Natural Science Report, Ochanomizu University, Vol. 33, No. 1~2, 1982 37
お茶の水女子大学 自然科学報告 第33巻 第1~2号

Split Graphs with Dilworth Number Three

Chiê Nara

The Doctoral Research Course in Human Culture,
Ochanomizu University

(Received September 10, 1982)

§1. Introduction.

For a set S and a binary relation $\leqslant$ on S, the pair $(S, \leqslant)$ is called a *preordered set* if the relation is a preorder, i.e., a reflexive and transitive relation, and a subset S_0 of S is called *incomparable* if for any pair $\{x, y\}$ of S_0 it holds neither $x \leqslant y$ nor $y \leqslant x$. Then *Dilworth number* of the preordered set $(S, \leqslant)$ is defined by the maximum cardinality of incomparable subsets of S, which is equal to the minimum number of chains of the preorder $\leqslant$ covering S ([1]).

For a simple graph G we denote the vertex set by $V(G)$ and the neighborhood of a vertex v by $N(v)$ or $N_G(v)$. Let S be a subset of $V(G)$. Then the *vicinal preorder* $\leqslant$ is defined on S by

$$u \leqslant v \quad \text{if and only if} \quad N(u) \subset N(v) \cup \{v\}$$

for $u, v \in S$ which is in fact a preorder, and the Dilworth number of $(S, \leqslant)$ is written by $\nabla_G(S)$. *Dilworth number of* G, denoted by $\nabla(G)$, is defined by the Dilworth number of the preordered set $(V(G), \leqslant)$.

Especially if the vertex set $V(G)$ is decomposed into two subsets I_G and K_G such that the induced subgraph $\langle I_G \rangle$ and $\langle K_G \rangle$ are a descrete graph and a complete graph, respectively, G is called a *split graph* and denoted by $G = (I_G, K_G)$. It is easy to see that we have $u \leqslant v$ for any $u \in I_G$ and any $v \in K_G$ of a split graph G and hence it holds $\nabla(G) = \max\{\nabla_G(I_G), \nabla_G(K_G)\}$.

A characterization of split graphs with Dilworth number two is obtained by S. Foldes and P. L. Hammer ([2]). The aim of this paper is to give a characterization of split graphs with Dilworth number three.

* Present Address: Department of Mathematics, Musashi Institut of Technology, Tamazutsumi, Setagaya-ku, Tokyo 158, Japan.

§ 2. k-critical graphs.

In this section let $G=(I_G, K_G)$ be a split graph. We denote the cardinality of a set X, the degree of a vertex v and the edge set by $|X|$, $d_G(v)$ and $E(G)$ respectively, and set

$$N_G^*(v)=N_G(v)\cap I_G \quad \text{and} \quad d_G^*(v)=|N_G^*(v)|$$

for $v\in K_G$.

For two graphs G_1 and G_2, we write $G_1\prec G_2$ if G_2 has an induced subgraph isomorphic to G_1 and we shall identify G_1 with the induced subgraph isomorphic to G_1 if there is no fear of confusion.

For a positive integer $k\geqq 2$ a split graph $H=(I_H, K_H)$ is called *k-critical* if it satisfies $\nabla_H(I_H)\geqq k$ and $\nabla_{H-v}(I_{H-v})\leqq k-1$ for any vertex $v\in V(H)$, and we denote by $\mathfrak{F}_k$ the set of k-critical split graphs. Then the split graph G satisfies $\nabla_G(I_G)\geqq k$ if and only if G has an induced subgraph isomorphic to a graph of $\mathfrak{F}_k$.

PROPOSITION 1([2]). *The set $\mathfrak{F}_3$ is equal to the set $\{H_1, H_2, H_3\}$ of Figure* 1.

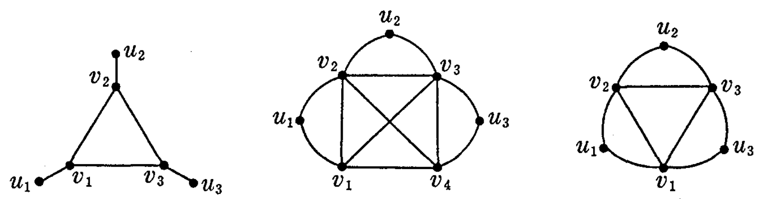


Figure 1.

In what follows we shall suppose that vertices of H_i $(i=1, 2, 3)$ are named by Figure 1.

We shall show three lemmas.

LEMMA 1. *If G is k-critical, then*

(i) *$\nabla_G(I_G)=k$ and $|I_G|=k$ hold and any pair of two vertices of I_G is incomparable,*

(ii) *$\nabla_{G-v}(I_{G-v})=k-1$ holds for any $v\in I_G$, so $G-v$ has an induced subgraph isomorphic to a graph of $\mathfrak{F}_{k-1}$,*

(iii) *$d_G^*(v)\leqq k-1$ holds for all $v\in K_G$.*

PROOF. (i) By $\nabla_G(I_G)\geqq k$ there is a subset $S_0\subset I_G$ which is incomparable and hence the induced subgraph $G_0=\langle S_0\cup K_G\rangle$ satisfies $\nabla_{G_0}(S_0)=k$ and $G_0\in\mathfrak{F}_k$, so it must be $G=G_0$ by the k-criticalness of G.

(ii) is true since $|I_G-\{v\}|=k-1$ and $I_G-\{v\}$ is incomparable in $G-v$.

Let's prove (iii). If $d_G^*(v)\geqq k$ for some $v\in K_G$, then v is adjacent to all vertices of I_G by (i). Hence any pair of two vertices of I_G is incomparable in G if and only if so is in $G-v$, which leads $\nabla_{G-v}(I_{G-v})=k$ and contradicts the k-criticalness of G.

LEMMA 2. *Let G be k-critical. If G has a vertex $v \in K_G$ with $d_G^*(v)=1$, say $N_G^*(v)=\{u\}$, then there is a vertex $w \in I_G-\{u\}$ satisfying*

$$N_G(u)-\{v\} \subset N_G(w) .$$

PROOF. Let w be any vertex of $I_G-\{v\}$. Assume $N_G(u)-\{v\} \not\subset N_G(w)$. Since the pair of u and w is incomparable, it holds $N_G(w) \not\subset N_G(u)$ and hence $N_G(w) \not\subset N_G(u)-\{v\}$. By $N_G^*(v)=\{v\}$ we get $N_{G-v}(w)=N_G(w)$ and $N_{G-v}(u)=N_G(u)-\{v\}$, which implies $N_{G-v}(w) \not\subset N_{G-v}(u)$ and $N_{G-v}(u) \not\subset N_{G-v}(w)$. Thus we can obtain $\nabla_{G-v}(I_{G-v})=k$, which is a contradiction.

Let's define a split graph $\hat{G}=(I_{\hat{G}}, K_{\hat{G}})$ as follows: the vertex sets are $V(\hat{G})=V(G)$, $I_{\hat{G}}=I_G$ and $K_{\hat{G}}=K_G$ and it holds

$$uv \in E(\hat{G}) \quad \text{if and only if} \quad uv \notin E(G)$$

for $u \in I_G$ and $v \in K_G$.

LEMMA 3. *We have $\nabla_{\hat{G}}(I_{\hat{G}})=\nabla_G(I_G)$. Moreover $\hat{G}$ is k-critical if and only if G is k-critical.*

PROOF. For any two vertices u and v of I_G, we have

$$N_{\hat{G}}(u) \subset N_{\hat{G}}(v) \quad \text{if and only if} \quad N_G(u) \supset N_G(v) .$$

Hence a subset of $I_{\hat{G}}$ is incomparable in $\hat{G}$ if and only if so is it in G, which implies $\nabla_{\hat{G}}(I_{\hat{G}})=\nabla_G(I_G)$. Similarly we can get that for any $v \in V(\hat{G})$

$$\nabla_{\hat{G}-v}(I_{\hat{G}-v}) \leqq k-1 \quad \text{if and only if} \quad \nabla_{G-v}(I_{G-v}) \leqq k-1 .$$

Thus the lemma has been proved.

§3. 4-critical graphs.

In the rest of this paper, let $G=(I_G, K_G)$ be a 4-critical split graph, i.e., $G \in \mathfrak{F}_4$. Then by (ii) of Lemma 1 $G-v$ has an induced subgraph isomorphic to a graph of $\mathfrak{F}_3$ for any $v \in I_G$ and hence by Proposition 1 it holds $G-v \succ H_i$ (some $i=1, 2$, or 3).

We shall use the symbol $G \cong G'$ if G is isomorphic to a graph G'.

PROPOSITION 2. *If there is a vertex $v \in K_G$ with $d_G^*(v)=1$, say $N_G^*(v)=\{u\}$, and if $G-u$ has an induced subgraph isomorphic to H_1 of Figure 1, then G is isomorphic to G_1, G_2 or G_3 of Figure 2.*

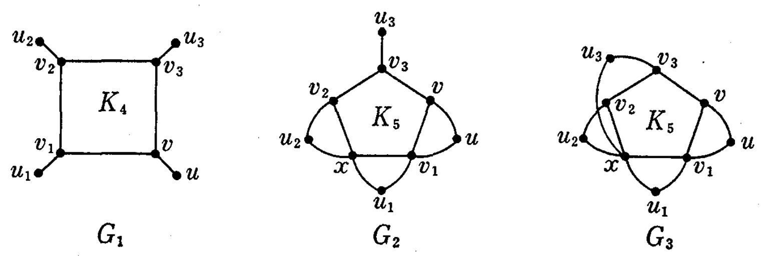


Figure 2.

Proof. By Lemma 2, there is a vertex $w \in I_G - \{u\} = \{u_1, u_2, u_3\}$ such that $N_G(u) \cap K_{H_1} \subset N_G(w) \cap K_{H_1}$. We can assume $w = u_1$ without loss of generality. Put $S = N_G(u) \cap K_{H_1}$. Then $|S| \leqq 1$ follows from $|N_G(u_1) \cap K_{H_1}| = 1$. It is easy to see that $G \cong G_1$ if $|S| = 0$. Let $|S| = 1$. Then $S = N_G(u_1) \cap K_{H_1} = \{v_1\}$. Since the pair $\{u, u_1\}$ is incomparable in G, there is a vertex $x \in K_G$ not adjacent to u but to u_1 and it satisfies $x \notin K_{H_1}$.

Now $d_G^*(x) \leqq 3$ holds by (iii) of Lemma 1. If $d_G^*(x) = 1$ holds, we have $\nabla_{G-v_1}(I_{G-v_1}) = 4$, which contradicts the 4-criticalness of G. Hence $d_G^*(x) = 2$ or 3, which implies that G is isomorphic to G_2 or G_3 respectively.

Proposition 3. *If there is a vertex $v \in K_G$ with $d_G^*(v) = 1$, say $N_G^*(v) = \{u\}$, and if $G - u$ has H_2 as an induced subgraph, then G is isomorphic to G_2, G_3 of Figure 2, G_4, G_5 or G_6 of Figure 3.*

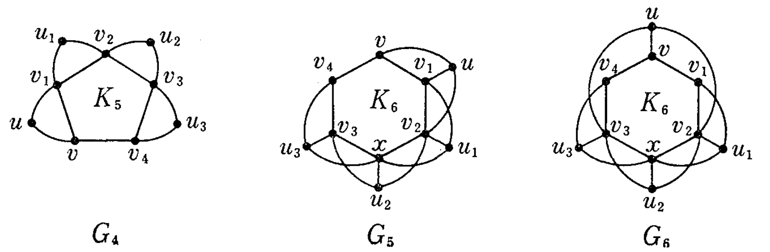


Figure 3.

Proof. Let u_i ($i = 1, 2$ or 3) be a vertex of $I_G - \{u\}$ satisfying $N_G(u) \cap K_{H_2} \subset N_G(u_i) \cap K_{H_2}$, which is guaranteed by Lemma 2. Put $S = N_G(u) \cap K_{H_2}$. Then $|S| \leqq 2$ since $|N_G(u_i) \cap K_{H_2}| = 2$. It is easy to see that $G \cong G_2$ if $|S| = 0$ and $G \cong G_3$ or G_4 if $|S| = 1$. Let $|S| = 2$. Then $S = N_G(u_i) \cap K_{H_2}$ holds and there is a vertex $x \in N_G(u_i) - N_G(u)$ by the incomparableness of the pair $\{u, u_1\}$, which satisfies $x \notin K_{H_2}$.

Let's prove $d_G^*(x) = 3$. Assume $d_G^*(x) \leqq 2$ and $i = 1$. If $d_G^*(x) = 1$, then $\nabla_{G-v_1}(I_{G-v_1}) = 4$ and if $d_G^*(x) = 2$, say $N_G^*(x) = \{u_1, z\}$, then $\nabla_{G-v_j}(I_{G-v_j}) = 4$ ($j = 2$ or 4 according to $z = u_2$ or u_3) holds. These contradict the 4-criticalness of G. We can similarly lead a contradiction in the case of $d_G^*(x) \geqq 2$ and $i = 3$. Finally assume $d_G^*(x) \leqq 2$ and $i = 2$. If $d_G^*(x) = 1$, then $\nabla_{G-v_2-v_3}(I_{G-v_2-v_3}) = 4$ and if $d_G^*(x) = 2$, say

$N_G^*(x)=\{u_2, z\}$, then $\nabla_{G-v_j}(I_{G-v_j})=4$ ($j=2$ or 3 according to $z=u_1$ or u_3) holds. These contradict the 4-criticalness of G. Hence we get $d_G^*(x)=3$.

Therefore we can conclude $G\cong G_5$ for $i=1$ or 3 and $G\cong G_6$ for $i=2$.

PROPOSITION 4. *If there is a vertex* $v\in K_G$ *with* $d_G^*(v)=1$, *say* $N_G^*(v)=\{u\}$, *and if* $G-u$ *has* H_3 *as an induced subgraph, then* G *is isomorphic to* G_7, G_8, G_9 *or* G_{10} *of Figure* 4.

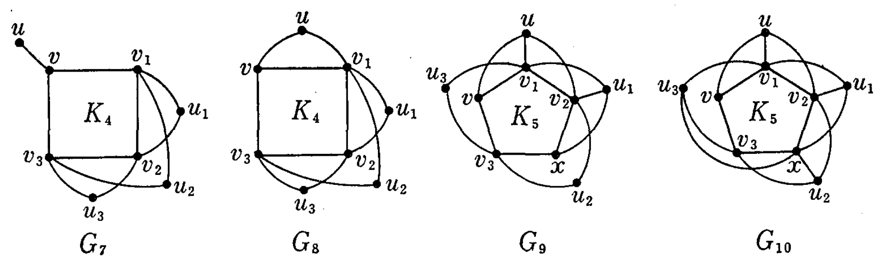


Figure 4.

PROOF. Let u_i ($i=1, 2$ or 3) be a vertex of I_G-u satisfying $N_G(u)\cap K_{H_3}\subset N_G(u_i)\cup K_{H_3}$, which is guaranteed by Lemma 2. We can assume $i=1$ without loss of generality. Put $S=N_G(u)\cap K_{H_3}$. Then $|S|\leqq 2$ since $|N_G(u_1)\cup K_{H_3}|=2$. It is easy to see that $G\cong G_7$ or G_8 if $|S|=0$ or 1, respectively. Let $|S|=2$. Then $S=N_G(u_1)\cap K_{H_3}=\{v_1, v_2\}$. Since the pair $\{u, u_1\}$ is incomparable, there is a vertex $x\in N_G(u_1)-N_G(u)$, which satisfies $x\notin K_{H_3}$.

It must be $d_G^*(x)=1$ or 3. For, if we assume $d_G^*(x)=2$ and put $N_G^*(x)=\{u_1, u_i\}$ ($i=2$ or 3), then $\nabla_{G-v_j}(I_{G-v_j})=4$ ($j=2$ or 1 according to $i=2$ or 3) holds and contradicts the 4-criticalness of G. Hence we get $G\cong G_9$ or G_{10} for $d_G^*(x)=1$ or 3, respectively.

PROPOSITION 5. *If* G *satisfies the condition*

$$(*)\qquad d_G^*(v)=2 \quad \text{for all} \quad v\in K_G\ ,$$

then G *is isomorphic to* G_{11} *or* G_{12} *of Figure* 5.

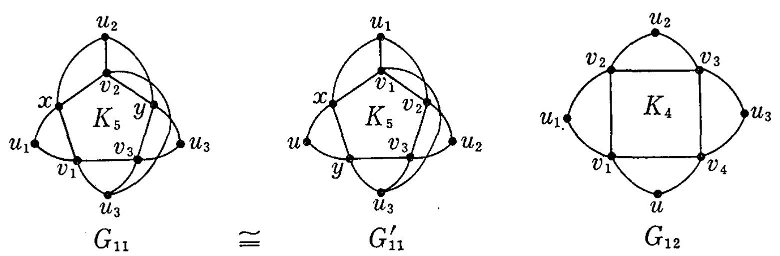


Figure 5.

PROOF. Let u be any fixed vertex of I_G. Then $G-u$ has H_1, H_2 or H_3 as an induced subgraph by (ii) of Lemma 1. If $G-u\succ H_1$

holds, by the condition (*) we get $K_{H_1} \subset N_G(u)$. Considering the incomparableness of $\{u, u_1\}$ there is a vertex $x \in N_G(u_1) - N_G(u)$, which satisfies $x \in K_{H_1}$. By (*) we may put $N_G^*(x) = \{u_1, u_2\}$ without loss of generality. Since the pair $\{u, u_3\}$ is also incomparable, there is another vertex $y \in N_G(u_3) - N_G(u)$ satisfying $y \notin K_{H_1}$. It holds $N_G^*(y) = \{u_3, u_i\}$ ($i=1$ or 2) by (*), which implies $G \cong G_{11}$.

If $G - u \succ H_2$, then by (*) we get $\{v_1, v_4\} \subset N_G(u)$ and $G \cong G_{12}$.

If $G - u \succ H_3$, then the set $N_G(u) \cap K_{H_3}$ is empty by (*). Since the pair $\{u, u_1\}$ is incomparable, there is a vertex $x \in N_G(u) - N_G(u_1)$ satisfying $x \notin K_{H_3}$. By (*) it holds $N_G^*(x) = \{u, u_i\}$ ($i=2$ or 3). Hence by the incomparableness of $\{u, u_i\}$, there is another vertex $y \in N_G(u) - N_G(u_i)$ satisfying $y \notin K_{H_3}$. Therefore $G \cong G'_{11} \cong G_{11}$ by (*).

Now we shall introduce 4 more graphs $G_{13} \sim G_{16}$ by the following Figure 6.

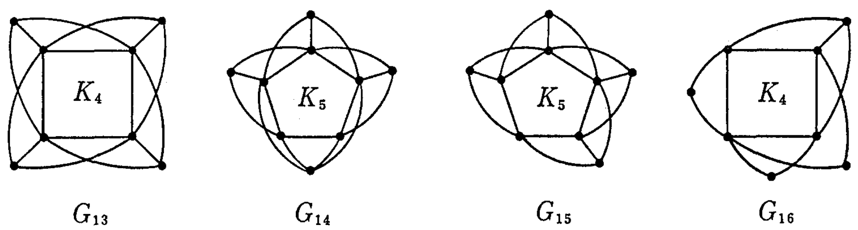


Figure 6.

Then, using the above propositions and the relations $\hat{G}_1 \cong G_{13}$, $\hat{G}_2 \cong G_{14}$, $\hat{G}_3 \cong G_{10}$, $\hat{G}_4 \cong G_{15}$, $\hat{G}_5 \cong G_5$, $\hat{G}_6 \cong G_6$, $\hat{G}_7 \cong G_{16}$, $\hat{G}_8 \cong G_8$, $\hat{G}_9 \cong G_9$, $\hat{G}_{10} \cong G_3$, $\hat{G}_{11} \cong G_{11}$ and $\hat{G}_{12} \cong G_{12}$, we get the following

THEOREM 1. *A split graph $H = (I_H, K_H)$ satisfies $\nabla_H(I_H) \leqq 3$ if and only if H has no induced subgraph isomorphic to $G_1, G_2, \cdots, G_{15}$ or G_{16}.*

PROOF. The contraposition of the theorem is that $\nabla_H(I_H) \geqq 4$ if and only if H has an induced subgraph isomorphic to $G_1, G_2, \cdots, G_{15}$ or G_{16}. Then it is enough to prove that $\mathfrak{F}_4$ is equal to the set $\{G_1, \cdots, G_{16}\}$. It is easy to see that G_i ($1 \leqq i \leqq 16$) is contained in $\mathfrak{F}_4$.

Let's prove the converse. Let H be a graph of $\mathfrak{F}_4$. By (iii) of Lemma 1 we have $d_H^*(v) \geqq 3$ for all $v \in K_H$. If $d_H^*(v) = 1$ holds for some $v \in K_H$ or if $d_H^*(v) = 2$ for all $v \in K_H$, we have already proved in Proposition 2, 3, 4 and 5 that H is isomorphic to $G_1, G_2, \cdots, G_{11}$ or G_{12}. Hence we can assume that there is a vertex $v \in K_H$ with $d_H^*(v) = 3$. Considering the split graph $\hat{H}$, we get $d_{\hat{H}}^*(v) = 1$ and $\hat{H} \in \mathfrak{F}_4$ by Lemma 3. Therefore $\hat{H}$ is isomorphic to $G_1, G_2, \cdots, G_9$ or G_{10} by Proposition 2, 3 and 4, which implies that H is isomorphic to $G_{13}, G_{14},$

G_{10}, G_{15}, G_5, G_6, G_{16}, G_8, G_9 or G_3 respectively. This completes the proof of Theorem 1.

The complement H^c of a split graph H is also a split graph and $\nabla_{H^c}(I_{H^c}) = \nabla_H(K_H)$. So using $\nabla(H) = \max\{\nabla_H(I_H), \nabla_H(K_H)\}$, we get

THEOREM 2. *A split graph H satisfies $\nabla(G) \leqq 3$ if and only if G and G^c has no induced subgraph isomorphic to $G_1, G_2, \cdots, G_{15}$ or G_{16}.*

References

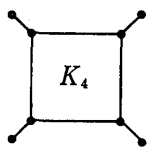
[1] R. P. Dilworth, A decomposition theorem for partially ordered sets, Ann. of Math. **51** (1950), 161-166.

[2] S. Foldes and P. L. Hammer, Split graphs having Dilworth number two, Can. J. of Math., Vol. **29**, no. 3 (1977), 666-672.

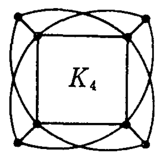
ANNEX

4-critical split graphs

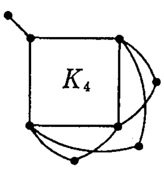
8 *vertices*



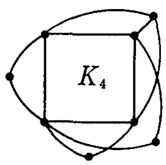
G_1



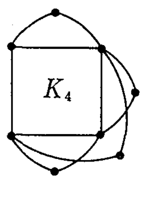
$\widehat{G_1} \cong G_{13}$



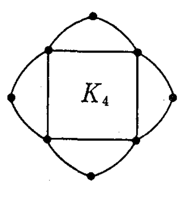
G_7



$\widehat{G_7} \cong G_{16}$

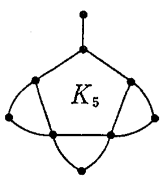


$G_8 \cong \widehat{G_8}$

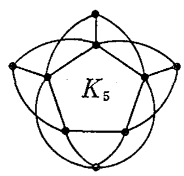


$G_{12} \cong \widehat{G_{12}}$

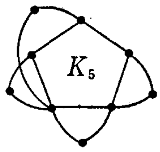
9 *vertices*



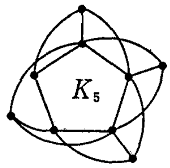
G_2



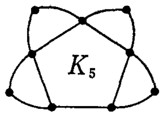
$\widehat{G_2} \cong G_{14}$



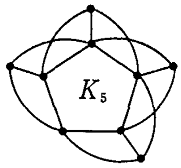
G_3



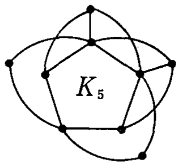
$\widehat{G_3} \cong G_{10}$



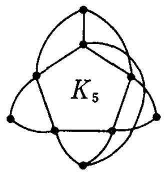
G_4



$\widehat{G_4} \cong G_{15}$

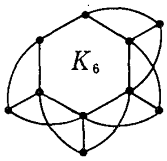


$G_9 \cong \widehat{G_9}$

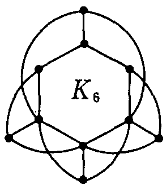


$G_{11} \cong \widehat{G_{11}}$

10 *vertices*



$G_5 \cong \widehat{G_5}$



$G_6 \cong \widehat{G_6}$