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Orthogonal Decomposition of Curvature Type Tensors

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Introduction. In this paper, we study the algebraic properties of curvature type tensors of a Riemannian space. Taking the contraction c on B^p (see § 1), we first give the orthogonal decomposition of B^p as $\sum(\mathrm{Ker}\, c^{r+1})\cap(\mathrm{Ker}\, c^r)^{\perp}$. This corresponds to the decomposition structure given by K. Nomizu [4] or R. S. Kulkarni [5]. Next in §§ 3, 4, we consider the first Bianchi identity. Generalizing it, we define the operator $\mathfrak{T}$ on B^p. Then the strict curvature type tensors (those satisfying the first Bianchi identity) are just the elements of Ker $\mathfrak{T}$. Since $\mathfrak{T}$ is a self-adjoint operator on B^p, we can determine all eigenvalues of $\mathfrak{T}$ and obtain the eigen-space decomposition of B^p. This shows that there exist many kinds of curvature type tensors which are not strict. In the last section we take skew-symmetric $2p$-forms (p even) as an example of curvature type tensors. It follows that they coincide with the eigenvectors of the maximum eigenvalue of $\mathfrak{T}$.

1. Contraction of curvature type tensors. Let (M, g) be an n-dimensional Riemannian space with the Riemannian metric tensor $g=(g_{ij})$. We denote by Λ^p the vector space of differential forms of degree p on M, and put $\Lambda=\sum\Lambda^p$. Λ is a graded algebra with respect to the exterior product $\wedge$. The element ω of the tensor space $D^{p,q}=\Lambda^p\otimes\Lambda^q$ has the tensorial component $\omega_{i_1\cdots i_p,j_1\cdots j_q}$ (we often write it as ω_{I_p,J_q} for short) which is skew symmetric for each i and j. Let $\omega=\alpha_1\otimes\beta_1\in D^{p,q}$ and $\eta=\alpha_2\otimes\beta_2\in D^{r,s}$, then the product is defined by $\omega\wedge\eta=(\alpha_1\wedge\alpha_2)\otimes(\beta_1\wedge\beta_2)\in D^{p+r,q+s}$. The space $D=\sum D^{p,q}$ becomes a graded algebra with this product. For $\omega=(\omega_{I_p,J_q})\in D^{r,s}$ and $\eta=(\eta_{I_r,J_s})\in D^{r,s}$, we have

$$(\omega\wedge\eta)_{I_{p+r},J_{q+s}}$$
$$=\frac{1}{p!\,q!\,r!\,s!}\sum_{\alpha,\beta}\varepsilon(\alpha)\varepsilon(\beta)\omega_{i_{\alpha_1}\cdots i_{\alpha_p},j_{\beta_1}\cdots j_{\beta_q}}\eta_{i_{\alpha_{p+1}}\cdots i_{\alpha_{p+r}},j_{\beta_{q+1}}\cdots j_{\beta_{q+s}}},$$

where α and β run through all permutations of degree $p+r$ and $q+s$ respectively, and $\varepsilon(\ )$ denotes the sign of the permutation. Let

$\omega, \eta \in D^{p,q}$ then the inner product $\langle \omega, \eta \rangle$ is defined by

$$\langle \omega, \eta \rangle = \frac{1}{p!\,q!} \sum \omega_{i_1 \cdots i_p, j_1 \cdots j_q} \eta_{a_1 \cdots a_p, b_1 \cdots b_q} g^{i_1 a_1} \cdots g^{i_p a_p} g^{j_1 b_1} \cdots g^{j_q b_q} .$$

If the types of degree of ω and η are different, then we set $\langle \omega, \eta \rangle = 0$. A curvature type tensor ω of degree p is the element $\omega = \omega(_{I_p, J_p}) \in D^{p,p}$ which satisfies

$$\omega_{I_p, J_p} = \omega_{J_p, I_p} .$$

Denote by B^p the space of curvature type tensors of degree p. The space $B = \sum B^p$ is too a graded algebra.

A contraction operator $c: D^{p,q} \to D^{p-1,q-1}$ is defined by

$$(c\omega)_{I_{p-1}, J_{q-1}} = \sum g^{ab} \omega_{aI_{p-1}, bJ_{q-1}}$$

for $\omega = (\omega_{I_p, J_q}) \in D^{p,q}$. For $\omega \in D^{0,q} + D^{p,0}$, we set $c\omega = 0$.

The Riemannian metric tensor $g = (g_{ij})$ is an element of B^1, and we put $g\omega = g \wedge \omega \in D^{p+1,q+1}$ for any $\omega \in D^{p,q}$. Then $g: D^{p,q} \to D^{p+1,q+1}$ is a linear mapping, and the local expression of $g\omega$ for $\omega = (\omega_{I_p, J_q}) \in D^{p,q}$ is given by

$$(g\omega)_{I_{p+1}, J_{q+1}} = \sum_k \sum_h (-1)^{k+h} g_{i_k j_h} \omega_{I_{p+1}(\hat{k}), J_{q+1}(\hat{h})}$$

where $I_{p+1}(\hat{k})$ means the k-th index is deleted from the indices $(i_1 \cdots i_{p+1})$. The next lemma is immediate by the induction.

LEMMA 1-1. *Let r be a positive integer. Then we have*

$$cg^r\omega - g^r c\omega = r(n-p-q-r+1) g^{r-1}\omega ,$$
$$gc^r\omega - c^r g\omega = -r(n-p-q+r-1) c^{r-1}\omega$$

for $\omega \in D^{p,q}$.

LEMMA 1-2. *Let $\omega \in D^{p+1,q+1}$ and $\eta \in D^{p,q}$, then*

$$\langle c\omega, \eta \rangle = \langle \omega, g\eta \rangle .$$

PROOF. Taking $\eta = (\eta_{I_p, J_q}) \in D^{p,q}$ we have

$$\begin{aligned}
\langle \omega, g\eta \rangle &= \frac{1}{(p+1)!\,(q+1)!} \sum \omega_{I_{p+1}, J_{q+1}} \sum (-1)^{k+h} g_{i_k j_h} \eta_{I_{p+1}(\hat{k}), J_{q+1}(\hat{h})} \\
&= \frac{1}{(p+1)!\,(q+1)!} \sum (-1)^{k+h} \omega_{i_k I_{p+1}(\hat{k}), j_h J_{q+1}(\hat{h})} \\
&\quad \times (-1)^{k+h} g_{i_k j_h} \eta_{I_{p+1}(\hat{k}), J_{q+1}(\hat{h})} \\
&= \frac{1}{(p+1)!\,(q+1)!} \sum (c\omega)_{I_{p+1}(\hat{k}), J_{q+1}(\hat{h})} \eta_{I_{p+1}(\hat{k}), J_{q+1}(\hat{h})} \\
&= \frac{1}{p!\,q!} \sum (c\omega)_{I_p, J_q} \eta_{I_p, J_q} = \langle c\omega, \eta \rangle .
\end{aligned}$$

COROLLARY 1-3. *The mapping* $g\colon D^{p,q}\to D^{p+1,q+1}$ *is injective if* $n-p-q>0$.

PROOF. Let $\omega\in D^{p,q}$ satisfy $g\omega=0$. From Lemmas 1-1, 1-2 we have

$$\begin{aligned}\langle gc\omega-cg\omega,\,\omega\rangle&=\langle c\omega,\,c\omega\rangle-\langle g\omega,\,g\omega\rangle\\&=-(n-p-q)\langle\omega,\,\omega\rangle\ .\end{aligned}$$

Hence if $n-p-q>0$, $\langle c\omega,\,c\omega\rangle=\langle\omega,\,\omega\rangle=0$ holds, and consequently $\omega=0$ follows.

DEFINITION. The linear operator $\sigma\colon D^{p,q}\to D^{p+1,q+1}$ is defined by

$$\sigma\theta=\sum_{r\geqq 1}\frac{(-1)^{r-1}}{r!\prod_{k=0}^{r-1}(m+k)}g^{r}c^{r-1}\theta$$

for $\theta\in D^{p,q}$, where we put $m=n-p-q>0$. The sum is finite because $c^{N}\theta=0$ for large N.

LEMMA 1-4. *For* $\theta\in D^{p,q}$, *we have, if* $n-p-q>0$

$$\begin{aligned}c\sigma\theta&=\theta\ ,\\\sigma cg\theta&=g\theta\ .\end{aligned}$$

PROOF. Since $c^{r-1}\theta\in D^{p-r+1,q-r+1}$, we have

$$\begin{aligned}c\sigma\theta&=\sum\frac{(-1)^{r-1}}{r!\prod_{k=0}^{r-1}(m+k)}(g^{r}c^{r}\theta+r(m+r-1)g^{r-1}c^{r-1}\theta)\\&=\sum\frac{(-1)^{r-1}}{r!\prod_{k=0}^{r-1}(m+k)}g^{r}c^{r}\theta-\sum\frac{(-1)^{r-2}}{r!\prod_{k=0}^{r-2}(m+k)}g^{r-1}c^{r-1}\theta+\frac{1}{m}m\theta\\&=\theta\ .\end{aligned}$$

By the similar way, we have for $cg\theta\in D^{p,q}$

$$\begin{aligned}\sigma cg\theta&=\sum\frac{(-1)^{r-1}}{r!\prod_{k=0}^{r-1}(m+k)}g^{r}c^{r}g\theta\\&=\sum\frac{(-1)^{r-1}}{r!\prod_{k=0}^{r-1}(m+k)}g^{r}(gc^{r}\theta+r(m+r-1)c^{r-1}\theta)\\&=\sum\frac{(-1)^{r-1}}{r!\prod_{k=0}^{r-1}(m+k)}g^{r+1}c^{r}\theta+\sum\frac{(-1)^{r-1}}{(r-1)!\prod_{k=0}^{r-2}(m+k)}g^{r}c^{r-1}\theta+g\theta\\&=g\theta\ .\end{aligned}$$

As an easy result, we have the following

COROLLARY 1-5. *The mapping* $c: D^{p,q} \to D^{p-1,q-1}$ *is surjective if* $p+q<n$.

2. Orthogonal decomposition of $D^{p,q}$ with respect to the contraction c. Let $\omega \in D^{p,q}$. The mapping conf: $D^{p,q} \to D^{p,q}$ is defined by

$$\mathrm{conf}\,\omega = \omega - \sigma c \omega \ .$$

This coincides with the mapping con in Kulkarni [5]. We have from Lemma 1-4 that for $\omega \in D^{p,q}$

$$c(\mathrm{conf}\,\omega) = 0 \ ,$$
$$\mathrm{conf}\,(g\omega) = 0 \ .$$

According to Kulkarni, we call effective the elements of $\mathrm{Ker}\, c$ and set $E^{p,q} = \{\omega \in D^{p,q};\ c\omega = 0\}$.

LEMMA 2-1. conf: $D^{p,q} \to D^{p,q}$ *satisfies*

$$\mathrm{conf}^2 = \mathrm{conf} \ ,$$
$$\langle \mathrm{conf}\,\omega, \eta \rangle = \langle \omega, \mathrm{conf}\,\eta \rangle \ ,$$

for ω *and* $\eta \in D^{p,q}$. *Moreover we have*

$$\mathrm{Im\,conf} = \mathrm{Ker}\, c \ , \qquad \mathrm{Ker\,conf} = \mathrm{Im}\, g \ .$$

PROOF. The last relations are evident from Lemma 1-4 and the fact that $\mathrm{Im}\,\sigma \subset \mathrm{Im}\, g$. Next we have

$$\mathrm{conf}\,(\mathrm{conf}\,\omega) = \mathrm{conf}\,\omega - \sigma c\,\mathrm{conf}\,\omega = \mathrm{conf}\,\omega \ ,$$

and

$$\begin{aligned} \langle \mathrm{conf}\,\omega, \eta \rangle &= \langle \omega, \eta \rangle - \sum \frac{(-1)^{r-1}}{r!\prod_{k=0}^{r-1}(m+k)} \langle g^r c^r \omega, \eta \rangle \\ &= \langle \omega, \eta \rangle - \sum \frac{(-1)^{r-1}}{r!\prod_{k=0}^{r-1}(m+k)} \langle c^r \omega, c^r \eta \rangle \end{aligned}$$

which is symmetric with respect to ω and η. Hence

$$\langle \mathrm{conf}\,\omega, \eta \rangle = \langle \omega, \mathrm{conf}\,\eta \rangle$$

holds.

THEOREM 2-2. $D^{p,q}$ *is decomposed as the following*:

$$D^{p,q}=C_0^{p,q}\oplus C_1^{p,q} \quad C_0^{p,q}=\{\omega\in D^{p,q};\ \mathrm{conf}\ \omega=0\}\ ,$$
$$C_1^{p,q}=\{\omega\in D^{p,q};\ \mathrm{conf}\ \omega=\omega\}\ .$$

We have $C_0^{p,q}=\mathrm{Im}\, g$ *and* $C_1^{p,q}=\mathrm{Ker}\, c$.

PROOF. Since conf is symmetric with respect to $\langle\ ,\ \rangle$ and satisfies $\mathrm{conf}^2=\mathrm{conf}$, the eigenvalues of conf are 0 and 1. The eigenspaces $C_0^{p,q}$ and $C_1^{p,q}$ are orthogonal and the space $D^{p,q}$ is decomposed as their direct sum. The last assertion is obvious.

As a consequence of the above lemmas, we obtain the following exact sequences if $n-p-q>0$:

$$(0)\to D^{p-1,q-1}\xrightarrow{g}D^{p,q}\xrightarrow{\mathrm{conf}}D^{p,q}\xrightarrow{c}D^{p-1,q-1}\to(0)\ .$$

Since the relations $C_1^{p,q}=E^{p,q}$ and $C_0^{p,q}=g(D^{p-1,q-1})$ hold, we have $D^{p,q}=E^{p,q}\oplus g(D^{p-1,q-1})$, and hence

$$D^{p,q}=E^{p,q}\oplus gE^{p-1,q-1}\oplus g^2E^{p-2,q-2}\oplus\cdots$$

is obtained. We denote by $A^{\perp}$ the orthogonal complement of the space A. Then it is easy to prove

$$g^rE^{p-r,q-r}=\mathrm{Ker}\, c^{r+1}\cap(\mathrm{Ker}\, c^r)^{\perp}$$

for any integer $r\geqq 0$, because we have $c^{r+1}(g^rE^{p-r,q-r})=0$. Therefore the components $g^rE^{p-r,q-r}$ of the decomposition of $D^{p,q}$ are orthognal.

THEOREM 2-3. *The decomposition of* $D^{p+1,q+1}=\sum g^rE^{p-r+1,q-r+1}=\sum(\mathrm{Ker}\, c^{r+1})\cap(\mathrm{Ker}\, c^r)^{\perp}$ *can be written explicitly as*

$$\omega=\mathrm{conf}\ \omega+\sum\frac{g^r\ \mathrm{conf}\ c^r\omega}{r!\prod_{k=0}^{r-1}(m+r-1+k)}\ ,\quad m=n-p-q$$

for $\omega\in D^{p+1,q+1}$.

PROOF. We can put, for any $\omega\in D^{p+1,q+1}$,

$$\omega=\mathrm{conf}\ \omega+\sum a_rg^r\ \mathrm{conf}\ c^r\omega$$

for some real numbers a_r. Then we have

$$c\omega=\sum r(m+r-1)a_rg^{r-1}\ \mathrm{conf}\ c^r\omega\ ,$$

and

$$c^r\omega=r!\prod_{k=0}^{r-1}(m+r-1+k)a_r\ \mathrm{conf}\ c^r\omega+g\eta$$

holds for some $\eta\in D^{p-r,q-r}$. Operating conf on each side, we get $a_r=1/(r!\prod_{k=0}^{r-1}(m+r-1+k))$, because $\mathrm{conf}^2=\mathrm{conf}$ and $\mathrm{conf}\ g=0$.

3. First Bianchi identity. We define the first Bianchi operator $\mathfrak{S}: D^{p,q}\to D^{p+1,q-1}$ according to Kulkarni [5] by

$$(\mathfrak{S}\omega)_{I_{p+1},J_{q-1}}=\sum(-1)^{k-1}\omega_{I_{p+1}(\hat{k}),i_kJ_{q-1}}$$

for $q\geqq 1$, and $\mathfrak{S}\omega=0$ for $\omega\in D^{p,0}$. The p-th curvature type tensor ω satisfying $\mathfrak{S}\omega=0$ is called strict. Clearly the strict curvature tensor is an extension of the first Bianchi identity for the Riemannian curvature tensor on a Riemannian space.

Taking the $*$-operator $*: \Lambda^p\to\Lambda^{n-p}$ we extend it to the mapping $*: D^{p,q}\to D^{n-p,n-q}$ as

$$(*\omega)_{I_{n-p},J_{n-q}}=\frac{1}{p!\,q!}\sum\det(g_{ij})g^{a_1a_1'}\cdots g^{a_pa_p'}g^{b_1b_1'}\cdots$$
$$g^{b_qb_q'}\varepsilon(a_1\cdots a_p\,i_1\cdots i_{n-p})\varepsilon(b_1\cdots b_q\,j_1\cdots j_{n-q})\omega_{a_1'\cdots a_p',b_1'\cdots b_q'}\,.$$

Then as usual, we have for $\omega\in D^{p,q}$

$$**\omega=(-1)^{(n+1)(p+q)}\omega\,.$$

We define $\bar{\mathfrak{S}}: D^{p,q}\to D^{p-1,q+1}$ by

$$\bar{\mathfrak{S}}\omega=(-1)^{n(p+q)-1}*\mathfrak{S}*\omega\,.$$

Then we get the local expression as

$$(\bar{\mathfrak{S}}\omega)_{I_{p-1},J_{q+1}}=\sum(-1)^{h-1}\omega_{j_hI_{p-1},J_{q+1}(\hat{h})}\,.$$

LEMMA 3-1. *We have*

$$\langle\mathfrak{S}\omega,\eta\rangle=\langle\omega,\bar{\mathfrak{S}}\eta\rangle$$

for $\omega\in D^{p,q}$, $\eta\in D^{p+1,q-1}$.

PROOF. Taking the components of the metric tensor as $g_{ij}=\delta_{ij}$ at a point, we calculate

$$\langle\mathfrak{S}\omega,\eta\rangle=\frac{1}{(p+1)!\,(q-1)!}\sum(-1)^{h-1}\omega_{I_{p+1}(\hat{h}),J_{q-1}}\eta_{I_{p+1},J_{q-1}}$$
$$=\frac{1}{p!\,(q-1)!}\sum\omega_{I_p',jJ_{q-1}}\eta_{jI_p',J_{q-1}}$$

and

$$\langle\omega,\bar{\mathfrak{S}}\eta\rangle=\frac{1}{p!\,q!}\sum(-1)^{h-1}\omega_{I_p,J_q}\eta_{j_hI_p,J_q(\hat{h})}$$
$$=\frac{1}{p!\,(q-1)!}\sum\omega_{I_p,jJ_{q-1}'}\eta_{jI_p,J_{q-1}'}$$
$$=\langle\mathfrak{S}\omega,\eta\rangle\,.$$

Since we have for $\omega \in D^{p,q}$, $\eta \in D^{r,s}$

$$(\mathfrak{S}\omega)_{I_{p+1},J_{q-1}} = \frac{(-1)^p}{p!} \sum_{\alpha} \varepsilon(\alpha)\omega_{i_{\alpha_1}\cdots i_{\alpha_p}, i_{\alpha_{p+1}}J_{q-1}} ,$$

$$(\mathfrak{S}\omega \wedge \eta)_{I_{p+r+1},J_{q+s-1}} = \frac{(-1)^p}{p!\,(q-1)!\,r!\,s!} \sum_{\alpha,\beta} \varepsilon(\alpha)\varepsilon(\beta)$$
$$\times \omega_{i_{\alpha_1}\cdots i_{\alpha_{p+1}}, j_{\beta_1}\cdots \beta_{q-1}} \eta_{i_{\alpha_{p+2}}\cdots i_{\alpha_{p+r+1}}, j_{\beta_q}\cdots j_{\beta_{q+s-1}}}$$

where α and β are the permutations of suitable orders, we obtain the following lemma.

LEMMA 3-2. *Let $\omega \in D^{p,q}$ and $\eta \in D^{r,s}$, then*

$$\mathfrak{S}(\omega \wedge \eta) = \mathfrak{S}\omega \wedge \eta + (-1)^{p+q}\omega \wedge \mathfrak{S}\eta .$$

LEMMA 3-3. *$\mathfrak{S}$ commutes with c, g, σ and* conf.

PROOF. We have

$$\begin{aligned}(\mathfrak{S}c\omega)_{I_p,J_{q-2}} &= \sum (-1)^{h-1}(c\omega)_{I_p(\hat{h}),i_hJ_{q-2}} \\ &= \sum (-1)^{h-1}\omega_{i_0I_p(\hat{h}),i_0i_hJ_{q-2}} \\ &= \sum (-1)^{h}\omega_{i_0I_p(\hat{h}),i_hi_0J_{q-2}} + \omega_{I_p,i_0i_0J_{q-2}} \\ &= (\mathfrak{S}\omega)_{i_0I_p,i_0J_{q-2}} \\ &= (c\mathfrak{S}\omega)_{I_p,J_{q-2}} .\end{aligned}$$

The metric tensor $g=(g_{ij}) \in B^1$ satisfies $\mathfrak{S}g=0$, and hence $\mathfrak{S}(g\omega)=g\mathfrak{S}\omega$ follows. The mappings σ and conf consist of the mappings g and c, and taking consideration of the coefficients of their expressions, we obtain

$$\mathfrak{S}\sigma = \sigma\mathfrak{S} ,$$
$$\mathfrak{S}\,\mathrm{conf} = \mathrm{conf}\,\mathfrak{S} .$$

COROLLARY 3-4. *$\mathfrak{S}$ leaves invariant each orthogonal component $g^rE^{p-r,q-r}$ of $D^{p,q}$. Especially, if $\omega \in D^{p,q}$ satisfies $\mathfrak{S}\omega=0$, then $\mathfrak{S}(g^r\,\mathrm{conf}\,c^r\omega)=0$ holds for any r.*

THEOREM 3-5. *Let $\omega \in B^p$. Then $\omega \in (\mathrm{Ker}\,\mathfrak{S})^\perp$ if and only if $\omega_{I_p,I_p}=0$ for any $I_p=(i_1, \cdots, i_p)$.*

PROOF. Taking a fixed subset $A=(a_1 \cdots a_p)$ of $(1, \cdots, n)$, we set

$$\omega^A_{I_p,J_p} = \sum_{\sigma,\tau} \varepsilon(\sigma)\varepsilon(\tau)\delta^{a_{\sigma_1}}_{i_1} \cdots \delta^{a_{\sigma_p}}_{i_p}\delta^{a_{\tau_1}}_{j_1} \cdots \delta^{a_{\tau_p}}_{j_p} .$$

Then ω^A is a p-th curvature type tensor. We have

$$(\mathfrak{S}\omega^A)_{I_{p+1},J_{p-1}} = \frac{1}{p!} \sum \varepsilon(\alpha)\varepsilon(\sigma)\varepsilon(\tau)\delta^{a_{\sigma_1}}_{i_{\alpha_1}} \cdots \delta^{a_{\sigma_p}}_{i_{\alpha_p}}\delta^{a_{\tau_1}}_{i_{\alpha_{p+1}}}\delta^{a_{\tau_2}}_{j_1} \cdots \delta^{a_{\tau_p}}_{j_{p+1}} .$$

In each terms of the summation there exists an integer k, $1\leqq k\leqq p$, such that $\sigma_k=\tau_1$ and hence it contains as a factor $\delta^{a_{\sigma_k}}_{i_{a_k}}\delta^{a_{\sigma_k}}_{i_{a_{p+1}}}$ which vanishes. Thus $\mathfrak{S}\omega^A=0$ holds. Then taking the metric tensor $g=(\delta_{ij})$ at a point, we have

$$\begin{aligned}\langle\omega, \omega^A\rangle &=\left(\frac{1}{p!}\right)^2 \sum \omega_{I_p,J_p}\omega^A_{I_p,J_p}\\ &=\omega_{A,A}\ .\end{aligned}$$

Hence if $\omega\in(\operatorname{Ker}\mathfrak{S})^\perp$, then $\omega_{A,A}=0$ for any $A=(a_1\cdots a_p)$. Conversely, let $\omega=\omega_1+\omega_2$ be an orthogonal decomposition such that $\omega_1\in\operatorname{Ker}\mathfrak{S}$ and $\omega_2\in(\operatorname{Ker}\mathfrak{S})^\perp$. We assume $\omega_{A,A}=0$. The above argument shows $(\omega_2)_{A,A}=0$, and hence $(\omega_1)_{A,A}=0$. It is well known that a strict curvature type tensor ω_1 satisfying $(\omega_1)_{A,A}=0$ must be 0, we conclude that $\omega=\omega_2\in(\operatorname{Ker}\mathfrak{S})^\perp$.

REMARK. Taking an orthonormal basis $e_1, \cdots, e_n$ of a tangent space, $\omega_{A,A}$ is sometimes called the sectional curvature $K_\omega(\sigma)$ of ω for the p-plane $\sigma=e_{a_1}\wedge\cdots\wedge e_{a_p}$.

A direct computation shows the following

LEMMA 3-6. *For* $\omega\in D^{p,q}$, we have

$$(\mathfrak{S}\bar{\mathfrak{S}}\omega)_{I_p,J_q}=p\omega_{I_p,J_q}-\sum_{h,k}\omega_{I_p(\overset{h}{\vee}{j_k}),J_q(\overset{k}{\vee}{i_h})}\ ,$$

$$(\bar{\mathfrak{S}}\mathfrak{S}\omega)_{I_p,J_q}=q\omega_{I_p,J_q}-\sum_{h,k}\omega_{I_p(\overset{h}{\vee}{j_k}),J_q(\overset{k}{\vee}{i_h})}\ ,$$

where $I_p(\overset{h}{\vee}{j_k})$ means the h-th number i_h is replaced with the number j_k. In particular, for $\omega\in D^{p,q}$ we have

$$(\mathfrak{S}\bar{\mathfrak{S}}-\bar{\mathfrak{S}}\mathfrak{S})\omega=(p-q)\omega\ .$$

For $\omega\in B^p$, we have $\mathfrak{S}\bar{\mathfrak{S}}\omega=\bar{\mathfrak{S}}\mathfrak{S}\omega$. We put $\mathfrak{T}=\mathfrak{S}\bar{\mathfrak{S}}\colon B^p\to B^p$. $\mathfrak{T}$ is a linear mapping and can be written locally as

$$(\mathfrak{T}\omega)_{I_p,J_p}=p\omega_{I_p,J_p}-\sum_{h,k}\omega_{I_p(\overset{k}{\vee}{j_h}),J_p(\overset{h}{\vee}{i_k})}\ .$$

LEMMA 3-7. $\operatorname{Ker}\mathfrak{T}=\operatorname{Ker}\mathfrak{S}=\operatorname{Ker}\bar{\mathfrak{S}}$.

PROOF. $\operatorname{Ker}\mathfrak{S}\subset\operatorname{Ker}\mathfrak{T}$ is trivial. Let $\mathfrak{T}\omega=0$ for $\omega\in B^p$, then $\langle\mathfrak{T}\omega, \omega\rangle=\langle\bar{\mathfrak{S}}\mathfrak{S}\omega, \omega\rangle=\langle\mathfrak{S}\omega, \mathfrak{S}\omega\rangle$ implies $\mathfrak{S}\omega=0$. The proof for $\bar{\mathfrak{S}}$ is similar.

Owing to the Lemmas 3-1, 3-3, we get

LEMMA 3-8. *$\mathfrak{T}$ commutes with the operators c, g and* conf, *and is a self-adjoint operator on B^p, i.e., it satisfies*

$$\langle \mathfrak{T}\omega, \eta\rangle = \langle \omega, \mathfrak{T}\eta\rangle$$

for $\omega, \eta \in B^p$.

COROLLARY 3-9. *$\mathfrak{T}$ leaves invariant the orthogonal components $g^r(E^{p-r,q-r} \cap B^{p-r})$ of B^p. Each eigenvalue of $\mathfrak{T}$ is a non-negative real number.*

PROOF. The first half is evident. Let λ and $\omega \neq 0$ be an eigenvalue and the eigenvector of $\mathfrak{T}$ then λ is real and we have

$$\lambda\langle \omega, \omega\rangle = \langle \mathfrak{T}\omega, \omega\rangle = \langle \mathfrak{S}\omega, \mathfrak{S}\omega\rangle$$

from which $\lambda \geqq 0$ is obtained.

4. Orthogonal decomposition of B^p with respect to $\mathfrak{T}$. We will determine all eigenvalues of $\mathfrak{T}$ on B^p. Any eigenvalue λ of $\mathfrak{T}$ is non-negative, and from Lemma 3-7 we know that the eigenvectors of $\lambda = 0$ are just the strict curvature type tensors. Since the elements of B^1 are strict by definition, we consider in the following under the condition $2 \leqq p < n/2$.

LEMMA 4-1. *For $\omega \in D^{p,q}$, we have*

$$(\mathfrak{S}^r\bar{\mathfrak{S}} - \bar{\mathfrak{S}}\mathfrak{S}^r)\omega = r(p-q+r-1)\mathfrak{S}^{r-1}\omega\ .$$

PROOF. When $r=1$ this reduces to Lemma 3-6. We assume that the equation is valid for $r \geqq 2$. Then we have

$$\begin{aligned}\mathfrak{S}^{r+1}\bar{\mathfrak{S}}\omega &= \mathfrak{S}^r(\bar{\mathfrak{S}}\mathfrak{S}\omega + (p-q)\omega)\\ &= \bar{\mathfrak{S}}\mathfrak{S}^{r+1}\omega + r(p-q+2+r-1)\mathfrak{S}^r\omega + (p-q)\mathfrak{S}^r\omega\\ &= \bar{\mathfrak{S}}\mathfrak{S}^{r+1}\omega + (r+1)(p-q+r)\mathfrak{S}^r\omega\end{aligned}$$

which shows the lemma is true for $r+1$.

LEMMA 4-2. *The only possible eigenvalues of $\mathfrak{T}$ on B^2 are* 0, 2, 6.

PROOF. Let $\omega \in D^{2,2}$. Then we have $\mathfrak{T}^2\omega = \mathfrak{S}^2\bar{\mathfrak{S}}^2\omega + 2\mathfrak{T}\omega$, and since $\bar{\mathfrak{S}}^3\omega = 0$,

$$\begin{aligned}\mathfrak{T}^3\omega &= \mathfrak{S}^3\bar{\mathfrak{S}}^3\omega + 8\mathfrak{T}^2\omega - 12\mathfrak{T}\omega\\ &= 8\mathfrak{T}^2\omega - 12\mathfrak{T}\omega\end{aligned}$$

follows. Taking λ and $\omega \in D^{2,2}$ such that $\mathfrak{T}\omega = \lambda\omega$, we get

$$(\lambda^3 - 8\lambda^2 + 12\lambda)\omega = 0$$

from which $\lambda = 0$ or 2 or 6 holds.

THEOREM 4-3. *The only possible eigenvalues of* $\mathfrak{T}$ *on* B^p *are the* $p+1$ *numbers* $\{r(r+1), 0\leqq r\leqq p\}$.

PROOF. It is sufficient to prove that $\mathfrak{T}$ has $p+1$ eigenvalues $\{r(r+1), 0\leqq r\leqq p\}$ on $D^{p,p}$. First we show by the induction that the equation

$$\mathfrak{T}^r\omega=\mathfrak{S}^r\bar{\mathfrak{S}}^r\omega+a_{r-1}^{(r)}\mathfrak{T}^{r-1}\omega+\cdots+a_1^{(r)}\mathfrak{T}\omega$$

holds for $\omega\in D^{p,p}$ and any integer $r\geqq 2$, where $a_i^{(r)}$ are certain real numbers. For $r=2$ or 3, the equation is given in the proof of Lemma 4-2. We assume it is true for r. Since $\mathfrak{T}^r\omega\in D^{p,p}$ and $\mathfrak{S}^r\omega\in D^{p-r,p+r}$, we have

$$\begin{aligned}\mathfrak{T}^{r+1}\omega&=\mathfrak{S}\bar{\mathfrak{S}}(\mathfrak{S}^r\bar{\mathfrak{S}}^r\omega)+\mathfrak{T}(\textstyle\sum a_i^{(r)}\mathfrak{T}^i\omega)\\&=\mathfrak{S}(\mathfrak{S}^r\bar{\mathfrak{S}}^{r+1}\omega+r(r+1)\mathfrak{S}^{r-1}\bar{\mathfrak{S}}^r\omega)+\textstyle\sum a_i^{(r)}\mathfrak{T}^{i+1}\omega\\&=\mathfrak{S}^{r+1}\bar{\mathfrak{S}}^{r+1}\omega+r(r+1)(\mathfrak{T}^r\omega-\textstyle\sum a_i^{(r)}\mathfrak{T}^i\omega)+\textstyle\sum a_i^{(r)}\mathfrak{T}^{i+1}\omega\\&=\mathfrak{S}^{r+1}\bar{\mathfrak{S}}^{r+1}\omega+(a_{r-1}^{(r)}+r(r+1))\mathfrak{T}^r\omega\\&\quad+\sum_{i=2}^{r-1}(a_{i-1}^{(r)}-r(r+1)a_i^{(r)})\mathfrak{T}^i\omega-r(r+1)a_1^{(r)}\mathfrak{T}\omega\\&=\mathfrak{S}^{r+1}\bar{\mathfrak{S}}^{r+1}\omega+\textstyle\sum a_i^{(r+1)}\mathfrak{T}^i\omega\ .\end{aligned}$$

Thus the equation is true for $r+1$. Moreover we see that

$$a_1^{(r+1)}=-r(r+1)a_1^{(r)}$$

holds for $r\geqq 2$, and $a_1^{(2)}=2$. Therefore

$$a_1^{(r+1)}=(-1)^{r+1}(r+1)r^2\cdots 3^2 2^2$$

is obtained. Taking λ and $\omega_p(\neq 0)\in D^{p,p}$ such that $\mathfrak{T}\omega_p=\lambda\omega_p$, we have as before the equation

$$(\lambda^{p+1}-a_p^{(p+1)}\lambda^p-\cdots-a_1^{(p+1)}\lambda)\omega_p=0\ .$$

Then the non-zero solutions $\lambda_1,\cdots,\lambda_p$ satisfy

$$(-1)^p\lambda_1\cdots\lambda_p=(-1)^p(p+1)p^2\cdots 3^2 2^2\ .$$

If μ is an eigenvalue of $\mathfrak{T}$ on $D^{p-1,p-1}$, then we have $\mathfrak{T}\omega_{p-1}=\mu\omega_{p-1}$ for some $\omega_{p-1}\in D^{p-1,p-1}$ and hence $\mathfrak{T}(g\omega_{p-1})=\mu g\omega_{p-1}$. Since g is injective for $p<n/2$, μ is an eigenvalue of $\mathfrak{T}$ on $D^{p,p}$. Thus we see that $\mu\,(\neq 0)$ is one of $\lambda_1,\cdots,\lambda_p$. Taking consideration of Lemma 4-2 we suppose that the non-zero eigenvalues of $\mathfrak{T}$ on $D^{p-1,p-1}$ are the $p-1$ numbers $r(r+1)$ $(1\leqq r\leqq p-1)$, then they are eigenvalues of $\mathfrak{T}$ on $D^{p,p}$ which are the same as $\lambda_1,\cdots,\lambda_{p-1}$. The last one λ_p is easily given by the above equation as $p(p+1)$. This proves the theorem.

COROLLARY 4-4. *If $\omega_p \in B^p$ satisfies $\mathfrak{T}\omega_p = p(p+1)\omega_p$, then ω_p is effective.*

PROOF. For such ω_p, it holds that

$$\mathfrak{T}(c\omega_p) = p(p+1)c\omega_p \ , \quad c\omega_p \in B^{p-1} \ .$$

However $p(p+1)$ is not contained in $\{r(r+1), 0 \leqq r \leqq p-1\}$ which are the possible numbers of eigenvalues of $\mathfrak{T}$ on B^{p-1}, hence we conclude $c\omega_p = 0$.

Let V^2_λ denote the eigenspaces of $\mathfrak{T}$ on B^2 for the eigenvalue λ. Then by the definition it follows

$$V^2_0 = \{\omega \in B^2;\ \omega_{ij,kh} = \omega_{ik,jh} - \omega_{jk,ih}\} \ ,$$
$$V^2_2 = \left\{\omega \in B^2;\ \omega_{ij,kh} = -\frac{1}{2}(\omega_{ik,jh} - \omega_{jk,ih})\right\} \ ,$$
$$V^2_6 = \left\{\omega \in B^2;\ \omega_{ij,kh} = \frac{1}{2}(\omega_{ik,jh} - \omega_{jk,ih})\right\} \ .$$

Making use of the property $\omega_{ij,kh} = \omega_{kh,ij}$, it is shown that if $\omega \in V^2_2$, then $\omega_{ij,kh} = -\omega_{ih,kj} = 0$, and if $\omega \in V^2_6$, then $\omega_{ij,kh} = -\omega_{ih,kj}$. Hence we have

$$V^2_2 = (0) \ ,$$
$$V^2_6 = \Lambda^4 \ .$$

THEOREM 4-5. *The eigenvalues of $\mathfrak{T}$ on B^2 are 0 and 6. V^2_0 is the space of the strict curvature type 2-tensors and V^2_6 is the space of the skew-symmetric 4-tensors.*

Theorem 4-3 states that the maximum eigenvalue of $\mathfrak{T}$ on B^p is possibly $p(p+1)$. We show

LEMMA 4-6. *For $\omega \in B^p$, we have* $\mathrm{alt}\,(\omega) \in B^p$ *and*

$$\mathfrak{T}(\mathrm{alt}\,(\omega)) = p(p+1)(\mathrm{alt}\,(\omega)) \ ,$$

and hence $\Lambda^{2p} \subset V^p_{p(p+1)}$.

PROOF. Direct computation shows

$$\begin{aligned}\mathfrak{T}(\mathrm{alt}\,(\omega))_{I_p,J_p} &= p(\mathrm{alt}\,(\omega))_{I_p,J_p} - \sum_{k,h} (\mathrm{alt}\,(\omega))_{I_p(\overset{k}{\vee}_{j_h}),J_p(\overset{h}{\vee}_{i_k})} \\ &= p(\mathrm{alt}\,(\omega))_{I_p,J_p} + \sum_{k,h} (\mathrm{alt}\,(\omega))_{I_p,J_p} \\ &= p(p+1)(\mathrm{alt}\,(\omega))_{I_p,J_p} \ .\end{aligned}$$

REMARK. If p is even, then $\Lambda^{2p} \subset B^p$ is in $V^p_{p(p+1)}$, but if p is odd, then $\Lambda^{2p} \cap B^p = (0)$ and hence $\mathrm{alt}\,(\omega) = 0$ holds.

Next we prove the main theorem.

THEOREM 4-7. *Let $p<n/2$, then the eigenvalues of $\mathfrak{T}$ on B^p are just $[p/2]+1$ numbers $\{2r(2r+1),\ 0\leqq r\leqq[p/2]\}$ and B^p is orthogonally decomposed with their eigenspaces.*

Let $\omega=(\omega_{I_p,J_p})\in B^p$. We define $\eta^{(r)}\in B^p$ for $0\leqq r\leqq p$ by

$$\eta^{(r)}_{I_p,J_p}=\sum_{A_r,H_r}\omega_{I_p\overset{\overset{a_1}{\vee}\ \cdots\ \overset{a_r}{\vee}}{(j_{h_1}\cdots j_{h_r})},\,J_p\overset{\overset{h_1}{\vee}\ \cdots\ \overset{h_r}{\vee}}{(i_{a_1}\cdots i_{a_r})}}$$

where the sum is taken over all A_r and H_r, $A_r=(a_1,\cdots,a_r)$ and $H_r=(h,\cdots,h_r)$ are any permutations of numbers taken from $(1,\cdots,p)$. Then we have

$$\begin{aligned}\sum_{a_{r+1},h_{r+1}}\eta^{(r)}_{I_p\overset{\overset{a_{r+1}}{\vee}}{(j_{h_{r+1}})},\,J_p\overset{\overset{h_{r+1}}{\vee}}{(i_{a_{r+1}})}}&=\sum_{A_{r+1},H_{r+1}}\omega_{I_p\overset{\overset{A_{r+1}}{\vee}}{(j_{H_{r+1}})},\,J_p\overset{\overset{H_{r+1}}{\vee}}{(i_{A_{r+1}})}}\\&\quad-2r(p-r)\sum_{A_r,H_r}\omega_{I_p\overset{\overset{A_r}{\vee}}{(j_{H_r})},\,J_p\overset{\overset{H_r}{\vee}}{(i_{A_r})}}\\&\quad+r^2(p-r+1)^2\sum_{A_{r-1},H_{r-1}}\omega_{I_p\overset{\overset{A_{r-1}}{\vee}}{(j_{H_{r-1}})},\,J_p\overset{\overset{H_{r-1}}{\vee}}{(i_{A_{r-1}})}}\ .\end{aligned}$$

We assume $\omega\in V^p_\lambda$. Then $\mathfrak{T}\omega=\lambda\omega$, and $\mathfrak{T}\eta^{(r)}=\lambda\eta^{(r)}$ holds. Since the left hand side of the above equation is $p\eta^{(r)}_{I_p,J_p}-(\mathfrak{T}\eta^{(r)})_{I_p,J_p}$, we have

$$(p-\lambda)\eta^{(r)}_{I_p,J_p}=\eta^{(r+1)}_{I_p,J_p}-2r(p-r)\eta^{(r)}_{I_p,J_p}+r^2(p-r+1)^2\eta^{(r-1)}_{I_p,J_p}$$

and consequently we obtain

$$\eta^{(r+1)}-(2r(p-r)+(p-\lambda))\eta^{(r)}+r^2(p-r+1)^2\eta^{(r-1)}=0\ .$$

Hence for small r, it holds

$$\begin{aligned}&\eta^{(0)}=\omega\ ,\\&\eta^{(1)}=(p-\lambda)\omega\ ,\\&\eta^{(2)}=((p-\lambda)^2+2(p-1)(p-\lambda)-p^2)\omega\ .\end{aligned}$$

We set $a_r=(2r+1)p-2r^2$, $b_r=r^2(p-r+1)^2$, and $A_r=a_r-\lambda$, $0\leqq r\leqq p$.

LEMMA 4-8. *We have for $r\geqq 3$,*

$$\eta^{(r)}=\{A_0\cdots A_{r-1}-\sum b_1A_2\cdots A_{r-1}+\sum b_1b_3A_4\cdots A_{r-1}-\cdots\}\omega\ ,$$

where the sum is taken all over the terms of the same type.

PROOF. We will show by the induction. We easily get

$$\eta^{(3)}=(A_0A_1A_2-b_1A_2-b_2A_0)\omega\ .$$

Suppose the lemma is true for r. Then we have

$$
\begin{aligned}
\eta^{(r+1)} &= A_r\eta^{(r)} - b_r\eta^{(r-1)} \\
&= \{A_0 \cdots A_r - \sum b_1 A_2 \cdots A_r + \sum b_1 b_3 A_4 \cdots A_r - \cdots\}\omega \\
&\quad - \{b_r A_0 \cdots A_{r-2} - \sum b_r b_1 A_2 A_{r-2} + \cdots\}\omega \\
&= \{A_0 \cdots A_r - \sum b_1 A_2 \cdots A_r + \sum b_1 b_3 A_4 \cdots A_r - \cdots\}\omega ,
\end{aligned}
$$

which shows that the lemma is true for $r+1$.

LEMMA 4-9. *Let $\omega \in B^p$ and $\eta^{(r)}$ is defined as above. Then*

$$
\frac{\eta^{(r)}}{(r!)^2} = \frac{\eta^{(p-r)}}{((p-r)!)^2} .
$$

PROOF. A straight calculation leads to

$$
\begin{aligned}
\eta^{(r)}_{I_p,J_p} &= \sum_{A_r, H_r} \omega_{\overset{\substack{A_r\\ \vee}}{I_p(j_{H_r})},\,\overset{\substack{H_r\\ \vee}}{J_p(i_{A_r})}} \\
&= (r!)^2 \sum_{\substack{a_1<\cdots<a_r\\ h_1<\cdots<h_r}} \omega_{i_1\cdots \overset{\substack{a_1\\ \vee}}{j_{h_1}} \cdots \overset{\substack{a_r\\ \vee}}{j_{h_r}} \cdots i_p,\, j_1 \cdots \overset{\substack{h_1\\ \vee}}{i_{a_1}} \cdots \overset{\substack{h_r\\ \vee}}{i_{a_r}} \cdots j_p} \\
&= (r!)^2 \sum_{\substack{a_1<\cdots<a_r\\ h_1<\cdots<h_r}} \omega_{j_{h_1}\cdots j_{h_r} i_{a_{r+1}} \cdots i_{a_p},\, i_{a_1} \cdots i_{a_r} j_{h_{r+1}} \cdots j_{h_p}} .
\end{aligned}
$$

Replacing the induces $j_{h_1}, \cdots, j_{h_r}$ to $(h_1, \cdots, h_r)$-position and $i_{a_1}, \cdots, i_{a_r}$ to $(a_1, \cdots, a_r)$-position, it becomes

$$
\begin{aligned}
&= (r!)^2 \sum_{\substack{a_{r+1}<\cdots<a_p\\ h_{r+1}<\cdots<h_p}} \omega_{\overset{\substack{a_1\\ \vee}}{i_{a_1}} \cdots \overset{\substack{a_r\\ \vee}}{i_{a_r}} \overset{\substack{a_{r+1}\\ \vee}}{j_{h_{r+1}}} \cdots \overset{\substack{a_p\\ \vee}}{j_{h_p}},\, \overset{\substack{h_1\\ \vee}}{j_{h_1}} \cdots \overset{\substack{h_r\\ \vee}}{j_{h_r}} \overset{\substack{h_{r+1}\\ \vee}}{i_{a_{r+1}}} \cdots \overset{\substack{h_p\\ \vee}}{i_{a_p}}} \\
&= (r!)^2 \frac{1}{((p-r)!)^2} \eta^{(p-r)}_{I_p,J_p} .
\end{aligned}
$$

PROOF OF THEOREM 4-7. By virtue of Theorem 4-5, the eigenvalues of $\mathfrak{T}$ on B^2 are $2r(2r+1)$, $r=0$ and 1. Therefore we suppose first that the eigenvalues of $\mathfrak{T}$ on B^{2p} are all $2r(2r+1)$, $0 \leqq r \leqq p$. Since each eigenvector $\omega \in B^{2p}$ of eigenvalue λ satisfies $g\omega(\neq 0) \in B^{2p+1}$ and $\mathfrak{T}(g\omega) = \lambda(g\omega)$, λ is an eigenvalue of $\mathfrak{T}$ on B^{2p+1}. Moreover by Lemma 4-8,

$$
\begin{aligned}
\eta^{(p+1)} &= \{A_0 \cdots A_p - \sum b_1 A_2 \cdots A_p + \cdots\}\omega \\
&= \{(-1)^{p+1}\lambda^{p+1} + (-1)^p(a_0 + \cdots + a_p)\lambda^p + [(p-1)]\}\omega
\end{aligned}
$$

holds, where $[(p-1)]$ shows the terms of degree less than $p-1$ in λ. On the other hand, from Lemma 4-9 we have

$$
\begin{aligned}
\eta^{(p+1)} &= \left(\frac{(p+1)!}{(p-1)!}\right)^2 \eta^{(p-1)} \\
&= p^2(p+1)^2\{(-1)^{p-1}\lambda^{p-1} + (-1)^p(a_0 + \cdots + a_{p-2})\lambda^{p-2} + [(p-3)]\}\omega
\end{aligned}
$$

and hence we obtain the equation

$$\lambda^{p+1}-(a_0+\cdots+a_p)\lambda^p+[(p-1)]=0\ .$$

Therefore the numbers of the different eigenvalues $\lambda_0, \cdots, \lambda_p$ of $\mathfrak{T}$ on B^{2p} are at most $p+1$ and they satisfy

$$\lambda_0+\cdots+\lambda_p=a_0+\cdots+a_p\ .$$

Next let ω' (resp. ω'') be the eigenvector of the eigenvalue μ (resp. ν) of $\mathfrak{T}$ on B^{2p+1} (resp. B^{2p+2}). Then the similar computation is valid for ω' and ω'', and hence μ and ν satisfy the equations

$$\mu^{p+1}-(a_0'+\cdots+a_p'-(p+1)^2)\mu^p+[(p-1)]=0\ ,$$
$$\nu^{p+2}-(a_0''+\cdots+a_{p+1}'')\nu^{p+1}+[(p)]=0\ ,$$

where we set $a_i'=(2i+1)(2p+1)-2i^2$, $a_i''=(2i+1)(2p+2)-2i^2$. Since the eigenvalues of $\mathfrak{T}$ on B^{2p} are also eigenvalues of $\mathfrak{T}$ on B^{2p+1} and the numbers of the eigenvalues of $\mathfrak{T}$ on B^{2p+1} are at most $p+1$, we conclude that $\mu_0, \cdots, \mu_p$ coincide with $\lambda_0, \cdots, \lambda_p$. We put $\nu_0=\lambda_0, \cdots, \nu_p=\lambda_p$ by the same reason. ν_i satisfy the equauion

$$\nu_0+\cdots+\nu_p+\nu_{p+1}=a_0''+\cdots+a_{p+1}''\ .$$

Therefore we have

$$\begin{aligned}\nu_{p+1}&=\sum a_i''-\sum\lambda_i\\&=\sum a_i''-\sum a_i\\&=2(p+1)(2(p+1)+1)\ .\end{aligned}$$

Thus it is shown that the eigenvalues of $\mathfrak{T}$ on B^{2p+2} are $2r(2r+1)$ $0\leqq r\leqq p+1$. This proves the theorem.

5. Skew-symmetric forms as curvature type tensors. By virtue of Lemma 4-6, $\omega\in\Lambda^{2p}$ satisfies $\mathfrak{T}\omega=p(p+1)\omega$. We prove that the converse is true. Let $p=2s$ be even in the following.

LEMMA 5-1. *Any $\omega\in B^p$ satisfies*

$$\mathrm{alt}\,(\mathfrak{T}\omega)=p(p+1)\,\mathrm{alt}\,(\omega)\ .$$

PROOF. Let $\omega=(\omega_{I_p,J_p})\in B^p$, then $\mathfrak{T}\omega=p\omega-\eta^{(1)}$ and

$$\begin{aligned}(\mathrm{alt}\,(\eta^{(1)}))_{I_{2p}}&=\frac{1}{(2p)!}\sum\varepsilon(\sigma)\eta^{(1)}_{i_{\sigma_1}\cdots i_{\sigma_p},i_{\sigma_{p+1}}\cdots i_{\sigma_{2p}}}\\&=\frac{1}{(2p)!}\sum\varepsilon(\sigma)\sum_{a,k}\omega_{i_{\sigma_1}\cdots \overset{\overset{a}{\vee}}{i_{\sigma_{p+k}}}\cdots i_{\sigma_p},i_{\sigma_{p+1}}\cdots \overset{\overset{k}{\vee}}{i_{\sigma_a}}\cdots i_{\sigma_{2p}}}\ .\end{aligned}$$

If we put $\sigma'=(\sigma_1\cdots\overset{\overset{a}{\vee}}{\sigma_{p+k}}\cdots\overset{\overset{p+k}{\vee}}{\sigma_a}\cdots\sigma_{2p})$, then $\varepsilon(\sigma)=-\varepsilon(\sigma')$ and hence we have

$$(\mathrm{alt}\,(\eta^{(1)}))_{I_{2p}} = \frac{-1}{(2p)!} \sum \varepsilon(\sigma') \sum_{k,a} \omega_{i_{\sigma'_1} \cdots i_{\sigma'_p}, i_{\sigma'_{p+1}} \cdots i_{\sigma'_{2p}}}$$
$$= -p^2 (\mathrm{alt}\,(\omega))_{I_{2p}} .$$

It follows that $\mathrm{alt}\,(\mathfrak{T}\omega) = p\,\mathrm{alt}\,(\omega) + p^2\,\mathrm{alt}\,(\omega) = p(p+1)\,\mathrm{alt}\,(\omega)$.

COROLLARY 5-2. *If $\omega \in B^p$ satisfies $\mathfrak{T}\omega = \lambda\omega$, $\lambda < p(p+1)$, then*

$$\mathrm{alt}\,(\omega) = 0 .$$

PROOF. By Lemma 5-1, it is easily shown that $\lambda(\mathrm{alt}\,(\omega)) = p(p+1)\,\mathrm{alt}\,(\omega)$ and hence $\mathrm{alt}\,(\omega) = 0$.

Taking the eigenspace V^p_λ for eigenvalue λ of $\mathfrak{T}$ on B^p, we have $B^p = \sum_r V^p_{2r(2r+1)}$ and $\Lambda^{2p} \subset V^p_{p(p+1)}$.

LEMMA 5-3. *If $\omega \in B^p$ satisfies $\mathfrak{T}\omega = p(p+1)\omega$, then we have*

$$\eta^{(r)} = (-1)^r p^2 (p-1)^2 \cdots (p-r+1)^2 \omega .$$

PROOF. When $r=1$, we have $\eta^{(1)} = p\omega - \mathfrak{T}\omega = -p^2\omega$. We assume the lemma is true for $r \geqq 2$. Then making use of the successive relations of $\eta^{(r)}$ we have

$$\begin{aligned}\eta^{(r+1)} &= (2r(p-r) + p - (p^2+p))\eta^{(r)} - r^2(p-r+1)^2 \eta^{(r-1)} \\ &= (-1)^{r+1} p^2 (p-1)^2 \cdots (p-r+1)^2 (p^2 - 2pr + r^2)\omega \\ &= (-1)^{r+1} p^2 \cdots (p-r)^2 \omega\end{aligned}$$

which shows the lemma is true for $r+1$.

LEMMA 5-4. *For $\omega \in B^p$ we have*

$$\mathrm{alt}\,(\omega) = \frac{(p!)^2}{(2p)!} \sum \frac{(-1)^r}{(r!)^2} \eta^{(r)} .$$

PROOF. First for $\eta^{(r)}$ of $\omega \in B^p$, we have

$$\begin{aligned}\eta^{(r)}_{i_1 \cdots i_p, i_{p+1} \cdots i_{2p}} &= (r!)^2 \sum_{\substack{1 \leqq k_1 < \cdots < k_r \leqq p \\ 1 \leqq a_1 < \cdots < a_r \leqq p}} \omega_{i_1 \cdots \overset{a_1}{\underset{\vee}{i_{p+k_1}}} \cdots \overset{a_r}{\underset{\vee}{i_{p+k_r}}} \cdots i_p, i_{p+1} \cdots \overset{k_1}{\underset{\vee}{i_{a_1}}} \cdots \overset{k_r}{\underset{\vee}{i_{a_r}}} \cdots i_{a_{2p}}} \\ &= (-1)^r (r!)^2 \sum (-1)^{\sum a_i + \sum k_i} \omega_{i_1 \overset{\wedge}{\cdots} i_p i_{p+k_1} \cdots i_{p+k_r}, i_{p+1} \overset{\wedge}{\cdots} i_{2p} i_{a_1} \cdots i_{a_r}} .\end{aligned}$$

For any permutation $\sigma = (\sigma_1 \cdots \sigma_{2p})$ of $1, \cdots, 2p$, the pairs $(\sigma_1 \cdots \sigma_p)$ and $(\sigma_{p+1} \cdots \sigma_{2p})$ have the numbers $1 < \cdots < p < p+k_1 < \cdots < p+k_r$, $a_1 < \cdots < a_r < p+1 < \cdots < 2p$ for some r. Thus we have

$$\begin{aligned}&\sum \varepsilon(\sigma) \omega_{i_{\sigma_1} \cdots i_{\sigma_p}, i_{\sigma_{p+1}} \cdots i_{\sigma_{2p}}} \\ &\quad = (p!)^2 \sum_{\substack{k_1 < \cdots < k_r \\ a_1 < \cdots < a_r}} \varepsilon(1 \overset{\wedge}{\cdots} p\ p+k_1 \cdots p+k_r\ a_1 \cdots a_r\ p+1 \overset{\wedge}{\cdots} 2p) \\ &\qquad \times \omega_{i_1 \overset{\wedge}{\cdots} i_p i_{p+k_1} \cdots i_{p+k_r}, i_{a_1} \cdots i_{a_r} i_{p+1} \overset{\wedge}{\cdots} i_{2p}}\end{aligned}$$

$$=(p!)^2 \sum \frac{(-1)^r}{(r!)^2} \eta^{(r)}_{i_1\cdots i_p, i_{p+1}\cdots i_{2p}} .$$

Since $(-1)^r=(-1)^{p-r}$ and $\eta^{(r)}/(r!)^2=\eta^{(p-r)}/((p-r)!)^2$, we obtain

$$\begin{aligned}\operatorname{alt}(\omega) &= \frac{1}{(2p)!} \sum \varepsilon(\sigma)\omega_{i_{\sigma_1}\cdots i_{\sigma_p}, i_{\sigma_{p+1}}\cdots i_{\sigma_{2p}}} \\ &= \frac{(p!)^2}{(2p)!} \sum \frac{(-1)^2}{(r!)^2} \eta^{(r)} .\end{aligned}$$

THEOREM 5-5. *If $\omega \in B^p$ satisfies $\mathfrak{T}\omega=p(p+1)\omega$, then $\omega \in \Lambda^{2p}$.*

PROOF. If $\mathfrak{T}\omega=p(p+1)\omega$, then we have by Lemmas 5-3, 5-4

$$\begin{aligned}\operatorname{alt}(\omega) &= \frac{(p!)^2}{(2p)!} \sum_{r=0}^{p} \frac{(-1)^r}{(r!)^2} \eta^{(r)} \\ &= \frac{1}{\binom{2p}{p}} \sum_{r=0}^{p} \left(\frac{p(p-1)\cdots(p-r+1)}{r!}\right)^2 \omega \\ &= \frac{1}{\binom{2p}{p}} \sum_{r=0}^{p} \binom{p}{r}^2 \omega = \omega ,\end{aligned}$$

which shows ω is a skew-symmetric $2p$-form.

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