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Projective Normality and the Defining Equations of an Elliptic Ruled Surface with Negative Invariant

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Let C be a complete nonsingular curve over an algebraic closed field k. Let X be a ruled surface over C, that is a complete nonsingular surface with a surjective morphism $\pi: X \to C$ such that each fibre is isomorphic to $\boldsymbol{P}^1$ and π admits a section. When C is an elliptic curve, X is called an elliptic ruled surface. From now on we consider elliptic ruled surfaces over a fixed elliptic curve C.

The present paper is a continuation of [2], and our notations are the same as used in that paper.

REMARK. The explanation of C_0 in the introduction of [2] should read as follows. We fix a section C_0 with $\mathcal{O}_X(C_0) \cong \mathcal{O}_{P(\mathscr{E})}(1)$, where $\mathscr{E}$, which is explaned later, is a normalized locally free sheaf of rank 2 on C.

THEOREM ([1, V, 2.21], [2, Th. 3.3]). *Let X be an elliptic ruled surface with non-negative invariant e and $D \sim nC_0 + \mathfrak{b}f$ a divisor on X. Then*

(1) *D is ample if and only if $n \geqq 1$ and $\deg \mathfrak{b} > ne$;*

(2) *D is very ample if and only if $n \geqq 1$ and $\deg \mathfrak{b} \geqq ne+3$. In this case D is normally generated and $I(D) = \mathrm{Ker}\,[S\Gamma(D) \to \bigoplus_{j \geqq 0} \Gamma(jD)]$ is generated by its elements of degree 2 and 3.*

In this paper a similar result for X with negative invariant will be proved. For the fixed elliptic curve C, such a surface X is unique up to isomorphism and $e=-1$. It is known that $D \sim nC_0 + \mathfrak{b}f$ *on X is ample if and only if $n \geqq 1$ and $\deg \mathfrak{b} > (1/2)ne$* ([1, V, 2.21]). The result, which will be proved in §3, is as follows: *D is very ample if and only if $n=1$ and $\deg \mathfrak{b} \geqq 2$; or $n \geqq 2$ and $\deg \mathfrak{b} \geqq (ne+3)/2$. In this case D is normally generated and $I(D)$ is generated by its elements of degree 2 and 3.*

As in the previous paper [2], our main tool for the proof is cohomology of a divisor on X. In [2] the dimension $h^i(D)$ of the i-th cohomology group $H^i(X, D)$ was computed partially, using the following exact sequences (#1), (#2) and their long exact sequences.

$$(\#1) \qquad 0 \to D - C_0 \to D \to D|_{C_0} \cong D \otimes \mathcal{O}_C \to 0 .$$
$$(\#2) \qquad 0 \to D - yf \to D \to D|_{\pi^{-1}(y)} \cong D \otimes \mathcal{O}_{P^1} \to 0 , \quad \text{where} \quad y \in C .$$

In §2 we will compute $h^i(D)$ for $D \sim nC_0 + \mathfrak{b}f$ by using the following exact sequence

$$(\#3) \qquad 0 \to D - Y \to D \to D|_Y \to 0 ,$$

where Y is an elliptic curve on X which is numerically equivalent to $2C_0 - f$. The sequence (#3) and its resulting cohomology sequence will play an important role in this paper. The existence of such a curve Y is proved by using a technique of an elementary transformation of elliptic ruled surfaces. In the first section we will study an elementary transformation of X with center one point.

§1. An elementary transformation.

We begin by recalling some notations and some definitions. Let π: $X \cong \boldsymbol{P}(\mathscr{E}) \to C$ be an elliptic ruled surface, where $\mathscr{E}$ is a normalized locally free sheaf of rank 2 on C. Then $-\mathfrak{e} = -\bigwedge^2 \mathscr{E}$ has degree e that is the invariant of X. A section is said to be minimal when its self-intersection number is equal to $-e$. We fix a section C_0 such that $\mathcal{O}_X(C_0) \cong \mathcal{O}_{P(\mathscr{E})}(1)$, which is minimal. A divisor D on X is written $nC_0 + \mathfrak{b}f$ up to linear equivalence, where n is an integer and $\mathfrak{b}$ is a divisor on C.

First we remark on minimal sections. When $e > 0$; or X corresponds to an indecomposable locally free sheaf $\mathscr{E}$ of degree 0; or X is isomorphic to $\boldsymbol{P}^1 \times C$, any minimal section is linearly equivalent to $\mathcal{O}_{X_0}(C_0)$. Furthermore in the former two cases, C_0 is unique since $h^0(C_0) = 1$. Next if $X \cong \boldsymbol{P}(\mathcal{O}_C \oplus \mathfrak{e})$ with invariariant 0, where $\mathfrak{e} \not\sim \mathcal{O}_C$, then a minimal section is linearly equivalent to $\mathcal{O}_X(C_0)$ or $\mathcal{O}_X(C_0 - \mathfrak{e}f)$. Since $h^0(C_0) = h^0(C_0 - \mathfrak{e}f) = 1$, we denote by $C_\mathfrak{e}$ the minimal section other than C_0. Finally when $e = -1$, $|C_0 + \mathfrak{b}f|$ contains a minimal section if and only if $deg\, \mathfrak{b} = 0$. In this case $h^0(C_0 + \mathfrak{b}f) = 1$.

Now we study an elementary transformation of X with center a point $P \in X$. It is easy to see that the elementary transform X' of X is also an elliptic ruled surface ([1, V, 5.7.1]). Our result is as follows: *if P is on a minimal section, then the strict transform of the section is also minimal on X' with invariant $e+1$. In the other case, X' has the invariant $e-1$* (Prop. 1.1, 1.2, 1.3, 1.4).

Let L_y be $\pi^{-1}(y)$ where $y=\pi(P)$. Let $f\colon \widetilde{X}\to X$ be the monoidal transformation with center P and E the exceptional curve of f. If Z is a curve on X, we denote by $\widetilde{Z}$ [resp. Z'] the strict transform of Z via f [resp. elm_P]. Let $g\colon \widetilde{X}\to X'$ be the morphism sending $\widetilde{L}_y$ to a point Q, then we get the elementary transformation $elm_P= g\cdot f^{-1}$.

PROPOSITION 1.1. *Let $X\cong \boldsymbol{P}(\mathcal{O}_C\oplus\mathfrak{e})$ be an elliptic ruled surface with $e\geqq 1$ and P a point on X with $\pi(P)=y$.*

(1) *If $P\in C_0$, then $X'\cong \boldsymbol{P}(\mathcal{O}_C\oplus(\mathfrak{e}-y))$ and C_0' is the minimal section of X'* (Fig. 1).

(2) *If $P\notin C_0$ and $e\geqq 2$, then $X'\cong \boldsymbol{P}(\mathcal{O}_C\oplus(\mathfrak{e}+y))$ and C_0' is the minimal section of X'.*

PROOF. When $P\in C_0$, we see that $C_0'^2=\widetilde{C}_0^2=C_0^2-1$. Since a section with negative self-intersection number is minimal, the invariant of X' is $e+1$. If Z is a section on X such that $Z^2=e$, then $Z\sim C_0-\mathfrak{e}f$ ([2, Prop. 3.1 (a)]). Note $Z\cdot C_0=0$, then we get $Z'\sim C_0'-(\mathfrak{e}-y)f$ and $Z'^2=e+1$. Hence we see that $X'\cong \boldsymbol{P}(\mathcal{O}_C\oplus(\mathfrak{e}-y))$. Considering elm_Q, we get (2) easily.

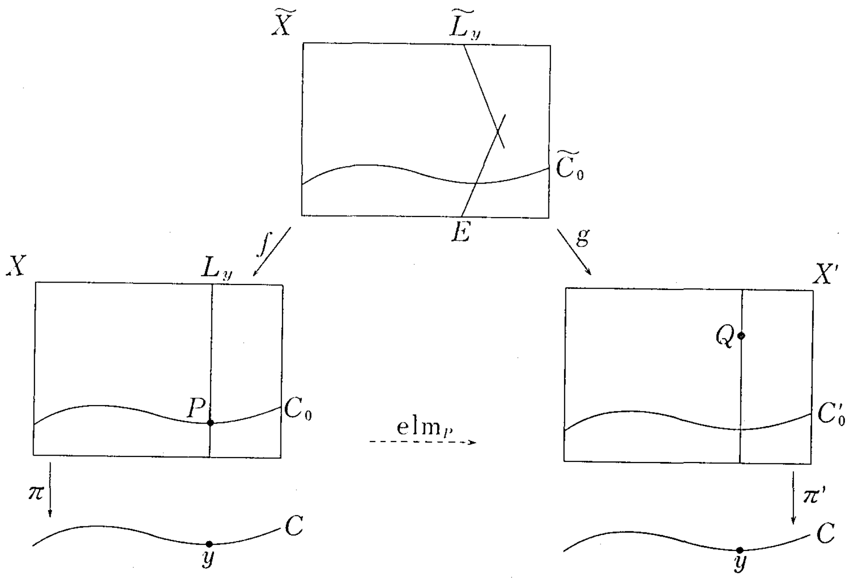

Fig. 1.

If $P\notin C_0$ and $e=1$, then X' is an elliptic ruled surface with invariant 0. But in this case we have to study more carefully.

Let Z_1 and Z_2 be any two members of $|C_0-\mathfrak{e}f|$. Then $Z_1\cap Z_2$ is one point on $L_{-\mathfrak{e}}$. We denote it by R, which is $Bs|C_0-\mathfrak{e}f|$.

PROPOSITION 1.2. *Let X, P and R be as above.*

(1) *If $y\not\sim -\mathfrak{e}$, then $X'\cong \boldsymbol{P}(\mathcal{O}_C\oplus(\mathfrak{e}+y))$. The minimal sections*

on X' are C_0' and C_1', where C_1 is the section which is passing through P and linearly equivalent to $C_0-\mathfrak{e}f$ (Fig. 2).

(2) *If $P=R$, then $X'\cong \boldsymbol{P}(\mathcal{O}_C\oplus\mathcal{O}_C)$.*

(3) *If $y\sim -\mathfrak{e}$ and $P\neq R$, then X' corresponds to an indecomposable normalized sheaf of degree 0 and the minimal section of X' is C_0'.*

PROOF. We only prove (1). The assertions (2) and (3) can be proved in the same manner. Since $h^0(C_0-\mathfrak{e}f)=h^0(\mathcal{E}\otimes\mathcal{O}_C(-\mathfrak{e}))=2$, there exists a member of $|C_0-\mathfrak{e}f|$ that passes P. Because $P\notin C_0$ and $y\not\sim -\mathfrak{e}$, there exists a unique section $C_1\in|C_0-\mathfrak{e}f|$ that passes P. We can easily compute $C_1'\sim C_0'-(\mathfrak{e}+y)f$ and $C_1'^2=0$. In general if Z is a section which passes [resp. dose not pass] through P, then $Z'^2=Z^2-1$ [resp. Z^2+1]. Hence for a section $Z\in|C_0-\mathfrak{e}f|$ not passing through P, $Z'^2=2$. For any other section Z which is not linearly equivalent to C_0 nor $C_0-\mathfrak{e}f$, $Z^2\geqq e+2=3$ by [2, Prop. 3.1(a)]. So we see that C_0' and C_1' are the minimal sections on X' and $X'\cong \boldsymbol{P}(\mathcal{O}_C\oplus(\mathfrak{e}+y))$.

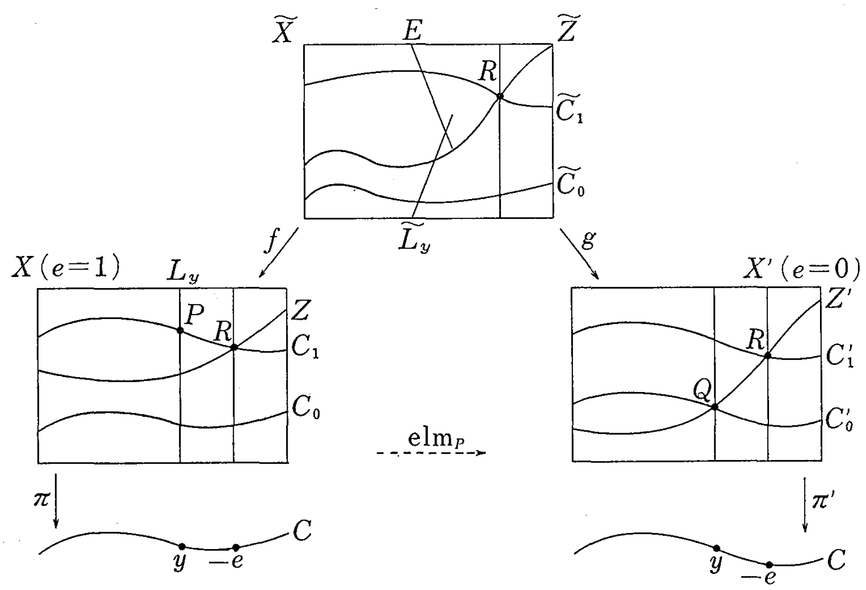

Fig. 2.

The assertion (2) of the following proposition is avairable in §2.

PROPOSITION 1.3. *Let $X\cong\boldsymbol{P}(\mathcal{O}_C\oplus\mathfrak{e})$ be an elliptic ruled surface with $e=0$ and $\mathfrak{e}\not\sim\mathcal{O}_C$. Put $y=\pi(P)$.*

(1) *If $P\in C_0$ [resp. $C_{\mathfrak{e}}$], then $X'\cong\boldsymbol{P}(\mathcal{O}_C\oplus(\mathfrak{e}-y))$ [resp. $\boldsymbol{P}(\mathcal{O}_C\oplus(-\mathfrak{e}-y))$] with invariant 1 and the minimal section of X' is C_0' [resp. $C_{\mathfrak{e}}'$].*

(2) *If $P\notin C_0\cup C_{\mathfrak{e}}$, then X' has the invariant -1. For every $\mathfrak{b}\in Pic^1(C)$ such that $\mathfrak{b}$ is neither y nor $y-\mathfrak{e}$, there exists a unique*

section $Z_{\mathfrak{b}} \in |C_0+\mathfrak{b}f|$ *passing through* P. *The set of minimal sections on* X' *are the union of* C_0', C_e' *and* $Z_{\mathfrak{b}}'$*'s* (Fig. 3).

PROOF. The assertions can be proved in the same manner as Proposition 1.2. We will only give a remark on $Z_{\mathfrak{b}}$ in (2).

Every section $Z \in |C_0+\mathfrak{b}f|$ intersects with C_0 and C_e at $S \in L_{\mathfrak{e}+\mathfrak{b}}$ and $T \in L_{\mathfrak{b}}$ respectively. These two points are $Bs|C_0+\mathfrak{b}f|$. On the other hand $h^0(C_0+\mathfrak{b}f)=h^0(\mathscr{E}\otimes\mathscr{O}_C(\mathfrak{b}))=2$, so there exists a member of $|C_0+\mathfrak{b}f|$ passing through P. Because $P \notin C_0 \cup C_e$, $\mathfrak{b} \not\sim y$ and $\mathfrak{b} \not\sim y-\mathfrak{e}$, such a member is unique, which is $Z_{\mathfrak{b}}$. Clearly C_0', C_e' and $Z_{\mathfrak{b}}'$ have their self-intersection number 1.

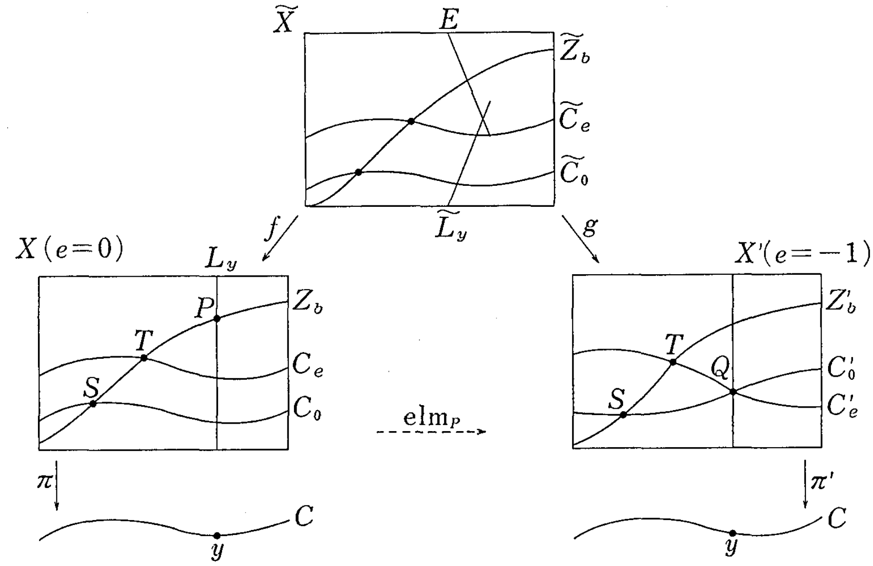

Fig. 3.

Similarly examining elm_P of X with $e=-1$, we get the following proposition but omit its proof.

PROPOSITION 1.4. *Let* X *be an elliptic ruled surface with* $e=-1$ *and* $C_{\mathfrak{b}}$ *the minimal section which is linearly equivalent to* $C_0+\mathfrak{b}f$ *for every* $\mathfrak{b} \in \mathrm{Pic}^0(C)$.

(1) *If there is only one minimal section passing through* P, *then* X' *corresponds to an indecomposable normalized sheaf of degree* 0.

(2) *If there are two minimal sections* $C_{\mathfrak{b}}$ *and* $C_{\mathfrak{b}'}$ *passing through* P, *then* $X' \cong \boldsymbol{P}(\mathscr{O}_C \oplus (\mathfrak{b}-\mathfrak{b}'))$ *with invariant* 0, *where* $\mathfrak{b} \not\sim \mathfrak{b}'$.

§2. Dimension of $H^i(X, D)$.

This section is devoted to a computation of $h^i(D)$ of a divisor D on an elliptic ruled surface with invariant -1. But for a while let X be $\boldsymbol{P}(\mathcal{O}_C\oplus\mathfrak{e})$ with invariant 0.

LEMMA 2.1. *Let $\mathfrak{e}$ be a divisor on C such that $2\mathfrak{e}\sim\mathcal{O}_C$. Then on $X\cong\boldsymbol{P}(\mathcal{O}_C\oplus\mathfrak{e})$ there exists an irreducible reduced curve that is linearly equivalent to $2C_0$. Such a curve is elliptic.*

PROOF. Since $h^0(2C_0)=h^0(S^2(\mathcal{O}_C\oplus\mathfrak{e}))=2$, there exists an effective divisor $Y\sim 2C_0$. Assume that Y is reducible. Say $Y=Y_1\cup Y_2$, then $Y_1\sim C_0+\mathfrak{b}f$ and $Y_2\sim C_0-\mathfrak{b}f$ for some divisor $\mathfrak{b}$ of degree 0. Since $h^0(C_0+\mathfrak{b}f)=h^0(\mathfrak{b}\oplus(\mathfrak{b}+\mathfrak{e}))>0$, $\mathfrak{b}$ must be linearly equivalent to $\mathcal{O}_C$ or $\mathfrak{e}$. Hence we get that $Y_1=Y_2=C_0$ or $Y_1=Y_2=C_e$. So a general Y of $|2C_0|$ is irreducible.

Now we show that Y is elliptic. The restriction map $\pi|_Y\colon Y\to C$ is a finite morphism of degree 2. Hence Hurwitz formula implies that the genus of the normalization of Y is not less than 1. On the other hand, by adjunction formula $2p_a(Y)-2=2C_0\cdot(2C_0+K)$ we get $p_a(Y)=1$, where K is the canonical divisor $-2C_0+\mathfrak{e}f$. This means that the genus of the normalization of Y is not grater than 1. From these we see that Y is a nonsingular curve of genus 1. q.e.d.

Now we consider an elementary transformation of X with center a point $P\in Y$, where X and Y are as above. Since Y intersects with neither C_0 nor C_e, $X'=elm_PX$ is an elliptic ruled surface with invariant -1 and C_0' is a minimal section by Proposition 1.3. It is easily computed that Y' is linearly equivalent to $2C_0'-yf$. Hence we get an elliptic curve Y' that is numerically equivalent to $2C_0'-f$. We have just proved the following corollary.

COROLLARY 2.2. *The numerical equivarent class of $2C_0-f$ on X with $e=-1$ contains an irreducible nonsingular curve, which is an elliptic curve.*

PROPOSITION 2.3. *Let $D\sim nC_0+\mathfrak{b}f$ be a divisor on X and m the degree of $\mathfrak{b}$.*

(1) *If $n\geqq 0$ and $m>-n/2$, then $h^1(D)=h^2(D)=0$ and $h^0(D)=(1/2)(n+1)(2m+n)$.*

(2) *If $n\geqq 0$ and $m<-n/2$, then $h^0(D)=h^2(D)=0$.*

(3) *If $n=-1$, then $h^0(D)=h^1(D)=h^2(D)=0$.*

PROOF. We have already known (3) and the that $h^2(D)$ is vanishing under the assumption $n\geqq 0$ in the previous paper. Let $Y\sim 2C_0-\mathfrak{y}f$ be an elliptic curve in Corollary 2.2, where $\mathfrak{y}$ is some divisor of degree 1. Consider the following exact sequence

$$(\#3)\qquad 0\to(n-2)C_0+(\mathfrak{b}+\mathfrak{y})f\to nC_0+\mathfrak{b}f\to nC_0+\mathfrak{b}f|_Y\to 0\,.$$

Since $deg(D|_Y)=n+2m$ is grater than 0, $H^1(D|_Y)$ is vanishing and we get that $H^1(D-Y)\to H^1(D)$ is surjective. Repeat this replacing D by $D-Y$, then we obtain the following surjections:

$$\begin{cases} H^1\left(D-\dfrac{n}{2}Y\right)\to H^1(D)\,, & \text{if } n \text{ is even ;} \\ H^1\left(D-\dfrac{n+1}{2}Y\right)\to H^1(D)\,, & \text{if } n \text{ is odd .} \end{cases}$$

In n is even, then $H^1(D-(n/2)Y)=H^1((\mathfrak{b}+(n/2)\mathfrak{y})f)\cong H^1(C, \mathfrak{b}+(n/2)\mathfrak{y})$, which is zero. Hence $H^1(D)$ vanishes. If n is odd, then $H^1(D-((n+1)/2)Y)=H^1(-C_0+(\mathfrak{b}+((n+1)/2)\mathfrak{y})f)$, which is zero by (3). So we get $H^1(D)=0$. Therefore we conclude that $h^0(D)$ is equal to $\chi(D)$, which is computed by the Riemann-Roch formula $\chi(D)=(1/2)D\cdot(D-K)+1+p_a(X)$, where $p_a(X)=\chi(\mathcal{O}_X)-1$ is equal to -1. Similarly we can prove (2).

§3. Projective normality of D on X.

PROPOSITION 3.1 ([2, Prop. 3.1, 3.2]). *Let $D\sim C_0+\mathfrak{b}f$ be a divisor on X.*

(1) *$|D|$ contains a section if and only if $deg\,\mathfrak{b}\geqq 0$.*

(2) *$|D|$ has no base points if and only if $deg\,\mathfrak{b}\geqq 1$.*

(3) *D is very ample if and only if $deg\,\mathfrak{b}\geqq 2$. In this case D is normally generated and $I(D)$ is generated by its elements of degree 2 and 3.*

LEMMA 3.2. *If $n\geqq 1$ and $deg\,\mathfrak{b}\geqq(2-n)/2$, then a divisor $D\sim nC_0+\mathfrak{b}f$ is free from base points.*

PROOF. We prove the result by induction on n. First we remark that a divisor $\mathfrak{b}f$ with $deg\,\mathfrak{b}\geqq 2$ is free from base points. When $n=1$, the lemma holds by Proposition 3.1. We assume $n\geqq 2$ and put $m=deg\,\mathfrak{b}$. Let Y be as in the proof of Proposition 2.3. Since $H^1(D-Y)$ vanishes by Proposition 2.3 (1), we get the following exact sequence

$$(\sharp 4) \qquad 0\to\Gamma(D-Y)\to\Gamma(D)\to\Gamma(D|_Y)\to 0\,.$$

We see that D is free from base points when both $D-Y$ and $D|_Y$ are free from base points. For $deg(D|_Y)=n+2m$ is grater than or equal to 2, $D|_Y$ is free from base points. If n is odd, then by the induction hypothesis that $\Gamma(D-Y)$ has no base points, we get that so dose $\Gamma(D)$. If n is even and $n+2m>2$, then by the first remark and the induction hypothesis, we get that $\Gamma(D)$ has no base points.

To complete the proof, we have only to prove that $D \sim 2C_0 + \mathfrak{b}f$ with $m=0$ is free from base points. We show that for every point P on X there exists a member of $|D|$ which dose not contain P. If P is not on C_0 nor the section $C_0+\mathfrak{b}f$, then we have nothing to do. Let P be on C_0, then the minimal section $C_0+(\pi(P)-\mathfrak{e})f$ intersects with C_0 at P and any other minimal sections do not pass through P. Hence for $\mathfrak{b}' \in Pic^0(C)$ in general position, the union of two minimal sections $C_0+\mathfrak{b}'f$ and $C_0+(\mathfrak{b}-\mathfrak{b}')f$ is what we want. Similarly when P is on $C_0+\mathfrak{b}f$, we can find a member of $|D|$ which does not contain P.

LEMMA 3.3. *Let $D \sim nC_0+\mathfrak{b}f$ be a divisor on X such that $n \geqq 2$ and $deg\,\mathfrak{b} \geqq (3-n)/2$. If the map β: $\Gamma(D)\otimes\Gamma(D)\to\Gamma(2D)$ is surjective, then D is normally generated.*

PROOF. By Lemma 3.2, $\Gamma(D)$ has no base points. By Proposition 2.3 (1), for $t \geqq 2$ both $h^1((t-1)D)$ and $h^2((t-2)D)$ are zero. An application of the generalized lemma of Castelnuovo in [3] to the map β_t: $\Gamma(tD)\otimes\Gamma(D)\to\Gamma((t+1)D)$ yields that β_t is surjective. By the assumption of the surjectivity of β and by ampleness of D we get that D is normally generated.

THEOREM 3.4. *A divisor $D \sim nC_0+\mathfrak{b}f$ on X is very ample if and only if $n=1$ and $deg\,\mathfrak{b} \geqq 2$: or $n \geqq 2$ and $deg\,\mathfrak{b} \geqq (3-n)/2$. In this case D is normally generated and $I(D)$ is generated by its elements of degree 2 and 3.*

PROOF. Let Y and m be as in the proof of Lemma 3.2. We have only to prove the case when $n \geqq 2$. If D is very ample, then $D|_Y$ is also very ample. Hence $deg(D|_Y)=n+2m$ must be grater than or equal to 3. To prove the converse we show that D is normally generated. By Lemma 3.3 it is sufficient to prove the surjectivity of the map β.

In case $n=2$ or 3, we note that the assumption $n+2m \geqq 3$ means $n+m \geqq 3$. We can consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \Gamma(D-C_0)\otimes\Gamma(D) & \to & \Gamma(D)\otimes\Gamma(D) & \to & \Gamma(D|_{C_0})\otimes\Gamma(D) & \to 0 \\ & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & \\ 0 \to & \Gamma(2D-C_0) & \to & \Gamma(2D) & \to & \Gamma(2D|_{C_0}) & \to 0\,. \end{array}$$

The rows are exact because $H^1(D-C_0)$ and $H^1(2D-C_0)$ are vanishing by Proposition 2.3 (1). Since $deg\,(D|_{C_0})=n+m \geqq 3$, $D|_{C_0}$ is normally generated. Hence γ is surjective. An application of the generalized lemma of Castelnuovo in [3] to the map α yields its surjectivity. In fact $\Gamma(D-C_0)$ has no base points by Lemma 3.2, and $H^1(D-(D-C_0))=0$

by Proposition 2.3 (1) and $H^2(D-2(D-C_0))=H^2((2-n)C_0-\mathfrak{b}f)$ is zero by Proposition 2.3 (2), (3). The surjectivity of α and γ implies that of β.

In case $n\geqq 4$, we use induction on n. Tensoring the exact sequence (#4) in the proof of Lemma 3.2 with $\Gamma(D)$ and $\Gamma(D-Y)$ we get the following commutative diagrams with exact rows.

$$\begin{array}{ccccccccc}
0\to & \Gamma(D-Y)\otimes\Gamma(D) & \to & \Gamma(D)\otimes\Gamma(D) & \to & \Gamma(D|_Y)\otimes\Gamma(D) & \to 0 \\
 & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & \\
0\to & \Gamma(2D-Y) & \to & \Gamma(2D) & \to & \Gamma(2D-Y|_Y) & \to 0
\end{array}$$

$$\begin{array}{ccccccccc}
0\to & \Gamma(D-Y)\otimes\Gamma(D-Y) & \to & \Gamma(D)\otimes\Gamma(D-Y) & \to & \Gamma(D|_Y)\otimes\Gamma(D-Y) & \to 0 \\
 & \downarrow{\scriptstyle\beta'} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma'} & \\
0\to & \Gamma(2D-2Y) & \to & \Gamma(2D-Y) & \to & \Gamma(2D-Y|_Y) & \to 0
\end{array}$$

Since $deg(D|_Y)=n+2m\geqq 3$, $D|_Y$ is normally generated. Hence γ is surjective. We claim that γ' is surjective. Since both $deg(D|_Y)$ and $deg(D-Y|_Y)$ are grater than or equal to 3, $\Gamma(D|_Y)\otimes\Gamma(D-Y|_Y)\to\Gamma(2D-Y|_Y)$ is surjective ([4, Prop. 1.10]). Furthermore there is a surjection $\Gamma(D-Y)\to\Gamma(D-Y|_Y)$, because $H^1(D-Y)$ is vanishing. Therefore γ' is surjective. By the induction hypothsis that β' is surjective, so is α. It follows that β is surjective as required.

For the rest of the theorem on generators of $I(D)$, see [2, 69–70 pp.]. The same proof will work. q.e.d.

We summarize our results in Fig. 4, where a lattice point (n, m) means a divisor which is numerically equivalent to nC_0+mf.

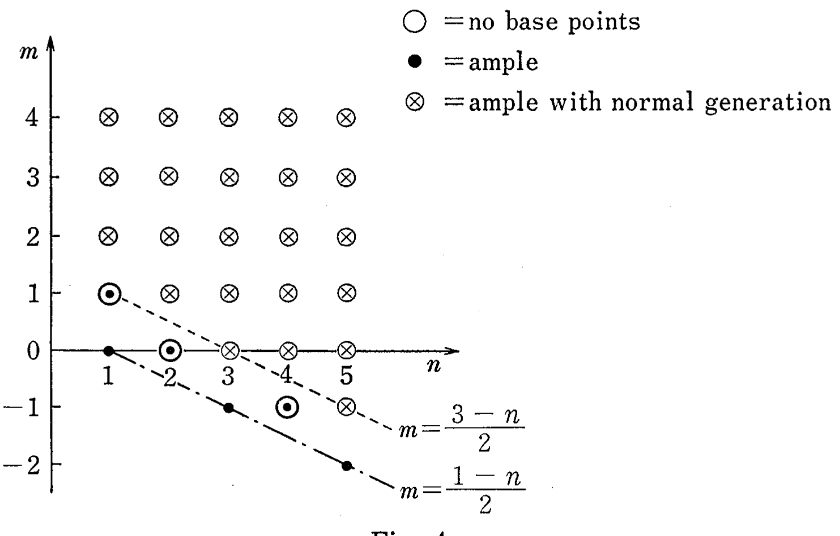

Fig. 4.

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