On p-indicators in Ext$(Q/Z, T)$ with Infinitely Many Gaps

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§ 1. Introduction.

All groups considered in this paper are abelian groups. Notations and terminology follow [1] and further details may be found in [2] and [3]. In [1], we investigated p-indicators in Ext$(Q/Z, T)$ without gaps and consequently p-indicators with finitely many gaps, where T is a given reduced p-group. Our aim in this paper is not only to examine p-indicators of elements belonging to Ext$(Q/Z, T)$ with infinitely many gaps, but also to clarify the relations between the groups T, $\varprojlim_{\sigma<\rho} T/p^\sigma T$ and Ext$(Q/Z, T)$ for some ordinal ρ.

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§ 2. On $\varprojlim_{\sigma<\rho} G/p^\sigma G$.

PROPOSITION 1. *Let p be a prime and ρ be an ordinal which is cofinal with ω. If G is an abelian group whose p-length is ρ and $p^\rho G=0$, then for $\alpha<\rho$,*

$$\varprojlim_{\sigma<\rho} p^\alpha G/p^\sigma G = p^\alpha \varprojlim_{\sigma<\rho} G/p^\sigma G .$$

That is, $\varprojlim_{\sigma<\rho} G/p^\sigma G$ is a p-isotype subgroup of $\prod_{\sigma<\rho} G/p^\sigma G$.

PROOF. If we consider $\{p^\alpha G: \alpha<\rho\}$ as a neighborhood system of 0 in G, then G is a Hausdorff group satisfying first axiom of countability and $\varprojlim_{\sigma<\rho} G/p^\sigma G$ is the completion of G by this topology. So, we write $\varprojlim_{\sigma<\rho} G/p^\sigma G=\hat{G}$. By Lemma 37.1 in [2], $p^\alpha(G/p^\sigma G)=p^\alpha G/p^\sigma G$. Consequently we have

$$\begin{aligned} p^\alpha \hat{G} &\subset (p^\alpha \prod_{\sigma<\rho} G/p^\sigma G) \cap \hat{G} \\ &= (\prod_{\sigma<\rho} p^\alpha G/p^\sigma G) \cap \hat{G} \\ &\subset \widehat{p^\alpha G} . \end{aligned}$$

To prove the converse, we need following lemma.

LEMMA 1. *If p, ρ and G are the same as in Proposition* 1, *then* $\widehat{pG}=p\hat{G}$.

PROOF. Let $x\in\varprojlim_{\sigma<\rho} pG/p^{\sigma}G$ and let $\sigma_1, \sigma_2, \cdots$ be an increasing sequence of ordinals where $\sup\sigma_i=\rho$. Let σ_i-th coordinate of x be $px_i+p^{\sigma_i}G$. We may suppose $\sigma_1=1$. Let j be the least integer such that $px_j+p^{\sigma_j}G\neq 0$. Clearly $j>1$. Put $x'_{j-1}=x_j$. Next, we can write $px_{j+1}-px_j=py'_j$ where $y'_j\in p^{\sigma_{j-1}}G$ since $px_{j+1}-px_j\in p^{\sigma_j}G$. Put $x'_j=x'_{j-1}+y'_j$. Repeating this procedure, $x'\in\varprojlim_{\sigma<\rho} G/p^{\sigma}G$ is uniquely determined whose σ_i-th coordinate is $x'_i+p^{\sigma_i}G$ and $px'=x$. Therefore $\widehat{pG}\subset p\hat{G}$.

Let $\alpha<\rho$. Since the p-length of $p^{\alpha}G$ is cofinal with ω, $\widehat{p^{\alpha+1}G}\subset p\cdot\widehat{p^{\alpha}G}$ by Lemma 1. Now, by transfinite induction, we can conclude $\widehat{p^{\alpha}G}\subset p^{\alpha}\hat{G}$ for $\alpha<\rho$. This completes the proof of Proposition 1.

COROLLARY TO PROPOSITION 1. *If p, ρ and G are the same as in Proposition* 1, *then $\hat{G}/G$ is p-divisible.*

PROOF. G is dense in $\hat{G}$ with respect to the neighborhood system $\{\widehat{p^{\alpha}G} : \alpha<\rho\}$. For $x\in\hat{G}$, $(x+p\hat{G})\cap G=(x+\widehat{pG})\cap G\neq\phi$ implies $\hat{G}/G$ is p-divisible.

PROPOSITION 2. *Let p be a prime and ρ be an ordinal which is cofinal with ω. If T is a reduced p-group whose p-length is $\geqq\rho$, then*

(I) $\varprojlim_{\sigma<\rho} G/p^{\sigma}G\cong G/p^{\rho}G$ *where* $G=\mathrm{Ext}(Q/Z, T)$.

(II) $\varprojlim_{\sigma<\rho} T/p^{\sigma}T$ *can be embedded in $G/p^{\rho}G$ as an isotype subgroup.*

PROOF. Corollary to Proposition 1 implies that $(\varprojlim_{\sigma<\rho} G/p^{\sigma}G)/(G/p^{\rho}G)$ is divisible. By 54(F) in [2], $\varprojlim_{\sigma<\rho} G/p^{\sigma}G$ is a reduced cotorsion group. By 54(B) in [2], the fact that $G/p^{\rho}G$ is cotorsion implies that $(\varprojlim_{\sigma<\rho} G/p^{\sigma}G)/(G/p^{\rho}G)$ is reduced. Therefore $\varprojlim_{\sigma<\rho} G/p^{\sigma}G\cong G/p^{\rho}G$.

Applying Proposition 1 to $T/p^{\rho}T$, we know that $\varprojlim_{\sigma<\rho} T/p^{\sigma}T$ is an isotype subgroup of $\prod_{\sigma<\rho} T/p^{\sigma}T$. Since $T/p^{\sigma}T\cong(T+p^{\sigma}T)/p^{\sigma}T$ is an isotype subgroup of $G/p^{\sigma}G$, $\prod_{\sigma<\rho} T/p^{\sigma}T$ can be embedded in $\prod_{\sigma<\rho} G/p^{\sigma}G$ as an isotype subgroup. Therefore $\varprojlim_{\sigma<\rho} T/p^{\sigma}T$ can be embedded in $\varprojlim_{\sigma<\rho} G/p^{\sigma}G\subset\prod_{\sigma<\rho} G/p^{\sigma}G$ as an isotype subgroup.

§ 3. On p-indicators with infinitely many gaps in Ext$(Q/Z, T)$.

PROPOSITION 3. *Let p be a prime and T be a reduced p-group. If* $x\in$

$\operatorname{Ext}(Q/Z, T)$ *has p-indicator* $(\sigma_0, \sigma_1, \sigma_2, \cdots)$ *with infinitely many gaps, then* $x \in \operatorname{Ext}(Q/Z, p^{\sigma_0}T)$.

PROOF. According to the direct decomposition of $p^{\sigma_0}\operatorname{Ext}(Q/Z, T)$ in Proposition 56.4 in [2], we write $x = x_1 + x_2$ where $x_1 \in \operatorname{Ext}(Q/Z, p^{\sigma_0}T)$ and $x_2 \in p^{\sigma_0}\operatorname{Ext}(Q/Z, T/p^{\sigma_0}T)$. x's p-indicator in $p^{\sigma_0}\operatorname{Ext}(Q/Z, T)$ is $(0, \sigma_1', \sigma_2', \cdots)$ where $\sigma_i = \sigma_0 + \sigma_i'$ and gaps occur in the same place as in $(\sigma_0, \sigma_1, \sigma_2, \cdots)$. Suppose $x_2 \neq 0$. Since $p^{\sigma_0}\operatorname{Ext}(Q/Z, T/p^{\sigma_0}T)$ is reduced and torsion free, the p-indicator of x exceeds that of x_2 somewhere. Therefore $x_2 = 0$.

PROPOSITION 4. *Let p be a prime and T be a reduced p-group. Let $\sigma_0, \sigma_1, \sigma_2, \cdots$ be a strictly increasing sequence of ordinals with infinitely many gaps. That is, for a strictly increasing sequence of integers $0 \leqq k_1 < k_2 < k_3 < \cdots$, gaps occur after each σ_{k_i}. Then, the necessary and sufficient condition for $(\sigma_0, \sigma_1, \sigma_2, \cdots)$ to be a p-indicator of some element belonging to $\varprojlim_{\sigma < \sup \sigma_i} T/p^\sigma T$ is that the σ_{k_i}-th Ulm-Kaplansky invariant of T is not equal to 0 for every i.*

PROOF. Suppose $x \in \varprojlim_{\sigma < \sup \sigma_i} T/p^\sigma T$ has p-indicator $(\sigma_0, \sigma_1, \cdots)$. By Proposition 1, we can observe the p-indicator of x in $\prod T/p^\sigma T$. Write the σ_{k_i}-th coordinate of x, $x_i + p^{\sigma_{k_i}}T$. $h(p^{k_i+1}x) > \sigma_{k_i}+1$ implies $p^{k_i+1}x_{k_i+1} = px'$ for some $x' \in p^{\sigma_{k_i}+1}T$. $p^{k_i}x_{k_i+1} - x' \neq 0$ and belongs to $p^{\sigma_{k_i}}T[p]$ follow $h(p^{k_i}x) = \sigma_{k_i}$.

To prove the sufficiency, we need following lemma.

LEMMA 2. *Let $\sigma_0, \sigma_1, \sigma_2, \cdots$ be a strictly increasing sequence of ordinals where the gaps occur after each σ_{k_i}. Then there exists an increasing sequence of integers $i(1) < i(2) < i(3) < \cdots$ such that $\sigma_{k_{i(n)}} + k_{i(n+1)} \leqq \sigma_{k_{i(n+1)}}$.*

PROOF. Put $i(1) = 2$ and $i(n+1) = i(n) + k_{i(n)}$. Gaps occur $i(n+1) - i(n)$ times between $\sigma_{k_{i(n)}}$ and $\sigma_{k_{i(n+1)}}$. Therefore

$$\sigma_{k_{i(n+1)}} \geqq \sigma_{k_{i(n)}} + k_{i(n+1)} - k_{i(n)} + i(n+1) - i(n)$$

$$= \sigma_{k_{i(n)}} + k_{i(n+1)}\,.$$

Now, suppose $p^{\sigma_{k_i}}T[p] \neq 0$ for $i = 1, 2, \cdots$. For $x_{i(1)}$ we can choose an element in T which has p-indicator $(\sigma_0, \sigma_1, \cdots, \sigma_{k_{i(2)}}, \infty)$. Next, let x be in T with p-indicator $(\sigma_{k_{i(2)}+1}, \cdots, \sigma_{k_{i(3)}}, \infty)$. $\sigma_{k_{i(1)}} + k_{i(2)} \leqq \sigma_{k_{i(2)}}$ implies that $x = p^{k_{i(2)}+1}x'$ for some $x' \in p^{\sigma_{k_{i(1)}}}T$ and $p^j x' \in p^{\sigma_{j+1}}T$ for $j = 1, \cdots, k_{i(2)}$. If we put $x_{i(1)} + x' = x_{i(2)}$, $(\sigma_0, \sigma_1, \cdots, \sigma_{k_{i(3)}}, \infty)$ is p-indicator of $x_{i(2)}$ and $x_{i(2)} - x_{i(1)} \in p^{\sigma_{k_{i(1)}}}T$. Repeating this procedure, we get a series of elements in T $x_{i(1)}, x_{i(2)}, \cdots$ where $x_{i(j)}$'s p-indicator is $(\sigma_0, \sigma_1, \cdots, \sigma_{k_{i(j+1)}}, \infty)$ and $x_{i(j+1)} - x_{i(j)} \in p^{\sigma_{k_{i(j)}}}T$. Thus we can construct an element $x \in \varprojlim_{\sigma < \sup \sigma_i} T/p^\sigma T$ whose $\sigma_{k_{i(j)}}$-th coordinate is $x_{i(j)} + p^{\sigma_{k_{i(j)}}}T$ and whose p-indicator is $(\sigma_0, \sigma_1, \sigma_2, \cdots)$. This completes the proof of Proposition 4.

PROPOSITION 5. *Let p be a prime and T be a reduced p-group. Let σ_0, σ_1, σ_2, $\cdots$ be a strictly increasing sequence of ordinals with infinitely many gaps. Then the necessary and sufficient condition for $(\sigma_0, \sigma_1, \sigma_2, \cdots)$ to be a p-indicator of some element belonging to* $\mathrm{Ext}(Q/Z, T)$ *is that the σ_{k_i}-th Ulm-Kaplansky nivariant of T is not equal to* 0 *for every i.*

PROOF. The necessity follows 103, (ii) in [3] since the torsion part of $\mathrm{Ext}(Q/Z, T)$ is T. Put $\sup \sigma_i = \rho$. If $x + p^\rho G$ has p-indicator $(\sigma_0, \sigma_1, \sigma_2, \cdots)$ in $G/p^\rho G$, then x has also p-indicator $(\sigma_0, \sigma_1, \sigma_2, \cdots)$ by Lemma 37.1 in [2]. Thus the sufficient part of Proposition 5 is an immediate consequence of proposition 2, (II) and Proposition 4.

References

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[2] L. Fuchs, Infinite Abelian Groups, Vol. 1, Academic Press, New York, 1970.
[3] L. Fuchs, Infinite Abelian Groups, Vol. 2, Academic Press, New York, 1973.