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On Existence of Green Operator and Positive Superharmonic Functions

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§ 0. Introduction.

In her previous paper [3], the author treated an operator L which is a generalization of the elliptic differential operators of the form $\Delta+\sum_{i=1}^{n} a_i(x)\partial/\partial x_i$ and, by means of an axiomatic treatment, proved that a Green operator associated with L exists if a nonconstant positive L-harmonic function exists.

The purpose of the present paper is to prove, under the axiomatic treatment as in [3], that there exists a Green operator associated with L if and only if a nonconstant positive L-superharmonic function exists. This fact is well known in the case of usual Riemann surfaces [1], [5].

§ 1. Preliminaries.

Let X be a locally compact, σ-compact and connected Hausdorff space, $C(X)$ be the set of all real-valued continuous functions on X and $C_0(X)$ be the set of all functions in $C(X)$ with compact support.

Let L be a (generally unbounded) linear operator of a linear subspace $\mathfrak{D}(L)$ of $C(X)$ into $C(X)$ satisfying the following conditions: $\mathfrak{D}(L)\cap C_0^+(D)$ is dense in $C_0^+(D)$ with respect to the supremum norm for any subdomain D of X, any constant c belongs to $\mathfrak{D}(L)$ and $Lc=0$, and L is a local operator; we further assume that L satisfies the axioms (α), (β), (γ) and (δ) mentioned in [3]; these axioms respectively correspond to maximum principle, Harnack property, the solvability of the "equation" $(\lambda-L)u=f$ in $\mathfrak{D}(L)\cap\overline{C_0(D)}$ and the property known as Weyl's lemma in the case of usual Laplacian. As for the precise meaning of those properties, one may refer to § 1 of the author's previous paper [3]. For example, the axiom (α) is mentioned as follows:

MAXIMUM PRINCIPLE. If $Lu\geqq 0$ and u is nonconstant in a domain D, then u dose not take its maximum in the interior of D.

DEFINITION 1. A subdomain D of X is called a *regular domain* if the closure $\bar{D}$ is compact and, for any $\varphi\in C(\partial D)$, there exists a solution $u\in\mathfrak{D}(L)\cap C(\bar{D})$ of

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the boundary value problem: $Lu=0$ in D and $u=\varphi$ on ∂D.

DEFINITION 2. A function u on a domain $D\subset X$ is said to be *L-harmonic* if $u\in\mathfrak{D}(L)$ and $Lu=0$ in D, and is said to be *L-superharmonic* if it satisfies the following three conditions:

i) $-\infty<u\leqq\infty$ and $u\not\equiv\infty$ in D,

ii) u is lower semicontinuous in D,

iii) if V is a regular domain with its closure $\bar{V}\subset D$ and if w is continuous on $\bar{V}$, L-harmonic in V and satisfies $w\leqq u$ on ∂V, then $w\leqq u$ in V.

DEFINITION 3. A linear operator G of $C_0(X)$ into $C(X)$ is called a *Green operator* associated with L if, for any $f\in\mathfrak{D}(L)\frown C_0(X)$, $u=Gf$ belongs to $\mathfrak{D}(L)$ and satisfies $Lu=-f$ on X.

§ 2. Some properties of L-superharmonic functions.

PROPOSITION 1. *u is L-harmonic in D if and only if both u and $-u$ are L-superharmonic in D.*

PROOF. We first assume that u is L-harmonic in D. Then u and $-u$ obviously satisfy the conditions i) and ii) in the definition of L-superharmonicity. Let V be a regular domain with its closure $\bar{V}\subset D$, and assume that w is continuous on $\bar{V}$, L-harmonic in V and satisfies $w\leqq u$ on ∂V. Since $L(w-u)=0$ in V, $w-u\leqq 0$ in V by the maximum principle. Hence u is L-superharmonic. Similarly we may prove that $-u$ is L-superharmonic.

Next, we assume that both u and $-u$ are L-superharmonic. Clearly, u is continuous. For any regular domain V such that $\bar{V}\subset D$, there exists a function w continuous on $\bar{V}$, L-harmonic in V and satisfying $w=u$ on ∂V. As u is L-superharmonic, we have

$$u\geqq w \quad \text{in} \quad V. \tag{2.1}$$

On the other hand, $-w$ is continuous on $\bar{V}$ and L-harmonic in V, and $-w=-u$ on ∂V. As $-u$ is L-superharmonic, we have

$$u\leqq w \quad \text{in} \quad V. \tag{2.2}$$

(2.1) and (2.2) imply that $u=w$ in V; accordingly u is L-harmonic in V. Since V is taken arbitrarily in D, u is L-harmonic in D.

PROPOSITION 2. *Let u be L-superharmonic in D. If u takes its minimum at some point in D, then u is constant in D.*

PROOF. We put $a=\min_{x\in D}u(x)$ and $A=\{x\,|\,u(x)=a\}$. It is clear that A is closed in D by the lower semicontinuity of u. We assume $A\subsetneqq D$, and consider a point $x_0\in\partial A$ and a regular domain V satisfying $x_0\in V\subset\bar{V}\subset D$. Since A is closed, $u(x_1)>a$ for a suitable $x_1\in\partial V\backslash A$. We take b such as $u(x_1)>b>a$ and put $U=\{x \mid u(x)>b\}$. Then there exists a continuous function v on ∂V satisfying that $v=a$ on $\partial V \backslash U$, $v(x_1)=b$ and $a \leqq v \leqq b$ on ∂V. For V is a regular domain, there exists w which is continuous on $\bar{V}$, harmonic in V and satisfies $w=v$ on ∂V. Since w is not constant, w does not take its minimum in the interior of V by the maximum principle. Hence

(2.3) $$w>a \quad \text{in the interior of} \quad V.$$

On the other hand, it holds that $w \leqq u$ on ∂V by the construction of w. From this fact and the L-superharmonicity of u,

(2.4) $$w \leqq u \quad \text{on} \quad V.$$

Since $V \frown A \neq \emptyset$, (2.3) and (2.4) contradict each other. Hence $A=D$ and accordingly u is constant in D.

§ 3. Main results.

THEOREM 1. *Under the condition that $u \in \mathfrak{D}(L_D)$, u is L-superhormonic in D if and only if $Lu \leqq 0$ in D.*

PROOF. Let u be L-superharmonic in D, and suppose that there exists $x_1 \in D$ for which $Lu(x_1)>0$. We may take an open set U such that $U \ni x_1$ and $Lu>0$ in U, and consider a regular domain V such that $x_1 \in V \subset \bar{V} \subset U$. Then there exists a function w which is continuous on $\bar{V}$, L-harmonic in V and satisfies $w=u$ on ∂V. From the L-superharmonicity of u, it follows that $u \geqq w$ in V. Hence $u-w$ attains its maximum in the interior of V. On the other hand $L(u-w)=Lu>0$ in V. Hence, by the maximum principle, $u-w$ is constant and accordingly $Lu=0$ in V; this contradicts the assumption $Lu(x_1)>0$.

Next assume that $Lu \leqq 0$ in D and we consider an arbitrary regular domain V such that $\bar{V} \subset D$. We consider a function w continuous on $\bar{V}$, L-harmonic in V and satisfying $w \leqq u$ on ∂V. Then $L(w-u) \geqq 0$ in V and hence, by the maximum principle, it holds that $w-u \leqq 0$ in V. Hence u is L-superharmonic in D.

Let $\{D_n\}_{n=0,1,2,\ldots}$ be an exhaustion of X (§ 4 in [3]) and, for each n, ω_n be the function L-harmonic in $D_n \backslash \bar{D}_0$, continuous on $\bar{D}_n \backslash D_0$ and satisfying that $\omega_n =0$ on ∂D_0 and $=1$ on ∂D_n. Then, as is shown in [3], $\omega=\lim_{n\to\infty} \omega_n$ exists and the function ω satisfies either $\omega \equiv 0$ on $X \backslash \bar{D}_0$ or $\omega>0$ everywhere on $X \backslash \bar{D}_0$. We prepare the following:

LEMMA. *Let u be a positive L-superharmonic function on X and let V be an open set such that $V \frown \bar{D}_0=\emptyset$. If $\omega \equiv 0$ in $X \backslash \bar{D}_0$, then*

$$\inf_{\partial V} u \leqq \inf_{V} u .$$

PROOF. We set $M_1=\inf_{\partial V} u$, $M_2=\inf_{V} u$, and suppose that $M_1>M_2$. Then

$$(M_2-M_1)\omega_n+M_1 \leqq M_1 \leqq u \quad \text{on} \quad \partial V \frown D_n$$

and

$$(M_2-M_1)\omega_n+M_1=M_2\leqq u \quad \text{on} \quad V\cap\partial D_n.$$

Since $u-\{(M_2-M_1)\omega_n+M_1\}$ is L-superharmonic, we have

$$(M_2-M_1)\omega_n+M_1\leqq u \quad \text{on} \quad D_n\cap V$$

by Proposition 2. Passing to the limit as $n\to\infty$, we get $M_1\leqq u$ on V; this contradicts the assumption: $M_1>M_2=\inf_V u$.

THEOREM 2. *In order that a Green operator associated with L exists, it is necessary and sufficient that the space X admits a positive nonconstant L-superharmonic function.*

PROOF. We first prove that the condition is necessary. Suppose that there exists a Green operator G. For any $f\in C_0(X)\cap\mathfrak{D}(L)$ which is nonnegative and is not equal to 0 identically, we obtain by Theorem 1 in [3] that $Gf\in\mathfrak{D}(L)$ and $L(Gf)=-f\leqq 0$. It holds that $Gf\geqq 0$ as G is a positive operator, and Gf is nonconstant since $f\not\equiv 0$. Thus Gf is a nonconstant positive L-superharmonic function by Theorem 1.

Next we prove that the condition is sufficient. Let u be a nonconstant positive L-superharmonic function on X. Then there exists M such that $u(x_1)<M<u(x_2)$ for some $x_1, x_2\in X$. Let U be a non-empty open set such that $\bar{U}$ is compact and is contained in $\{x\,|\,u(x)>M\}$, and put $V=X\backslash\bar{U}$. We take an exhaustion $\{D_n\}_{n=0,1,2\cdots}$ such that $\bar{D}_0\subset U$ and define the functions ω_n and ω as mentioned above. Then, since $x_1\in V$ and $u(x_1)<M\leqq\inf_{\bar{U}} u$, we have

$$\inf_V u<\inf_{\bar{U}} u\,.$$

Since $\bar{U}\supset\partial V$, we get

$$\inf_V u<\inf_{\partial V} u\,.$$

Hence we obtain $\omega>0$ by the previous lemma. Therefore, by using the same argument as mentioned in [3], we may prove that there exists a Green operator associated with L.

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