Natural Science Report, Ochanomizu University, Vol. 33, No. 1~2, 1982 27
お茶の水女子大学 自然科学報告 第33巻 第1~2号

On the Second Dual of a Simplex Space

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(Received September 10, 1982)

§ 1. Introduction.

A simplex space E can be expressed as a space $A_0(X)$ of continuous affine functions on a Choquet simplex X [6]. One can also identify it with a space $A_0(\overline{\partial X})$ of functions on $\overline{\partial X}$ (= the closure of the set ∂X of all extreme points of X), by using probability measures on X supported by $\overline{\partial X}$ [7]. AM space is a special case of a simplex space and it provides more information than a simplex space because of the property of lattice and algebra. The second dual E'' of a simplex space E is AM space and it can be expressed as a space of bounded affine functions on X. But E'' contains so many elements that it cannot be expressed isomorphically as a space of functions on $\overline{\partial X}$. So we investigate sublattices of E'' containing E, which can be identified with a space of functions on $\overline{\partial X}$. In § 3, we examine the smallest Banach sublattice E_1 of E'' containing E (Theorem 1). Although a simplex homomorphism keeps E_1 invariant (Theorem 2), E_1 is not necessarily T''-invariant for a positive operator T in E. So in § 4, we investigate sublattices of E'' containing E which is T''-invariant for any positive operator $T \in \mathfrak{L}(E)$ and obtain that a closed subspace E_2 of E'' generated by bounded upper semicontinuous affine functions on X is the desired one (Theorem 3). We also show that under some conditions, E_2 is the smallest one (Proposition 3).

§ 2. Basic results on a simplex space.

Let E be a simplex space [7], i.e., an ordered Banach space such that its dual space E' is an AL space and X be the set

$$\{x \in E';\ x \geqq 0,\ \|x\| \leqq 1\}$$

endowed with the weak*-topology. Then X is a simplex (in the sense of Choquet) [7] and E may be identified with $A_0(X)$, the space of all continuous affine functions on X vanishing at 0. For each

$x \in X$, there is a unique maximal representing measure μ_x on X supported by $\overline{\partial X}$ (the weak* closure of the set ∂X of extreme points of X). By using this measure, we may further identify E [8, Theorem 3.3] with the space $A_0(\overline{\partial X})$ $\left(=\left\{f \in C(\overline{\partial X});\ f(x)=\int f d\mu_x \text{ for all } x \in \overline{\partial X} \text{ and } f(0)=0\right\}\right)$. As functions on X, E'' may be identified with the space of all bounded affine functions on X vanishing at 0.

For f, $g \in E$, the least upper bound of f and g does not necessarily exist in E, but always exists in its second dual E''. We denote the least upper bound in E'' of f and g by $f \curlyvee g$. The subspace F of E'' is called a sublattice of E'' if f, $g \in F$ implies $f \curlyvee g \in F$.

Let f be a bounded function on X. The upper envelope $\hat{f}$ of f is defined by

$$\hat{f}(x)=\inf\{h(x);\ h \in A_0(X) \quad \text{and} \quad h \geqq f\}$$

for all $x \in X$. For $f_1, \cdots, f_n$ in E, the function $f_1 \vee \cdots \vee f_n$ defined by

$$(f_1 \vee \cdots \vee f_n)(x)=\max\{f_1(x), \cdots, f_n(x)\} \quad \text{for all } x \in X$$

is a convex continuous function on X. The upper envelope $(f_1 \vee \cdots \vee f_n)^\wedge$ of $f_1 \vee \cdots \vee f_n$ is upper semi-continuous affine on X since X is a simplex [2, 28.4] and so $(f_1 \vee \cdots \vee f_n)^\wedge$ belongs to E''. Therefore we have

LEMMA. *For $f_1, \cdots, f_n \in E$, we have*

$$f_1 \curlyvee \cdots \curlyvee f_n=(f_1 \vee \cdots \vee f_n)^\wedge .$$

PROOF. For $x \in X$, it holds that

$$(f_1 \curlyvee \cdots \curlyvee f_n)(x)=\sup\left\{\sum_{i=1}^n f_i(x_i);\ \sum_{i=1}^n x_i=x,\ x_i \in X\right\}$$

by [11, II.4.2]. Put $f=f_1 \vee \cdots \vee f_n$. Then by [6, Lemma 2.8], we have

$$\sup\left\{\sum_{i=1}^n f_i(x_i);\ \sum_{i=1}^n x_i=x,\ x_i \in X\right\} =\sup\{\mu(f);\ \mu \in P_x(X)\},$$

where $P_x(X)$ is the set of probability measures on X with barycenter x. Since f is a continuous function on X, we have

$$\hat{f}(x)=\sup\{\mu(f);\ \mu \in P_x(X)\}$$

by [10, Proposition 3.1]. Therefore we have

$$(f_1 \vee \cdots \vee f_n)\hat{} = \hat{f} = f_1 \curlyvee \cdots \curlyvee f_n \,. \qquad //$$

As for the absolute value $|f|_{E''}$ $(=f \curlyvee (-f))$ of $f \in E$ in E'', we have

COROLLARY. *Let $f \in E$. Then we have*

$$|f|_{E''} = (f \vee (-f))\hat{} \,.$$

§ 3. The smallest Banach sublattice of E''.

Though a simplex space E is not necessarily a Banach lattice, its second dual E'' is a Banach lattice (AM space). But E'' is too large compared with E. So we seek the smallest Banach sublattice of E'' containing E and characterize it. Let Y be the set

$$\{y \in E'''; \ y \geqq 0, \ \|y\| \leqq 1\}$$

endowed with the weak*-topology. Since E'' is an AM space with order unit, the set ∂Y is closed and E'' is isometrically isomorphic to $C_0(\partial Y)$ $(=\{f \in C(\partial Y); \ f(0)=0\})$. For $y, y' \in \partial Y$, define the equivalence relation $y \sim y'$ if $f(y)=f(y')$ holds for all $f \in E$. Let

$$E_1 = \{f \in E''; \ f(y)=f(y') \text{ if } y \sim y'\} \,.$$

Let $\{(x_\alpha, x'_\alpha, c_\alpha)\}_{\alpha \in \Delta}$ be a subset of $\overline{\partial X} \times \overline{\partial X} \times [0, 1]$ consisting of all the triple $(x_\alpha, x'_\alpha, c_\alpha)$ such that $f(x_\alpha)=c_\alpha f(x'_\alpha)$ holds for any $f \in E$. Let V be the space

$$\{f \in C(\overline{\partial X}); \ f(x_\alpha)=c_\alpha f(x'_\alpha) \quad \text{for all} \quad \alpha \in \Delta\} \,.$$

Though an element of V does not necessarily belong to E'', we have

THEOREM 1. *Let E be a simplex space and X be the set $\{x \in E'; \ x \geqq 0, \ \|x\| \leqq 1\}$ endowed with the weak*-topology. Let $S=\{(f_1 \vee \cdots \vee f_n)\hat{}; \ f_i \in E, \ n \in N\}$ and $E_1=\{f \in E''; \ f(y)=f(y') \text{ if } y \sim y'\}$, where $y \sim y'$ is defined above. Then*

i) *E_1 is the smallest Banach sublattice of E'' containing E.*

ii) *$(S-S)^a$ (= the norm closure of $S-S$ in E'') is equivalent to E_1.*

iii) *E_1 is isometrically isomorphic to V defined above.*

iv) *E_1 is the space of all bounded affine functions f on $\overline{\partial X}$ such that there exists $g \in V$ satisfying $g|\partial X = f|\partial X$.*[1]

PROOF. i) Since E'' can be identified with $C_0(\partial Y)$ and for $f, g \in E''$, $(f \curlyvee g)(y) = \max\{f(y), g(y)\}$ holds for all $y \in \partial Y$, it is easily

[1] $f|F$ denotes the restriction of f to F.

obtained by definition that E_1 is a Banach sublattice of E'' containing E.

Suppose F is a Banach sublattice of E'' such that $E_1 \supset F \supset E$. For $y, y' \in \partial Y$, define the equivalence relation $y \approx y'$ if $f(y)=f(y')$ holds for all $f \in F$. Then $F=\{f \in E''; f(y)=f(y') \text{ if } y \approx y'\}$ holds since F is a Banach sublattice. On the other hand, $E \subset F$ implies $E_1 \subset \{f \in E''; f(y)=f(y') \text{ if } y \approx y'\}$ i.e., $E_1 \subset F$. Therefore $E_1=F$ and E_1 is the smallest Banach sublattice of E'' containing E.

ii) $f, g \in S$ implies $f+g \in S$ by Theorem 28.4 (iv) in [5]. For $f, g \in S$, $f \curlyvee g \in S$ follows from Lemma. Let $f, g \in S-S$ i.e., $f=f_1-f_2$, $g=g_1-g_2$, $f_i, g_i \in S$ $(i=1, 2)$. Then by the relation

$$f \curlyvee g=(f_1+g_2) \curlyvee (g_1+f_2)-(f_2+g_2)\ ,$$

we have $f \curlyvee g \in S-S$. Let $f, g \in (S-S)^a$. Then for any $\varepsilon>0$, there exist $f_\varepsilon, g_\varepsilon \in S-S$ such that

$$\|f-f_\varepsilon\|<\varepsilon \quad \text{and} \quad \|g-g_\varepsilon\|<\varepsilon\ .$$

Therefore $\|f_\varepsilon \curlyvee g_\varepsilon - f \curlyvee g\| \leqq \varepsilon$, which means $f \curlyvee g \in (S-S)^a$. So $(S-S)^a$ is a Banach sublattice of E'' containing E. By the definition, the smallest is obvious. Therefore by i), $E_1=(S-S)^a$.

iii) For $f \in S$ i.e., $f=(f_1 \vee \cdots \vee f_n)^\wedge$, $f_i \in E$ $(i=1, \cdots, n)$, let $\phi(f)=(f_1 \vee \cdots \vee f_n) \,|\, \overline{\partial X}$. Then $\phi(f) \in V$ and $\|\phi(f)\|=\|f\|$. For $x \in \partial X$, $\phi(f)(x)=f(x)$ by [10, Proposition 3.1]. For $f, g \in S$, let $\phi(f-g)=\phi(f)-\phi(g)$. Then $\phi(f-g) \in V$ and $\|\phi(f-g)\|=\|f-g\|$. For $x \in \partial X$, $\phi(f-g)(x)=(f-g)(x)$. Let $f \in (S-S)^a$. Then there exists a sequence $\{f_n\} \subset S-S$ such that $\|f_n-f\| \to 0$. By the relation $\|f_n-f_m\|=\|\phi(f_n)-\phi(f_m)\|$, $\phi(f_n)$ is also a Cauchy sequence in V. Therefore $\phi(f_n)$ converges to an element $h \in V$. Let $h=\phi(f)$. Then $\|f\|=\|\phi(f)\|$ and $f(x)=\phi(f)(x)$ for all $x \in \partial X$. Moreover ϕ is a continuous one-to-one mapping of E_1 into V. It remains to prove that ϕ is onto. Since ϕ is isometric, $\phi((S-S)^a)$ is a closed sublattice of V. E separates points of $\overline{\partial X}$ and for any nonzero $x \in \overline{\partial X}$, there is $f \in E \subset E_1$ such that $f(x) \neq 0$. By Stone-Weierstrass theorem, we have $\phi((S-S)^a)=V$.

iv) In the proof of iii), we get the conclusion by putting $g=\phi(f)$ for $f \in E_1$. //

Let T be a positive operator in a simplex space E. Then E_1 is not necessarily T''-invariant. The following example shows that E_1 is not T''-invariant.

EXAMPLE. Let $E=\{f \in C([0, 1]); f(1/2)=(1/2)\{f(0)+f(1)\}\}$ and T be defined by

$$Tf(x)=\begin{cases} f(2x) & 0\leqq x\leqq \dfrac{1}{4} \\ \dfrac{1}{2}\left\{f\left(2x-\dfrac{1}{2}\right)+f\left(\dfrac{3}{2}-2x\right)\right\} & \dfrac{1}{4}\leqq x\leqq \dfrac{3}{4} \\ f(2x-1) & \dfrac{3}{4}\leqq x\leqq 1\,. \end{cases}$$

Then E is a simplex space, T is a positive operator and E_1 is the space of all bounded functions on [0, 1] such that f is continuous on $[0, 1/2)\cup(1/2, 1]$ with $f(1/2)=(1/2)\{f(0)+f(1)\}$. Put $f_1(x)=1-2x$ and $f_2(x)=2x-1$. Then $f_1, f_2\in E$ and $f_1\curlyvee f_2\in E_1$. But $T''(f_1\curlyvee f_2)\notin E_1$ And there is no extension $\tilde{T}$ of T to E_1 such that $\tilde{T}E_1\subset E_1$.

So we consider what kind of operator keeps E_1 invariant.

DEFINITION. *We call $T\in\mathfrak{L}(E)$ a simplex homomorphism*[2] *if for any f, $g\in E$ and any $x\in\partial X$, there exists $h\in E$ such that $h\geqq f$, g and $Th(x)=\max\{Tf(x), Tg(x)\}$.*

Then we have

THEOREM 2. *If T is a simplex homomorphism of a simplex space, then $T''E_1$ is contained in E_1.*

PROOF. For $f\in S$, i.e., $f=(f_1\vee\cdots\vee f_n)^\wedge$, $f_i\in E$, we have $(Tf_1\vee\cdots\vee Tf_n)^\wedge(x)=\max\{Tf_1(x), \cdots, Tf_n(x)\}$ for $x\in\partial X$. If $T'x\neq 0$, $T'x/\|T'x\|\in\partial X$, which implies that $f_1(T'x)\vee\cdots\vee f_n(T'x)=(f_1\vee\cdots\vee f_n)^\wedge(T'x)=f(T'x)$. Therefore $T''f(x)=f(T'x)=(Tf_1\vee\cdots\vee Tf_n)^\wedge(x)$. Since f is upper semi-continuous affine, $T''f$ is upper semi-continuous affine and so $T''f=(Tf_1\vee\cdots\vee Tf_n)^\wedge$ holds by [4, Korollar 2.4.5]. Hence $T''f\in S$. If $f\in S-S$, i.e., $f=f_1-f_2$, $f_i\in S$ $(i=1, 2)$, then $T''f=T''f_1-T''f_2\in S-S$. If $f\in\overline{S-S}$, we get $T''f\in\overline{S-S}$. Therefore $T''E_1$ is contained in E_1. //

§ 4. $\mathfrak{L}(E)_+$-invariant sublattice of E''.

Let F be a Banach sublattice of E'' containing E. Put $Z=\{z\in F'; z\geqq 0, \|z\|\leqq 1\}$. Let i; $X\to Y$ be the natural embedding and τ; $X\to Z$ be the restriction of Y to F defined by $\tau(x)=i(x)|F$ for all $x\in X$. Then we have

PROPOSITION 1. *$\tau(\partial X)$ is contained in ∂Z.*

PROOF. Let $\lambda\in\partial X$. Then since λ is a simplex homomorphism of E into $\boldsymbol{R}$, $i(\lambda)$ is a lattice homomorphism of E'' into $\boldsymbol{R}$. So for

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[2] F. Jellett called it a Riesz homomorphism [9].

any $f, g \in F \subset E''$, we have $(f \vee g)(i(\lambda)) = \max\{f(i(\lambda)), g(i(\lambda))\}$. Since F is a sublattice of E'', $f \vee g \in F$. So $i(\lambda)|F$ is a lattice homomorphism of F into $\boldsymbol{R}$. By Theorem III.9.1 in [11], we have $\tau(\lambda) = i(\lambda)|F \in \partial Z$. //

E_1 is not necessarily T''-invariant and ∂X is not necessarily $\sigma(E''', E'')$-dense in ∂Y as shown in the example [1, Example I.2.10]. So we investigate the family of a Banach sublattice F of E'' satisfying

I) F contains E and order unit of E''.

II) F is T''-invariant for any positive operator $T \in \mathfrak{L}(E)$.

III) $\tau(\partial X)$ is $\sigma(F', F)$-dense in ∂Z.

REMARK. Since a bounded affine function on $\overline{\partial X}$ has many extensions to bounded affine functions on X [1, Example I.2.10], E'' cannot be expressed isomorphically as a space of functions on $\overline{\partial X}$. But F can be expressed as a space of functions on $\overline{\partial X}$ by the condition III).

Let S_2 be the set of all bounded upper semi-continuous affine functions $\{f\}$ on X satisfying $f(0)=0$. Then we have

PROPOSITION 2. *Let f be a bounded upper semi-continuous function on $\overline{\partial X}$ such that $f(x) = \int f d\mu_x$ for all $x \in \overline{\partial X}$ and $f(0)=0$. Then there is a unique extension $\tilde{f} \in S_2$ such that $\tilde{f}|\overline{\partial X} = f$.*

PROOF. Put

$$g(x) = \begin{cases} f(x) & x \in \overline{\partial X} \\ c & x \in X \backslash \overline{\partial X}\,, \end{cases}$$

where $c = \inf\{f(x);\ x \in \overline{\partial X}\}$. Then g is an upper semi-continuous convex function on X. By [2, Lemma 1.2], $\hat{g}(x) = \mu_x(g) = \mu_x(f)$, since the maximal measure is supported by $\overline{\partial X}$. Let $\tilde{f} = \hat{g}$. Then $\tilde{f}$ is affine on X by the relation $\mu_{x+y} = \mu_x + \mu_y$ and so $\tilde{f}$ is the desired one. Uniqueness follows from [4, Korollar 2.4.5]. //

Let E_2 be the space $(S_2 - S_2)^a$ (norm closure in E''). For $\phi \in E''$, $\|\phi\| = \sup_{x \in X} |\phi(x)|$ is not necessarily equal to $\sup_{x \in \overline{\partial X}} |\phi(x)|$. But for $\phi \in E_2$, $\|\phi\| = \sup_{x \in \overline{\partial X}} |\phi(x)|$ holds. So by Proposition 2, E_2 can be expressed as a space of functions on $\overline{\partial X}$. Furthermore we have

THEOREM 3. *Let E_2 be the space $(S_2 - S_2)^a$ (norm closure in E''), where S_2 is the set of all bounded upper semi-continuous affine functions f on X satisfying $f(0)=0$. Then E_2 is a Banach sublattice of E'' satisfying* I)$\sim$III) *defined above.*

PROOF. For $f_1, f_2 \in S_2$, we shall show that $f_1 \curlyvee f_2 = (f_1 \vee f_2)\hat{}$. Since $f_1 \vee f_2$ is an upper semi-continuous convex function, $\mu_x(f_1 \vee f_2) = (f_1 \vee f_2)\hat{}(x)$ holds by [2, Lemma 1.2]. So we have that $(f_1 \vee f_2)\hat{}$ is upper semi-continuous affine on X by using the relation $\mu_{(x+y)/2} = 1/2(\mu_x + \mu_y)$ obtained from [5, II.28.4]. For $x \in X$,

$$(f_1 \curlyvee f_2)(x) = \sup\{(f_1 - f_2)(y) + f_2(x);\ 0 \leqq y \leqq x\} .$$

Put $K = \{y \in X; (f_1 - f_2)(y) \geqq 0\}$. Define the mapping $z; E \to R$ by $z(g) = \int g \cdot \chi_K d\mu_x$. Then $z \in E'$ and $0 \leqq z \leqq x$. So

$$\begin{aligned}(f_1 \curlyvee f_2)(x) &\geqq (f_1 - f_2)(z) + f_2(x) \\ &= \int (f_1 - f_2) \cdot \chi_K d\mu_x + f_2(x) \\ &= \int \{(f_1 - f_2) \vee 0\} d\mu_x + f_2(x) \\ &= \mu_x\{(f_1 - f_2) \vee 0\} + \mu_x(f_2) \\ &= \mu_x(f_1 \vee f_2) .\end{aligned}$$

So we have $f_1 \curlyvee f_2 \geqq (f_1 \vee f_2)\hat{}$. Since $(f_1 \vee f_2)\hat{}$ belongs to E'', we have $f_1 \curlyvee f_2 = (f_1 \vee f_2)\hat{}$.

For $f, g \in S_2 - S_2$, i.e., $f = f_1 - f_2$, $g = g_1 - g_2$, f_i, $g_i \in S_2$ $(i = 1, 2)$, we have

$$f \curlyvee g = \{(f_1 + g_2) \vee (g_1 + f_2)\}\hat{} - (f_2 + g_2) \in S_2 - S_2 .$$

For $f, g \in (S_2 - S_2)^a$, there exist $\{f_n\}, \{g_n\} \subset S_2 - S_2$ such that $\|f_n - f\| \to 0$ and $\|g_n - g\| \to 0$. Then $\|f_n \curlyvee g_n - f \curlyvee g\| \to 0$. Therefore $f \curlyvee g \in (S_2 - S_2)^a$. So E_2 is a Banach sublattice of E''.

I) By the definition of E_2, it is clear that E_2 contains E. The order unit $\boldsymbol{l}$ of E'' is expressed as follows: $\boldsymbol{l}(x) = \sup\{f(x); f \in E, \|f\| \leqq 1\}$. So $\boldsymbol{l}$ is lower semi-continuous affine on X and belongs to E_2.

II) For $f \in S_2$ and $x \in X$, we have

$$f(x) = \inf\{h(x);\ h \geqq f,\ h \in E\} .$$

So we have $T''f(x) = f(T'x) = \inf\{Th(x); h \geqq f, h \in E\}$ and $T''f$ is upper semi-continuous affine on X and $T''f \in S_2$. So $f \in E_2$ implies $T''f \in E_2$ in a similar way to the above proof.

III) Let $Z = \{z \in E_2';\ z \geqq 0,\ \|z\| \leqq 1\}$. Since E_2 is a Banach lattice with order unit, $\partial Z \backslash \{0\}$ is $\sigma(E_2', E_2)$-closed and E_2 is isomorphic to $C(\partial Z \backslash \{0\})$. $f \in S_2 - S_2$ and $\|f | \partial X\| \leqq 1/n$ imply $\|f\| \leqq 1/n$ by [4, Korollar 2.4.5]. So $f \in E_2$ and $f | \partial X = 0$ imply $f = 0$, that is, $f \in C(\partial Z \backslash \{0\})$ with $f | \tau(\partial X) = 0$ implies $f = 0$. Therefore $\tau(\partial X)$ is $\sigma(E_2', E_2)$ dense in ∂Z. //

The remaining problem is whether E_2 is the smallest Banach sublattice of E'' satisfying I)$\sim$III). In general, it is not true. For if E is a Banach lattice, E itself is the smallest Banach sublattice of E'' satisfying I)$\sim$III) and E is not equivalent to E_2. Nevertheless we have the following.

PROPOSITION 3. *Let E be a separable simplex space with order unit such that $\overline{\partial X}\backslash\partial X$ is a finite set $\{x_1, \cdots, x_n\}$ $(n\geqq 1)$. Then E_2 is the smallest Banach sublattice of E'' satisfying* I)$\sim$III).

PROOF. For $y_0\in\overline{\partial X}\backslash\partial X$, there are compact subsets K_1 and K_2 of ∂X such that $K_1\cap K_2=\phi$ and $\mu_{y_0}(K_i)>0$ $(i=1, 2)$. Let $U(y_0)$ be a neighborhood of y_0 in X such that $\overline{U(y_0)}\cap\overline{\partial X}\subset\partial X\cup\{y_0\}\backslash(K_1\cup K_2)$. Since E is separable, X is metrizable and so we consider the distance $d(x, y)$ of x and $y\in X$. Then there is a sequence $\{y_n\}$ in $\partial X\cap U(y_0)$ such that $d(y_0, y_n)<1/n$ $(n=1, 2, \cdots)$, since $y_0\in\overline{\partial X}\backslash\partial X$. Define the mapping k on $[0, 1]$ into X by $k(0)=y_0$ and

$$k(t)=n(n+1)\left\{\left(\frac{1}{n}-t\right)y_{n+1}+\left(t-\frac{1}{n+1}\right)y_n\right\} \quad \text{for} \quad \frac{1}{n+1}\leqq t\leqq\frac{1}{n} \quad (n=1, 2\cdots)\ .$$

By Theorem III.3.3 in [3], there is a function $g\in E$ such that $g|K_1>0$, $g|K_2<0$ and $g|(U(y_0)\cap\overline{\partial X})=0$. Let $g_0=g\vee(-g)$. Then $g_0(y_0)>0$. So let $f_0=(1/g_0(y_0))g_0$. Then $f_0\in E_1$, $f_0(y_0)=1$ and $f_0(x)=0$ for $x\in U(y_0)\cap\partial X$. Therefore $f_0(k(t))=0$ for $0<t\leqq 1$.

Each bounded upper semi-continuous affine function on $\overline{\partial X}$ is the uniform limit of simple affine functions g_m on $\overline{\partial X}$ such that

$$g_m(x)=\sum_{j=1}^{m} c_{mj}\int\chi_{A_{mj}}d\mu_x \quad \text{for all} \quad x\in\overline{\partial X}\ ,$$

where A_{mj} is a closed subset of $\overline{\partial X}$. Since f_0 belongs to the smallest Banach sublattice E_1 of E'', it is enough to show that for any closed subset A of $\overline{\partial X}$ there are positive operators T_1, $T_2\in\mathfrak{L}(E)$ such that

$$(*)\qquad \int\chi_A d\mu_x=(T_1''-T_2'')f_0(x) \quad \text{for all} \quad x\in\overline{\partial X}\ .$$

If $(\overline{\partial X}\backslash\partial X)\cap A\neq\phi$, put $(\overline{\partial X}\backslash\partial X)\cap A=\{x_1, \cdots, x_s\}$. Let K be a compact subset of $\partial X\backslash A$ such that

$$\mu_{x_i}(K)>(1-\mu_{x_i}(A))/2 \quad \text{for any} \quad i(1\leqq i\leqq s)\ .$$

If $(\overline{\partial X}\backslash\partial X)\cap A=\phi$, let $K=\phi$. Let

$$h(x)=\int\chi_A d\mu_x+2\int\chi_K d\mu_x \quad \text{for any} \quad x\in X\ .$$

Then h is an upper semi-continuous affine function on X and

$$\int \chi_A d\mu_x = h(x) - 2\int \chi_K d\mu_x \quad \text{for any} \quad x \in X .$$

Let $r>0$ be a number such that

$$r \leqq \min\{d(A, K), d(x_i, A), d(x_i, K); s+1 \leqq i \leqq n\}$$
$$r \leqq \min\{d(x_i, x_j); 1 \leqq i < j \leqq n\}$$

and

$$(**) \qquad \mu_{x_i}(\{x \in \overline{\partial X}; d(x, x_j) \leqq r\}) < \frac{1}{2n} \quad \text{for all} \quad i, j \ (1 \leqq i, j \leqq n) .$$

Put $\alpha_i = \mu_{x_i}(A) + 2\mu_{x_i}(K)$, $(1 \leqq i \leqq n)$

$$p_i(x) = \min\left\{1, \frac{1}{r} d(x, x_i)\right\}, \qquad (1 \leqq i \leqq n)$$

$$q_1(x) = \min\left\{1, \frac{1}{r} d(x, A)\right\}, \qquad q_2(x) = \min\left\{1, \frac{1}{r} d(x, K)\right\}$$

(if $(\overline{\partial X} \backslash \partial X) \cap A = \phi$, let $q_2(x) \equiv 1$) and

$$\begin{aligned} T_{A,K} f(x) = & \sum_{j=1}^{s} (\alpha_j - 1)(1 - p_j(x)) f(k(p_j(x))) \\ & + \sum_{j=s+1}^{n} \alpha_j (1 - p_j(x)) f(k(p_j(x))) \\ & + (1 - q_1(x)) f(k(q_1(x))) + 2(1 - q_2(x)) f(k(q_2(x))) \\ & + \sum_{j=1}^{n} \beta_j(f)(1 - p_j(x)) \quad \text{for} \quad f \in E , \end{aligned}$$

where $\beta_j(f)$ is the solution to the equations

$$\sum_{j=1}^{n} a_{ij} \beta_j(f) = b_i(f) \qquad (1 \leqq i \leqq n)$$

$$a_{ij} = \delta_{ij} - \int (1 - p_j(x)) d\mu_{x_i} \quad (\delta_{ij} \text{ is the Kronecker's delta})$$

$$\begin{aligned} b_i(f) = & \sum_{j=1}^{s} (\alpha_j - 1) \int (1 - p_j(x)) f(k(p_j(x))) d\mu_{x_i} \\ & + \sum_{j=s+1}^{n} \alpha_j \int (1 - p_j(x)) f(k(p_j(x))) d\mu_{x_i} \\ & + \int (1 - q_1(x)) f(k(q_1(x))) d\mu_{x_i} \\ & + 2\int (1 - q_2(x)) f(k(q_2(x))) d\mu_{x_i} - \alpha_i f(y_0) . \end{aligned}$$

Since the relation (**) implies det $(a_{ij}) \neq 0$ and $(a_{ij})^{-1} \geqq 0$, $\beta_i(f)$

can be obtained for any $f \in E_2$. $f \in E$ implies $T_{A,K}f \in E$ and $f \geqq 0$ means $b_i(f) \geqq 0$ and $\beta_i(f) \geqq 0$. So $T_{A,K}$ is a positive operator in E since b_i is a linear functional on E. Moreover we have $T''_{A,K}f_0 = h$ since $b_i(f_0) = 0$ implies $\beta_i(f_0) = 0$. By taking K and ϕ instead of A and K, we get $T_{K,\phi} \in \mathfrak{L}(E)$ instead of $T_{A,K}$ such that $T''_{K,\phi}f_0(x) = \int \chi_K d\mu_x$. By putting $T_1 = T_{A,K}$ and $T_2 = T_{K,\phi}$, we get the relation (*). Therefore E_2 is the smallest Banach sublattice of E'' satisfying I)$\sim$ III). //

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