for suitable $w_1, w_2 \in C_0^+(X) \cap \mathfrak{D}(L)$, the limit function u_λ in (2.2) is defined for any $w \in C_0(X) \cap \mathfrak{D}(L)$.

PROPOSITION 2.2. *u_λ is continuous on X.*

PROOF. Let $\{D_n\}_{n \geq 1}$ be a sequence of regular domains satisfying $\lim_{n \to \infty} D_n = X$. We fix a domain $D_0 \supset \operatorname{supp} w$ such that $\bar{D}_0$ is compact. Then for any $D_n \supset D_0$, we have

$$\| u_\lambda^{D_n} \| \leqq \frac{1}{\lambda} \| w \| . \tag{2.3}$$

As $(\lambda - L) u_\lambda^{D_n} = w$, $L u_\lambda^{D_n} = -w + \lambda u_\lambda^{D_n}$ and hence

$$\| L u_\lambda^{D_n} \| \leqq 2 \| w \| . \tag{2.4}$$

By (2.3), (2.4) and the axiom (β) in § 1, there exists a subsequence $\{u_\lambda^{D_n}\}$ which converges uniformly on every compact subset of D_0. Hence u_λ is continuous in D_0. As D_0 is arbitrary, u_λ is continuous on X.

By Proposition 2.2, the operator J_λ defined by

$$J_\lambda w = \lim_{D \uparrow X} J_\lambda^D w \tag{2.5}$$

is a bounded and positive linear operator of $C_0(X) \cap \mathfrak{D}(L)$ into $C_b(X)$ such that $\| J_\lambda \| \leqq \frac{1}{\lambda}$. Since $C_0^+(X) \cap \mathfrak{D}(L)$ is dense in $C_0^+(X)$, J_λ can be extended to a bounded and positive linear operator of $C_0(X)$ into $C_b(X)$, and we have

$$| J_\lambda w(x) | \leqq \frac{1}{\lambda} \| w \| \qquad \text{for any} \quad w \in C_0(X) .$$

Hence, for any fixed $\lambda > 0$ and $x \in X$, there exists a measure $\rho_\lambda^x(E)$ in X such that $\rho_\lambda^x(X) \leqq \frac{1}{\lambda}$ and

$$(J_\lambda w)(x) = \int_X w(y) d\rho_\lambda^x(y) \qquad \text{for any} \quad w \in C_0(X) .$$

For any $w \in C_b(X)$, we define

$$(J_\lambda w)(x) = \int_X w(y) d\rho_\lambda^x(y) , \tag{2.6}$$

and we have the following

PROPOSITION 2.3. *$(J_\lambda w)(x)$ is continuous on X for any $w \in C_b(X)$.*

PROOF. It is sufficient to prove this Proposition for $w \in C_b^+(X)$. Let $\{D_n\}_{n \geq 1}$ be a sequence of regular domains such that $\bar{D}_n \subset D_{n+1}$ and $\lim_{n \to \infty} D_n = X$, and $\{f_n\}$ be a sequence of functions in $C_0(X)$ satisfying that

$$\begin{cases} f_n(x)=1 & \text{for} \quad x\in \bar{D}_n\,, \\ 0\leqq f_n(x)\leqq 1 & \text{for} \quad x\in D_{n+1}-\bar{D}_n\,, \\ f_n(x)=0 & \text{for} \quad x\in X-D_{n+1}\,. \end{cases}$$

Since $C_0^+(D_{n+2})\cap\mathfrak{D}(L)$ is dense in $C_0^+(D_{n+2})$, there exists $w_n\in C_0^+(X)\cap\mathfrak{D}(L)$ such that $\operatorname{supp} w_n\subset D_{n+2}$ and $\|w_n-w\cdot f_n\|\leqq 1/n$. Then

$$\lim_{n\to\infty} w_n(x)=w(x) \qquad \text{at each point} \quad x\in X \tag{2.7}$$

and we have

$$\|w_n\|\leqq\|w\|+1\,. \tag{2.8}$$

It follows from (2.6), (2.7) and (2.8) that

$$\lim_{n\to\infty}(J_\lambda w_n)(x)=(J_\lambda w)(x) \qquad \text{for each point} \quad x\in X \tag{2.9}$$

by means of the bounded convergence theorem. We fix an arbitrary m. Then, for any $n>m$, the function $(J_\lambda^D w_n)(x)$ is continuous in x and increases monotonously as D increases and the limit function

$$(J_\lambda w_n)(x)=\lim_{D\uparrow X}(J_\lambda^D w_n)(x) \qquad (\text{cf. } (2.5)) \tag{2.10}$$

is also continuous by Proposition 2.2. Hence the convergence in (2.10) is uniform on $\bar{D}_m$ by Dini's theorem, and accordingly

$$|(J_\lambda^{D'_n}w_n)(x)-(J_\lambda w_n)(x)|<\frac{1}{n} \qquad \text{on} \quad \bar{D}_m$$

for a suitable regular domain $D'_n\supset D_{n+2}$. From this fact and (2.9), it follows that

$$\lim_{n\to\infty}(J_\lambda^{D'_n}w_n)(x)=(J_\lambda w)(x) \qquad \text{for each point} \quad x\in\bar{D}_m\,. \tag{2.11}$$

Since w_n is regarded as an element of $C_0(D'_n)\cap\mathfrak{D}(L)$, we have $(\lambda-L)J_\lambda^{D'_n}w_n=w_n$ and accordingly

$$\|LJ_\lambda^{D'_n}w_n\|\leqq\|\lambda J_\lambda^{D'_n}w_n\|+\|w_n\|\leqq 2(\|w\|+1)\,;$$

the last inequality follows from (2.8) and $\|J_\lambda^{D'_n}\|\leqq\frac{1}{\lambda}$. Therefore $\{J_\lambda^{D'_n}w_n\}_{n>m}$ and $\{LJ_\lambda^{D'_n}w_n\}_{n>m}$ are uniformly bounded. Hence, by the axiom (β) in § 1, a subsequence of $\{J_\lambda^{D'_n}w_n\}_{n>m}$ converges uniformly on $\bar{D}_m$. This fact and (2.11) imply that $(J_\lambda w)(x)$ is continuous on $\bar{D}_m$. Since m is arbitrary, the proof of this Proposition is complete.

PROPOSITION 2.4. *$C_0(X)\cap\mathfrak{D}(L)$ is contained in*

$$\mathfrak{R}_0(J_\lambda)=\{J_\lambda f\,|\,f\in C_0(X)\} \tag{2.12}$$

for any $\lambda>0$.

PROOF. We first prove that

$$\lambda\|v\| \leqq \|(\lambda-L)v\| \quad \text{for any} \quad v \in \overline{C_0(D)} \cap \mathfrak{D}(L) \tag{2.13}$$

where D is an arbitrary relatively compact subdomain of X. Let x_1 and x_2 be points in D for which $v(x_1)=\max_{x\in\bar{D}} v(x)$ and $v(x_2)=\min_{x\in\bar{D}} v(x)$. Then $Lv(x_1)\leqq 0$ (Lemma 1 in [3]), and accordingly

$$0 \leqq \lambda v(x_1) \leqq (\lambda-L)v(x_1) \leqq \|(\lambda-L)v\| .$$

Similarly we may show that $0 \geqq \lambda v(x_2) \geqq -\|(\lambda-L)v\|$. Hence we have (2.13). Now, for any $u\in C_0(X)\cap\mathfrak{D}(L)$, we consider $f=(\lambda-L)u$. Then $f\in C_0(X)$ and $\operatorname{supp} f \subset \operatorname{supp} u$. For any relatively compact domain $D \supset \operatorname{supp} u$, we may consider that $f\in C_0(D)$, and hence there exists a sequence $\{f_n\}\subset C_0(D)\cap\mathfrak{D}(L)$ such that

$$\lim_{n\to\infty} \|f_n - f\| = 0 . \tag{2.14}$$

By means of (2.13) and the fact that $\|J_\lambda^D\| \leqq \frac{1}{\lambda}$ and $J_\lambda^D=(\lambda-L)^{-1}$ (see § 1), we have

$$\begin{aligned}\lambda\|J_\lambda^D f - u\| &\leqq \lambda\|J_\lambda^D f - J_\lambda^D f_n\| + \lambda\|J_\lambda^D f_n - u\| \\ &\leqq \|f-f_n\| + \|(\lambda-L)(J_\lambda^D f_n - u)\| \\ &= \|f-f_n\| + \|f_n - f\| .\end{aligned}$$

Passing to the limit as $n\to\infty$, we obtain by (2.14) that

$$J_\lambda^D f = u . \tag{2.15}$$

Since D is arbitrary, we may take the limit as $D\uparrow X$ in (2.15) and we obtain by (2.5) that $u=J_\lambda f$ on X. Proposition 2.4 is thus proved.

THEOREM 1. *The family of operators* $\{J_\lambda\}_{\lambda>0}$ *in* $C_b(X)$ *satisfies the resolvent equation; namely, for any* $w\in C_b(X)$,

$$J_\lambda w - J_\mu w = (\mu-\lambda) J_\lambda J_\mu w \quad (\lambda,\ \mu>0) . \tag{2.16}$$

PROOF. It is sufficient to prove (2.16) for $w\in C_b^+(X)$. We first assume that $w\in C_0^+(X)$. Then, for any regular domain $D\supset \operatorname{supp} w$, we may consider that $w\in C_0^+(D)$. Since $\{J_\lambda^D\}_{\lambda>0}$ satisfies the resolvent equation in $\overline{C_0(D)}$, we have

$$J_\lambda^D w - J_\mu^D w = (\mu-\lambda) J_\lambda^D J_\mu^D w . \tag{2.17}$$

Let $\{D_n\}$ be a monotone increasing sequence of regular domains in X such that $\lim_{n\to\infty} D_n = X$, and define $J_\lambda^n = J_\lambda^{D_n}$. Then

$$\lim_{n\to\infty} J_\lambda^n w(x) = J_\lambda w(x) \tag{2.18}$$

by (2.5). Since J_μ^n is a positive operator and $J_\lambda^n w$ increases monotonously in n, we have $J_\lambda^n J_\mu^n w \leqq J_\lambda J_\mu^n w$. Similarly, the positivity of J_λ and the monotonicity of $J_\mu^n w$ in n imply that $J_\lambda J_\mu^n w \leqq J_\lambda J_\mu w$. Hence we get

$$J_\lambda^n J_\mu^n w \leqq J_\lambda J_\mu w .$$

We may assume $\mu>\lambda$ without loss of generality. Then it follows from (2.17) and the above inequality that

$$J_\lambda^n w - J_\mu^n w = (\mu-\lambda) J_\lambda^n J_\mu^n w \leqq (\mu-\lambda) J_\lambda J_\mu w .$$

Let $n \to \infty$, and we obtain by (2.18)

$$J_\lambda w - J_\mu w \leqq (\mu-\lambda) J_\lambda J_\mu w .$$

We shall prove the inverse inequality under the same assumption $\mu>\lambda$ as above. If $m<n$, we obtain from (2.17)

$$J_\lambda^n w = \{(\mu-\lambda) J_\lambda^n + I\} J_\mu^n w \geqq \{(\mu-\lambda) J_\lambda^n + I\} J_\mu^m w .$$

We fix m arbitrarily and let $n \to \infty$. Then we get by (2.18)

$$J_\lambda w \geqq \{(\mu-\lambda) J_\lambda + I\} J_\mu^m w .$$

Let $m \to \infty$, and we obtain

$$J_\lambda w \geqq \{(\mu-\lambda) J_\lambda + I\} J_\mu w$$

by means of (2.6), (2.18) and the bounded convergence theorem. Hence we have

$$J_\lambda w - J_\mu w \geqq (\mu-\lambda) J_\lambda J_\mu w .$$

Thus we get (2.16) for $w \in C_0^+(X)$. Now for any $w \in C_b^+(X)$, there exists a monotone increasing sequence $\{w_n\} \subset C_0^+(X)$ such that $\lim_{n\to\infty} w_n(x) = w(x)$ on X. Hence it follows from (2.6) and the bounded convergence theorem that

$$\lim_{n\to\infty} J_\mu w_n(x) = J_\mu w(x) , \qquad \|J_\mu w_n\| \leqq \frac{1}{\mu} \|w\|$$

and accordingly

$$\lim_{n\to\infty} J_\lambda J_\mu w_n(x) = J_\lambda J_\mu w(x) .$$

Therefore the equality (2.16) for $w \in C_b^+(X)$ follows from the same equality for $w_n \in C_0^+(X)$.

§ 3. Extension of the operator G.

In this §, we extend the operator G defined on $C_0(X)$ in § 1.

Let $\{\Phi(x, \cdot)\}_{x\in X}$ be the family of measures in X mentioned in § 1, and define

$$\mathfrak{D}(\bar{G}) = \left\{ v \in C_b(X) \,\Big|\, \sup_{x\in X} \int_X \Phi(x, dy) |v(y)| < \infty \right\} \tag{3.1}$$

and

$$(\bar{G}v)(x) = \int_X \Phi(x, dy) v(y) \qquad \text{for} \quad v \in \mathfrak{D}(\bar{G}) . \tag{3.2}$$

Then it is clear that $\bar{G}$ is an extension of G and that $(\bar{G}v)(x)$ is bounded on X

for any $v\in\mathfrak{D}(\bar{G})$. Hence $\bar{G}$ maps $\mathfrak{D}(\bar{G})$ into $C_b(X)$ by the following proposition.

PROPOSITION 3.1. *$(\bar{G}v)(x)$ is continuous on X for any $v\in\mathfrak{D}(\bar{G})$.*

PROOF. We may take a sequence $\{w_n\}\subset C_0(X)\cap\mathfrak{D}(L)$ such that

$$|w_n(x)|\leqq|v(x)| \quad\text{and}\quad \lim_{n\to\infty} w_n(x)=v(x) \quad\text{on}\quad X.$$

We define

$$u_n(x)=(Gw_n)(x)\equiv\int_X\Phi(x, dy)w_n(y) \qquad (n=1, 2, \cdots)$$

and

$$u(x)=(\bar{G}v)(x)\equiv\int_X\Phi(x, dy)v(y).$$

Then, by the dominated convergence theorem, we have

$$\lim_{n\to\infty} u_n(x)=u(x) \qquad \text{for every point} \quad x\in X. \tag{3.3}$$

On the other hand, it follows from the result of [3] that

$$-Lu_n=w_n \quad\text{and}\quad |u_n(x)|\leqq\int_X\Phi(x, dy)|v(y)|.$$

Accordingly $\{u_n\}$ and $\{Lu_n\}$ are uniformly bounded on X. Hence, by the axiom (β) in § 1, there exists a subsequence $\{u_{n_\nu}\}$ which converges uniformly on every compact subset of X, and

$$\lim_{\nu\to\infty} u_{n_\nu}(x)=u(x)=(\bar{G}v)(x) \qquad \text{by} \quad (3.3).$$

Hence $(\bar{G}v)(x)$ is continuous on X.

LEMMA 3.2. *For any $w\in\mathfrak{D}(\bar{G})$,*

$$|(J_\lambda w)(x)|\leqq(\bar{G}|w|)(x) \quad\text{and}\quad \lim_{\lambda\downarrow 0}(J_\lambda w)(x)=(\bar{G}w)(x) \quad\text{on}\quad X \tag{3.4}$$

where $|w|(x)=|w(x)|$.

PROOF. For any $w\in\mathfrak{D}(\bar{G})$, it is clear that w^+, w^- and $|w|\in\mathfrak{D}(\bar{G})$ and that (3.4) holds for w if it holds for w^+, w^- and $|w|$. Hence it is sufficient to prove (3.4) under the condition: $w\in\mathfrak{D}(\bar{G})$ and $w\geqq 0$.

First we assume that $w\in C_0^+(X)\cap\mathfrak{D}(L)$. For any regular domain $D\supset\operatorname{supp} w$, we have

$$0\leqq J_\lambda^D w\leqq G^D w \qquad \text{for any} \quad \lambda>0$$

from (1.3). Hence, by Proposition 2.1, (2.5) and (1.4), it follows that

$$0\leqq J_\lambda^D w\leqq J_\lambda w\leqq Gw.$$

Passing to the limit of each term as $\lambda\downarrow 0$, we obtain

$$0\leqq G^D w\leqq\lim_{\lambda\downarrow 0} J_\lambda w\leqq Gw.$$

Since $\lim_{D \uparrow X} G^D w = Gw$, the above inequality implies that $\lim_{\lambda \downarrow 0} J_\lambda w = Gw$. Thus (3.4) is proved for $w \in C_0^+(X) \cap \mathfrak{D}(L)$.

For any nonnegative function $w \in \mathfrak{D}(\bar{G})$, there exists a sequence $\{w_n\} \subset C_0^+(X) \cap \mathfrak{D}(L)$ such that

$$0 \leqq w_n(x) \leqq w(x) \quad \text{and} \quad \lim_{n \to \infty} w_n(x) = w(x) \quad \text{on} \quad X. \tag{3.5}$$

Then it follows from (2.6), (3.2) and the dominated convergence theorem that

$$\lim_{n \to \infty} (J_\lambda w_n)(x) = (J_\lambda w)(x), \qquad \lim_{n \to \infty} (G w_n)(x) = (\bar{G} w)(x). \tag{3.6}$$

Since $0 \leqq J_\lambda w_n \leqq G w_n$ as mentioned above, it follows from (3.5) and (3.6) that

$$0 \leqq J_\lambda w_n \leqq J_\lambda w \leqq \bar{G} w \qquad \text{for any} \quad \lambda > 0.$$

Passing to the limit of each term as $\lambda \downarrow 0$, we obtain

$$0 \leqq G w_n \leqq \lim_{\lambda \downarrow 0} J_\lambda w \leqq \bar{G} w.$$

This result and the second equality of (3.6) imply that $\lim_{\lambda \downarrow 0} J_\lambda w = \bar{G} w$. (3.4) is thus proved for nonnegative $w \in \mathfrak{D}(\bar{G})$.

LEMMA 3.3. *If $v \in \mathfrak{D}(\bar{G})$, then $J_\lambda v \in \mathfrak{D}(\bar{G})$ for any $\lambda > 0$.*

PROOF. It is sufficient to prove this lemma for $v \geqq 0$. Then, since $J_\lambda v \in C_0^+(X)$, there exists a monotone increasing sequence $\{w_n\} \subset C_0^+(X)$ such that

$$0 \leqq w_n(x) \leqq (J_\lambda v)(x) \quad \text{and} \quad \lim_{n \to \infty} w_n(x) = (J_\lambda v)(x) \quad \text{on} \quad X. \tag{3.7}$$

For any positive number $\mu < \lambda$,

$$0 \leqq J_\mu w_n \leqq J_\mu J_\lambda v = \frac{1}{\lambda - \mu} (J_\mu - J_\lambda) v \leqq \frac{1}{\lambda - \mu} J_\mu v \leqq \frac{1}{\lambda - \mu} \bar{G} v$$

by the resolvent equation and Lemma 3.2. Let $\mu \to 0$, and we have $0 \leqq G w_n \leqq \frac{1}{\lambda} \bar{G} v$, namely

$$0 \leqq \int_X \Phi(x, dy) w_n(y) \leqq \frac{1}{\lambda} (\bar{G} v)(x) \leqq \frac{1}{\lambda} \|\bar{G} v\|.$$

Let $n \to \infty$, and we obtain by (3.7) and the monotone convergence theorem that

$$0 \leqq \int_X \Phi(x, dy)(J_\lambda v)(y) \leqq \frac{1}{\lambda} \|\bar{G} v\|.$$

Hence we have $J_\lambda v \in \mathfrak{D}(\bar{G})$.

PROPOSITION 3.4. *If $v \in \mathfrak{D}(\bar{G})$, then*

$$\bar{G} J_\lambda v = J_\lambda \bar{G} v = \frac{1}{\lambda} (\bar{G} v - J_\lambda v) \qquad \textit{for any} \quad \lambda > 0. \tag{3.8}$$

PROOF. From the resolvent equation (2.16), it follows that

(3.9) $$J_\mu J_\lambda v = J_\lambda J_\mu v = \frac{1}{\lambda-\mu}(J_\mu v - J_\lambda v) \qquad (0<\mu<\lambda).$$

By means of the above two lemmas, the assumption $v \in \mathfrak{D}(\bar{G})$ implies that

$$J_\lambda v \in \mathfrak{D}(\bar{G}), \ \lim_{\mu\downarrow 0}(J_\mu J_\lambda v)(x) = (\bar{G}J_\lambda)v(x),$$

$$|(J_\mu v)(x)| \leqq (\bar{G}|v|)(x) \quad \text{and} \quad \lim_{\mu\downarrow 0}(J_\mu v)(x) = (\bar{G}v)(x) \quad \text{on} \quad X.$$

Accordingly $\lim_{\mu\downarrow 0}(J_\lambda J_\mu v)(x) = (J_\lambda \bar{G}v)(x)$ by (2.6) and the bounded convergence theorem. Hence, passing to the limit as $\mu\downarrow 0$ in (3.9), we obtain (3.8).

§ 4. The main results.

We define

$$E_0 = \left\{ \sum_{k=1}^{l} J_{\lambda_k} w_k \,\middle|\, w_k \in C_0(X)(1 \leqq k \leqq l);\ l = 1, 2, \cdots \right\},$$

and $E = \bar{E}_0$ (the closure of E_0 with respect to the norm in $C_b(X)$). Then E_0 is a Banach space as a closed linear subspace of the Banach space $C_b(X)$. By Proposition 2.4, we have

(4.1) $$E_0 \supset C_0(X) \cap \mathscr{D}(L)$$

and accordingly

(4.2) $$E \supset C_0(X).$$

We define an operator G_E in the Banach space E which is a restriction of the operator $\bar{G}$. Let

$$\mathfrak{D}(G_E) = \{v \in \mathfrak{D}(\bar{G}) \cap E \mid \bar{G}v \in E\}$$

and define

$$G_E v = \bar{G}v \qquad \text{for} \quad v \in \mathfrak{D}(G_E).$$

We shall later show that $\mathfrak{D}(G_E)$ is strongly dense in E.

The following proposition shows that every $J_\lambda(\lambda>0)$ may be regarded as an operator in E.

Proposition 4.1. *$J_\lambda u \in E$ for any $u \in E$.*

Proof. It is sufficient to prove this proposition for $u \in E_0$ since E_0 is dense in E and J_λ is a bounded operator. We may assume that $u = J_\mu w$, $w \in C_0(X)$. In case $\lambda \neq \mu$, we obtain from the resolvent equation (Theorem 1) that

$$J_\lambda u = \frac{1}{\mu-\lambda}(J_\lambda w - u) \in E_0 \subset E.$$

In case $\lambda = \mu$, we take a sequence $\{\lambda_n\}$ such that $\lambda_n > \lambda$ and $\lim_{n\to\infty} \lambda_n = \lambda$. Then $J_{\lambda_n} u \in E_0$ for any n as proved above. From the resolvent equation:

$$J_{\lambda_n} u - J_\lambda u = (\lambda - \lambda_n) J_{\lambda_n} J_\lambda u,$$

we get

$$\|J_{\lambda_n}u-J_\lambda u\| \leqq \frac{|\lambda-\lambda_n|}{\lambda_n\cdot\lambda}\|\lambda_n J_{\lambda_n}\|\cdot\|\lambda J_\lambda\|\cdot\|u\|$$

$$\leqq \frac{|\lambda-\lambda_n|}{\lambda^2}\|u\| \to 0 \quad \text{as} \quad n\to\infty .$$

Hence $J_\lambda u \in E$.

Now we are ready to show one of our main theorems.

THEOREM 2. *Let E be the Banach space defined above. Then,*

i) *the family of operators J_λ, $\lambda>0$, restricted to E satisfies the resolvent equation*

$$J_\lambda - J_\mu = (\mu-\lambda)J_\lambda J_\mu \qquad \textit{in} \quad E\,;$$

ii) $\operatorname*{s-lim}_{\lambda\uparrow\infty} \lambda J_\lambda = I$ *and* $\operatorname*{s-lim}_{\lambda\downarrow 0}\lambda J_\lambda = 0$ *in* E;

iii) *there exists a closed linear operator $\hat{G} = \operatorname*{s-lim}_{\lambda\downarrow 0} J_\lambda$ with domain $\mathfrak{D}(\hat{G})$ and range $\mathfrak{R}(\hat{G})$ both strongly dense in E such that the inverse operator $A=-\hat{G}^{-1}$ exists in such a way that $J_\lambda=(\lambda I-A)^{-1}$ and A is the infinitesimal generator of a uniquely determined semigroup $\{T_t\}_{t\geqq 0}$ of class (C_0) of bounded linear operators in E.*

PROOF. Part i) is an immediate consequence of Theorem 1 (§ 2) and Proposition 4.1. Part iii) follows from parts i), ii) and a result by K. Yosida [9] (see Theorem 1 and Remark 2 in [9]). In order to prove part ii), it is sufficient to show that

$$\lim_{\lambda\uparrow\infty}\|\lambda J_\lambda v - v\| = 0 \quad \text{and} \quad \lim_{\lambda\downarrow 0}\|\lambda J_\lambda v\|=0 \tag{4.3}$$

for $v\in E_0$ since $\bar{E}_0=E$ and $\|\lambda J_\lambda\|\leqq 1$ for any $\lambda>0$. Accordingly it is sufficient to show (4.3) for $v=J_\mu w$ where $\mu>0$ and $w\in C_0(X)$. It follows from the resolvent equation and (3.4) that

$$\|\lambda J_\lambda v-v\| = \|\lambda J_\lambda J_\mu w - J_\mu w\| = \|\mu J_\lambda J_\mu w - J_\lambda w\|$$

$$\leqq \frac{1}{\lambda}(\|\lambda J_\lambda\|\cdot\|\mu J_\mu\|\cdot\|w\| + \|\lambda J_\lambda\|\cdot\|w\|)$$

$$\leqq \frac{2}{\lambda}\|w\| \qquad \text{for any} \quad \lambda,\ \mu>0\,,$$

and

$$\|\lambda J_\lambda v\| = \|\lambda J_\lambda J_\mu w\| = \frac{\lambda}{\mu-\lambda}\|J_\lambda w - J_\mu w\|$$

$$\leqq \frac{\lambda}{\mu-\lambda}(\|J_\lambda w\|+\|J_\mu w\|) \leqq \frac{2\lambda}{\mu-\lambda}\|\bar{G}|w|\|$$

whenever $0<\lambda<\mu$. Hence we obtain (4.3). Theorem 2 is thus proved.

In the sequel, we shall investigate the relation between the operators $\hat{G}$ and G_E, and that between A and L.

LEMMA 4.2. *If* $v \in \mathfrak{D}(\bar{G}) \cap E$, *then* $v-\lambda J_\lambda v \in \mathfrak{D}(G_E) \cap \mathfrak{D}(\hat{G})$ *and* $G_E(v-\lambda J_\lambda v)=\hat{G}(v-\lambda J_\lambda v)=J_\lambda v$ *for any* $\lambda>0$.

PROOF. It follows from Lemma 3.3, Proposition 3.4 and Proposition 4.1 that the assumption $v \in \mathfrak{D}(\bar{G}) \cap E$ implies that $v-\lambda J_\lambda v \in \mathfrak{D}(\bar{G})$ and $\bar{G}(v-\lambda J_\lambda v)=J_\lambda v \in E$. Hence $v-\lambda J_\lambda v \in \mathfrak{D}(G_E)$ and $G_E(v-\lambda J_\lambda v)=J_\lambda v$. On the other hand, since $\operatorname{s-lim}_{\mu \downarrow 0} \mu J_\mu v=0$ from part ii) of Theorem 2, it follows from the resolvent equation that

$$\operatorname{s-lim}_{\mu \downarrow 0} J_\mu(v-\lambda J_\lambda v)=\operatorname{s-lim}_{\mu \downarrow 0} J_\lambda(v-\mu\lambda_\lambda v)=J_\lambda v\,.$$

Hence $v-\lambda J_\lambda v \in \mathfrak{D}(\hat{G})$ and $\hat{G}(v-\lambda J_\lambda v)=J_\lambda v$.

THEOREM 3. i) $\mathfrak{D}(G_E)$ *is strongly dense in* E.

ii) $\mathfrak{D}(G_E) \subset \mathfrak{D}(\hat{G})$ *and* $\hat{G}v=G_E v$ *for any* $v \in \mathfrak{D}(G_E)$, *namely* $\hat{G}$ *is an extention of* G_E.

iii) $\mathfrak{D}(L) \cap C_0(X) \subset \mathfrak{D}(A)$ *and* $Lu=Au$ *for any* $u \in \mathfrak{D}(L) \cap C_0(X)$, *namely* A *is an extension of* L *restricted to* $\mathfrak{D}(L) \cap C_0(X)$.

PROOF. i) By Lemma 3.3 and Lemma 4.2, $v-\lambda J_\lambda v \in \mathfrak{D}(G_E)$ for any $v \in E_0$. Since $\operatorname{s-lim}_{\lambda \downarrow 0}(v-\lambda J_\lambda v)=v$ by part ii) of Theorem 2 and since E_0 is strongly dense in E, we may conclude that $\mathfrak{D}(G_E)$ is strongly dense in E.

ii) For any $v \in \mathfrak{D}(G_E)$, we may see from Lemma 4.2 and Proposition 3.4 that $v-\lambda J_\lambda v \in \mathfrak{D}(\hat{G}) \cap \mathfrak{D}(G_E)$ and

$$\hat{G}(v-\lambda J_\lambda v)=G_E v-\lambda \bar{G} J_\lambda v=G_E v-\lambda J_\lambda G_E v\,.$$

Since $\operatorname{s-lim}_{\lambda \downarrow 0}(v-\lambda J_\lambda v)=v$ and $\operatorname{s-lim}_{\lambda \downarrow 0}(G_E v-\lambda J_\lambda G_E v)=G_E v$ from part ii) of Theorem 2 and since $\hat{G}$ is a closed operator, we may conclude that $v \in \mathfrak{D}(\hat{G})$ and $\hat{G}v=G_E v$.

iii) We first notice that $C_0(X) \subset \mathfrak{D}(\bar{G})$; this may be shown by the same argument as we have derived (5.2) in the proof of Theorem 2 in [3]. Hence $C_0(X) \subset \mathfrak{D}(\bar{G}) \cap E$ by (4.2). For any $u \in \mathfrak{D}(L) \cap C_0(X)$, we put $v=Lu$. Then

$$v \in C_0(X) \subset \mathfrak{D}(\bar{G}) \cap E\,. \tag{4.4}$$

We put $v_\lambda=\lambda u-v=(\lambda-L)u$. Then $u=J_\lambda v_\lambda$ as is shown in the proof of Proposition 2.4, and accordingly

$$u=J_\lambda(\lambda u-v)=\lambda J_\lambda u-J_\lambda v\,. \tag{4.5}$$

By part ii) of Theorem 2, we have

$$\operatorname{s-lim}_{\lambda \downarrow 0} \lambda J_\lambda u=0\,,$$

and, from (4.4) and by Lemma 3.2, we may see that

$$\lim_{\lambda \downarrow 0} J_\lambda v=\bar{G}v \qquad \text{(pointwise convergence on } X\text{)}\,.$$

Hence, passing to the limit as $\lambda \downarrow 0$ in (4.5), we obtain $u=-\bar{G}v$ and accordingly

$$\bar{G}v=-u\in C_0(X)\subset E\,. \tag{4.6}$$

From (4.4) and (4.6) follows that $v\in\mathfrak{D}(G_E)$ and $G_Ev=\bar{G}v$. Combining this result and part ii) just proved above, we obtain that $v\in\mathfrak{D}(\hat{G})$ and $u=-G_Ev=-\hat{G}v$. Since $A=-\hat{G}^{-1}$, we may see that $u\in\mathfrak{D}(A)$ and $Au=v=Lu$.

The proof of part iii) of the above theorem incidentally shows the following

PROPOSITION 4.3. *The set* $\mathfrak{R}_0(L)=\{Lu\,|\,u\in\mathfrak{D}(L)\cap C_0(X)\}$ *is contained in* $\mathfrak{D}(G_E)$ *and* $G_ELu=-u$ *for any* $u\in\mathfrak{D}(L)\cap C_0(X)$.

We see from Theorems 2 and 3 that $\mathfrak{D}(G_E)\subset\mathfrak{D}(\hat{G})\subset E$ and $\mathfrak{D}(G_E)$ is strongly dense in E. Proposition 4.3 shows that sufficiently many functions are contained in $\mathfrak{D}(G_E)$. A model of such family is the set of all functions of the form Lu with $u\in C_0^2(X)$ in the case where X is an n-dimensional euclidean domain and $L=\Delta+\sum_{i=1}^{n} a_i(x)\dfrac{\partial}{\partial x_j}$.

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