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On Balayage on Closed Sets

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Introduction. Let X be a locally compact Hausdorff space with a countable base and G be a continuous function-kernel on X. Further, assume that any non-empty open set is non-negligible. Then it is well-known that the following assertions are equivalent:

(i) G satisfies the domination principle,
(ii) G satisfies the balayage principle,
(iii) $\check{G}$ satisfies the domination principle,
(iv) $\check{G}$ satisfies the balayage principle. (cf. [4])

Here $\check{G}$ is the adjoint kernel of G defined

$$\check{G}(x, y) = G(y, x) \quad \text{for} \quad x, y \in X .$$

We recall that G satisfies the domination principle if $G\mu \leqq G\nu$ for $\mu \in M_K^+$ with $\int G\mu d\mu < +\infty$ and for $\nu \in M_K^+$ whenever the same inequality holds on the support supp μ of μ. We say that G satisfies the balayage principle if for every $\mu \in M_K^+$ and for every compact subset F of X, there is $\nu \in M_K^+$ with supp $\nu \subset F$ such that

$$G\nu \leqq G\mu \text{ on } X \quad \text{and} \quad G\nu = G\mu \text{ n.e. on } F ,$$

where "$G\nu = G\mu$ n.e. on F" means that the set $\{x \in F; G\nu(x) \neq G\mu(x)\}$ is negligible.

Further, with respect to familiar kernels, for example the Newton kernel in $\boldsymbol{R}^n (n \geqq 3)$ or the Green kernels on domains in $\boldsymbol{R}^n (n \geqq 3)$, every $\mu \in M_K^+$ is balayable on any closed set F, i.e., there exists $\nu \in M^+$ with supp $\nu \subset F$ such that $G\nu \leqq G\mu$ on X and $G\nu = G\mu$ n.e. on F.

On the other hand, we can find kernels with respect to which any $\mu \in M_K^+$ is not balayable on any closed set F, although they satisfy the domination principle (cf. [2]).

In this paper we shall ask necessary and sufficient conditions for a kernel G and a closed set F in order that any $\mu \in M_K^+$ is balayable on F.

§1. **Preliminary.** Throughout in this paper, let X be a locally compact Hausdorff space with a countable base and G be a continuous function-kernel, i.e., a continuous function from $X \times X$ into $\boldsymbol{R}^+ \cup \{+\infty\}$ in the extended sense satisfying $0 \leqq G(x, y) < +\infty$ if $x \neq y$ and $0 < G(x, y) \leqq +\infty$ if $x = y$. We denote by M^+ the set of all positive Radon measures on X and by M_K^+ the set of all positive Radon measures on X with compact support. For $\mu \in M^+$ and $x \in X$ put

$$G\mu(x) := \int G(x, y) d\mu(y)$$

and denote by $G\mu$ the function $x \mapsto G\mu(x)$. The adjoint kernel $\check{G}$ of G is the continuous function--kernel defined by

$$\check{G}(x, y) := G(y, x)$$

Further we use the following notations.

$\mathscr{E} := \left\{\mu \in M_K^+; \int G\mu d\mu < +\infty\right\}$,

$\mathscr{F} := \{\mu \in M_K^+; G\mu$ is finite and continuous$\}$,

$\check{\mathscr{F}} := \{\mu \in M_K^+; \check{G}\mu$ is finite and continuous$\}$,

$\mathscr{L} := \{\mu \in M_K^+; G\mu$ is locally bounded$\}$,

$\check{\mathscr{L}} := \{\mu \in M_K^+; \check{G}\mu$ is locally bounded$\}$.

For a closed set F

$M^+(F) := \{\mu \in M^+; \operatorname{supp} \mu \subset F\}$, where supp μ is the support of μ.

Similarly,

$$\check{\mathscr{F}}(F) := \{\mu \in \check{\mathscr{F}}; \operatorname{supp} \mu \subset F\}, \qquad \check{\mathscr{L}}(F) := \{\mu \in \check{\mathscr{L}}; \operatorname{supp} \mu \subset F)$$

and so on.

A Borel set B is said to be negligible if $\mu(B) = 0$ for all $\mu \in \mathscr{E}$. Let f, g be Borel measurable functions on X and F be a subset of X. We write

$$g \leqq f \text{ n.e. on } F \text{ (resp. } f = g \text{ n.e. on } F)$$

if the set $\{x \in F; f(x) < g(x)\}$ (resp. the set $\{x \in F; f(x) \neq g(x)\}$) is negligible.

Hereafter we assume that G satisfies the domination principle and each non- empty open set is non-negligible. Then $\check{G}$ also satisfies the domination principle and, both G and $\check{G}$ satisfy the balayage principle.

§ 2. A sublattice of $C(X)$.

We define

$$\check{S} := \{\check{G}\tau_1 \wedge \cdots \wedge \check{G}\tau_n; \tau_j \in \check{\mathscr{F}}, n \in N\},$$

where $\check{G}\tau_1 \wedge \cdots \wedge \check{G}\tau_n$ is min $\{\check{G}\tau_1, \check{G}\tau_2, \cdots, \check{G}\tau_n\}$. It is evident that $\check{S}$ is a min-stable convex cone in $C(X)$. Further we define, for $x \in X$,

$$A(x):=\{y\in X;\ a>0,\ u(x)=\mathrm{au}(x)\ \text{for every}\ u\in\check{S}\}\ .$$

Since $\check{G}$ satisfies the domination principle, it satisfies the continuity principle; for $\mu\in M_K^+$, $G\mu$ is finite and continuous everywhere whenever $G\mu$ is finite and continuous on supp μ. Using Lusin's theorem and the continuity principle for $\check{G}$, it is easy to see that for each non-negligible set N there is a non-zero measure $\mu\in\check{\mathscr{F}}$ with supp $\mu\subset N$ and that $A(x)=\{y\in X;\ b>0,\ G\varepsilon_y=bG\varepsilon_x\}$, where ε_x is the atomic measure at x. Since for every compact set K there is $\tau\in\check{\mathscr{F}}$ satisfying $\check{G}\tau\geqq 1$ on K, we can find $s_0\in C(X)$ with $s_0=\sum_{n=1}^{\infty}\check{G}\tau_n$ $(\tau_n\in\check{\mathscr{F}})$ and $s_0>0$ on X. Then it is easy to see that

$$A(x)=\{y\in X;\ u(y)/s_0(y)=u(x)/s_0(x)\quad\text{for all}\quad u\in\check{S}\}\ . \tag{2.1}$$

We define

$$A(K):=\bigcup_{x\in K}A(x)\quad\text{for a compact set } K\,.$$

Using (2.1) we can easily prove the following proposition.

PROPOSITION 1. *Let K be a compact subset of X and suppose that $y\notin A(K)$. Then there is a neighborhood $U(y)$ of y such that $U(y)\cap A(K)=\phi$.*

PROPOSITION 2. *Let K be a non-neigligible compact set and F be a compact set with $A(K)\cap F=\phi$. Then, for each $w_0\in\check{S}$ there exist $u_0,\ v_0\in S$ such that $0\leqq u_0-v_0\leqq w_0$ on X,*

$$u_0-v_0=w_0\ \textit{on}\ K\quad\textit{and}\quad u_0-v_0=0\ \textit{on}\ F\,. \tag{2.2}$$

PROOF. First, we shall prove that for each $x\in K$ there exist $u,\ v\in\check{S}$ such that

$$u-v=0\ \text{on}\ F\quad\text{and}\quad u(x)-v(x)>0\,. \tag{2.3}$$

Suppose that no pair of functions $u,\ v$ in $\check{S}$ satisfies (2.3). Then

$$u-v\geqq 0\ \text{on}\ F\quad\text{for}\ u,\ v\in\check{S}\ \text{implies}\ u(x)-v(x)\geqq 0\,. \tag{2.4}$$

In fact, if there are $u_1,\ v_1\in\check{S}$ such that $u_1-v_1\geqq 0$ on F and $u_1(x)-v_1(x)<0$, then $u=u_1$ and $v=u_1\wedge v_1$ satisfy (2.3) and this follows a contradiction. Thus $u-v\geqq 0$ on F for $u,\ v\in\check{S}$ implies $u(x)-v(x)\geqq 0$. Put

$$(\check{S}-\check{S})_F:=\{u_F-v_F;\ u\in\check{S},\ v\in\check{S}\}\ ,$$

where u_F and v_F are the restrictions of $u,\ v$ to F respectively. A positive linear functional $\Phi:\ u_F-v_F\mapsto u(x)-v(x)$ is a lattice homomorphism on the sublattice $(\check{S}-\check{S})_F$ of $C(F)$ and Φ in not zero. Therefore there exist positive real number b and $y\in F$ such that

$$v(y)=b\Phi(v) \quad \text{for all} \quad v\in(\check{S}-\check{S})_F\,. \qquad \text{(cf. Hilfssatz 4 in [1])}$$

Particularly

$$u(y)=b\Phi(v)=bu(x) \quad \text{for all} \quad u\in\check{S}\,.$$

Thus we have $y\in A(x)$ and this is a contradiction. Therefore there exist $u, v\in\check{S}$ satisfying (2.3). Since $\check{S}-\check{S}$ is a lattice, we can assume that $u-v\geqq 0$ on X. Since for any $x\in K$ we can find $u_x, v_x\in\check{S}$ such that $u_x-v_x\geqq 0$ on X, $u_x-v_x=0$ on F and $u_x(x)-v_x(x)>0$, there exist $u', v'\in\check{S}$ such that $u'-v'\geqq 0$ on X, $u'-v'=0$ on F and $u'-v'\geqq w_0$ on K. Then $(u'-v')\wedge w_0=u_0-v_0$ ($u_0\in\check{S}$, $v_0\in\check{S}$) is a function satisfying (2.2).

§ 3. **Balayage on closed sets.** Let F be a closed subset of X. We say that μ is balayable on F with respect to G if there exists $\nu\in M^+(F)$ such that $G\nu=G\mu$ n.e. on F and $G\nu\leqq G\mu$ on X, and say that ν is a balayaged measure of μ on F. Since X has a countable base, it is easy to prove the following proposition.

PROPOSITION 3. *Let F be a closed subset of X. If every $\mu\in M_K^+$ with supp $\mu\cap F=\phi$ is balayable on F, then every $\mu\in M_K^+$ is balayable on F.*

We fix an increasing sequence $\{K_n\}$ of compact sets with $X=\bigcup_{n=1}^\infty K_n$ and $K_n\subset K_{n+1}^i$ for every $n\in N$, where K_{n+1}^i is the interior of K_{n+1}. Since $X\backslash A(K_n)$ is open by Proposition 1, there is an increasing sequence $\{O_{nj}\}_j$ of compact sets satisfying $X\backslash A(K_n)=\bigcup_{j=1}^\infty O_{nj}$. Put

$$\begin{aligned}
&H_1:=K_1\,,\\
&H_2:=K_1\cup(K_2\cap O_{12})\,,\\
(3.1)\qquad &H_3:=K_1\cup(K_2\cap O_{13})\cup(K_3\cap O_{23})\,,\\
&\quad\vdots\\
&H_n:=K_1\cap(K_2\cap O_{1n})\cup(K_3\cap O_{2n})\cup\cdots\cup(K_n\cap O_{n-1,n})\,,\\
&\quad\vdots
\end{aligned}$$

Then $\{H_n\}$ is an increasing sequence of compact sets.

THEOREM 1. *Let F be a closed subset of X. The following three assertions are equivalent;*

(a) *Every $\mu\in M_K^+$ is balayable on F.*

(b) *ε_x is balayable on F for each $x\in CF$.*

(c) *Let $\{\sigma_n\}$ and $\{\tau_n\}$ be sequences of measures in $\check{\mathscr{L}}(F)$. Assume that $\check{G}\lambda\geqq\check{G}\sigma_n-\check{G}\tau_n\geqq 0$ n.e. on F for some $\lambda\in\check{\mathscr{L}}$ and $\check{G}\sigma_n-\check{G}\tau_n\geqq\check{G}\sigma_{n-1}-\check{G}\tau_{n-1}$ n.e. on F for every $n\in N$. Further, if $\lim_{n\to\infty}(\check{G}\sigma_n-\check{G}\tau_n)=$ n.e. on F, then $\lim_{n\to\infty}(\check{G}\sigma_n(x)-\check{G}\tau_n(x))=0$ for every $x\in CF$.*

PROOF. (a)$\Rightarrow$(b): Trivial.

(b)$\Rightarrow$(c): Let x be an arbitrary point in CF and ν_x be a balayaged measure of ν_x on F. Then $\nu_x(N)=0$ for any negligible set N. Since

$$\int \check{G}\lambda d\nu_x = \int G\nu_x d\lambda \leqq \int G\varepsilon_x d\lambda = \check{G}\lambda(x) < +\infty ,$$

we have, by the Lebesgue's convergence theorem,

$$\begin{aligned} 0 &= \int \lim_{n\to\infty} (\check{G}\sigma_n - \check{G}\tau_n) d\nu_x = \lim_{n\to\infty} \int (\check{G}\sigma_n - \check{G}\tau_n) d\nu_x \\ &= \lim_{n\to\infty} \int G\nu_x d(\sigma_n - \tau_n) = \lim_{n\to\infty} \int G\varepsilon_x d(\sigma_n - \tau_n) \\ &= \lim_{n\to\infty} (G\sigma_n(x) - G\tau_n(x)) . \end{aligned}$$

(c)$\Rightarrow$(a): It suffices to prove that every $\mu \in M_K^+$ with supp $\mu \cap F = \phi$ is balayable on F by Proposition 3. Let $\mu \in M_K^+$ be arbitrary with supp $\mu \cap F = \phi$ and $\{H_n\}$ be the increasing sequence of compact sets defined by (3.1). Put $J_n := H_n \cap F$ for each $n \in N$. Since G satisfies the domination principle, G satisfies the balayaged principle. Let ν_n be a balayaged measure of μ on J_n with respect to G. Then $G\nu_n = G\mu$ n.e. on J_n, $G\nu_n \leqq G\mu$ on X and supp $\nu_n \subset J_n$. Since $\{\nu_n\}$ is vaguely bounded, we can find a subsequence $\{\nu_{n_j}\}$ of $\{\nu_n\}$ which converges vaguely to $\nu \in M^+(F)$. We use also $\{\nu_n\}$ instead of $\{\nu_{n_j}\}$. Then

$$G\nu \leqq \liminf_{n\to\infty} G\nu_n \leqq G\mu \tag{3.2}$$

and hence

$$\int G\nu d\sigma \leqq \int G\mu d\sigma \quad \text{for every} \quad \sigma \in M_K^+ . \tag{3.3}$$

Especially, if

$$\int G\nu d\lambda = \int G\mu d\lambda \tag{3.4}$$

for every $\lambda \in \check{\mathscr{F}}(F)$, we have (3.4) for $\lambda \in \mathscr{E}(F)$ by Lusin's theorem and the continuity principle of $\check{G}$. Consequently $G\nu = G\mu$ n.e. on F. Thus it will be seen that ν is a balayaged measure of μ on F. We fix an arbiyrary $\lambda \in \check{\mathscr{F}}(F)$ to prove the equality (3.4). Since $\check{G}\lambda$ is continuous, we can find an increasing sequence $\{f_p\}$ of continuous functions with compact support satisfying

$$f_p = \check{G}\lambda \text{ on } K_p , \quad f_p = 0 \text{ on } CK_{p+1} \quad \text{and} \quad 0 \leqq f_p \leqq \check{G}\lambda \text{ on } X .$$

Since

$$\lim_{n\to\infty}\int f_p d\nu_n=\int f_p d\lambda ,$$

there is a sufficiently large number $n_p \geqq p$ such that

$$\int f_p d\nu > \int f_p d\nu_{n_p} - \frac{1}{p} . \tag{3.5}$$

We remark that

$$J_{n_p}=(K_1\cap F)\cup(K_2\cap F\cap O_{1,n_p})\cup\cdots\cup(K_p\cap F\cap O_{p-1,n_p})$$
$$\cup\cdots\cap(K_{n_p}\cap F\cap O_{n_{p-1},n_p}) .$$

Put

$$B_p:=J_{n_p}\cap K_p \quad\text{and}\quad B'_p:=J_{n_p}\backslash K_p .$$

Then both B_p and B'_p are compact and it holds that $A(B_p)\cap B'_p=\phi$. By Proposition 2, there exist $u_p, v_p \in \check{S}$ such that

$$0\leqq u_p-v_p\leqq \check{G}\lambda , \quad u_p-v_p=\check{G}\lambda \text{ on } B_p \quad\text{and}\quad u_p-v_p=0 \text{ on } B'_p .$$

Since $\check{S}-\check{S}$ is a lattice, we may assume that $u_p-v_p\leqq u_{p+1}-v_{p+1}$ for every $p\in N$. Both u_p and v_p belong to $\check{S}$, it is well-known that there are $\sigma_p\in M^+(J_{n_p})$ and $\tau_p\in M^+(J_{n_p})$ such that

$$\check{G}\sigma_p=u_p \text{ n.e. on } J_{p_p} , \quad \check{G}\sigma_p\leqq u_p \text{ on } X$$

and

$\check{G}\tau_p=v_p$ n.e. on J_{n_p}, $\check{G}\tau_p\leqq v_p$ on X (cf. Proof of Theorem 3 in [6], Proposition II.3 in [3]).

Then $\sigma_p\in\check{\mathscr{L}}$, $\tau_p\in\check{\mathscr{L}}$,

$$\check{G}\sigma_p-\check{G}\tau_p=\check{G}\lambda \text{ n.e. on } B_p \quad\text{and}\quad \check{G}\sigma_p-\check{G}\tau_p=0 \text{ n.e. on } B'_p . \tag{3.6}$$

Since $\check{G}$ satisfies the domination principle, it holds that $0\leqq\check{G}\sigma_p-\check{G}\tau_p\leqq\check{G}\lambda$ n.e. on X, $0\leqq\check{G}\sigma_p-\check{G}\tau_p\leqq\check{G}\lambda$ on CF and $\check{G}\sigma_p-\check{G}\tau_p\leqq\check{G}\sigma_{p+1}-\check{G}\tau_{p+1}$ n.e. on F. By (3.6) we have

$$\lim_{p\to\infty}(\check{G}\sigma_p-\check{G}\tau_p)=\check{G}\lambda \text{ n.e. on } \bigcup_{p=1}^{\infty} J_p . \tag{3.7}$$

If there is $y\in F$ satisfying $y\notin\bigcup_{p=1}^{\infty}J_p$ with $y\in A(z)$. Since $G(z, z)=G\varepsilon_z(z)=bG\varepsilon_y(z)<+\infty$, the set $\{z\}$ of one point is non-negligible and hence $\lim_{p\to\infty}(\check{G}\sigma_p(z)-\check{G}\tau_p(z))=\check{G}\lambda(z)$. Consequently $\lim_{p\to\infty}(\check{G}\sigma_p(y)-\check{G}\tau_p(y))=\check{G}\lambda(y)$. Thus we have

$$\lim_{p\to\infty}(\check{G}\sigma_p-\check{G}\tau_p)=\check{G}\lambda \text{ n.e. on } F .$$

By the assumption (c) we have

$$\lim_{p\to\infty}(\check{G}\sigma_p(x)-\check{G}\tau_p(x))=\check{G}\lambda(x) \quad \text{for every} \quad x\in CF\,.$$

From (3.5), (3.6) and Fatou's lemma it follows that

$$\begin{aligned}\int\check{G}\lambda d\nu&=\lim_{p\to\infty}\int f_p d\nu\geqq\liminf_{p\to\infty}\left(\int f_p d\nu_{n_p}-\frac{1}{p}\right)\\ &\geqq\liminf_{p\to\infty}\left(\int(\check{G}\sigma_p-\check{G}\tau_p)d\nu_{n_p}-\frac{1}{p}\right)\\ &=\liminf_{p\to\infty}\left(\int(\check{G}\sigma_p-\check{G}\tau_p)d\mu-\frac{1}{p}\right)\\ &\geqq\int\liminf_{p\to\infty}(\check{G}\sigma_p-\check{G}\tau_p)d\mu=\int\check{G}\lambda d\mu\,.\end{aligned}$$

Consequently, using (3.3), we have the equality (3.4) for every $\lambda\in\check{\mathscr{F}}(F)$. Thus we have proved that ν is a balayaged measure of μ on F.

We can prove the following theorem as the same method of the proof of Theorem 1.

THEOREM 1′. *Let F be a closed subset of X. The following assertions* (a′) *and* (c′) *are equivalent;*

(a′) *Every $\mu\in\mathscr{F}$ is balayable on F.*

(c′) *Let $\{\sigma_n\}$ and $\{\tau_n\}$ be sequences of measures in $\check{\mathscr{L}}(F)$. Assume that $0\leqq\check{G}\sigma_n-\check{G}\tau_n\leqq\check{G}\lambda$ n.e. on F for some $\lambda\in\check{\mathscr{L}}$ and $\check{G}\sigma_n-\check{G}\tau_n\geqq\check{G}\sigma_{n+1}-\check{G}\tau_{n+1}$ n.e. on F for every $n\in N$. If $\lim_{n\to\infty}(\check{G}\sigma_n-\check{G}\tau_n)=0$ n.e. on F, then $\lim_{n\to\infty}(\check{G}\sigma_n-\check{G}\tau_n)=0$ n.e. on X.*

PROPOSITION 4. *Let F be a non-negligible closed subset of F and μ be a measure in M_K^+. If μ is balayable on F, then, for every $\lambda\in\check{\mathscr{F}}$ and every positive real number ε, there exist $n_0\in N$, $\{\sigma_p\}\subset\check{\mathscr{L}}(F)$ and $\{\tau_p\}\subset\check{\mathscr{L}}(F\cap H_{n_0})$ such that*

$$(3.8)\qquad\begin{cases}0\leqq\check{G}\sigma_p-\check{G}\tau_p\leqq\check{G}\lambda \text{ n.e. on } X\,,\\ \lim_{p\to\infty}(\check{G}\sigma_p-\check{G}\tau_p)=\check{G}\lambda \text{ n.e. on } F\backslash A(K_{n_0})\,,\\ \lim_{p\to\infty}\int(\check{G}\sigma_p-\check{G}\tau_p)d\mu<\varepsilon\,.\end{cases}$$

PROOF. We use the increasing sequences $\{H_n\}_n$ and $\{O_{np}\}_p$ of compact sets defined in (3.1). Since, for a sufficient large number n, $A(F\cap H_n)\cap(F\cap O_{np})=\phi$ for every $p\in N$, we can find, by Proposition 2, sequences $\{u_{np}\}_p$ and $\{v_{np}\}_p$ in $\check{S}$ satisfying

$$u_{np}-v_{np}=\check{G}\lambda \text{ n.e. on } F\cap O_{np} \quad\text{and}\quad u_{np}-v_{np}=0 \text{ n.e. on } F\cap H_n\,.$$

Since $u_{np}\in\check{S}$ and $v_{np}\in\check{S}$, there are $\sigma'_{np}\in\check{\mathscr{L}}(F\cap(H_n\cup O_{np}))$ and $\tau_{np}\in\check{\mathscr{L}}(F\cap H_n)$ such that

$\check{G}\sigma'_{np}-\check{G}\tau_{np}\geqq\check{G}\lambda$ n.e. on $F\cap O_{np}$ and $\check{G}\sigma'_{np}-\check{G}_{np}=0$ n.e. on $F\cap H_n$.

We remark that $(\check{G}\sigma'_{np}-\check{G}\tau_{np})\wedge(\check{G}\lambda=\check{G}\sigma'_{np}\wedge(\check{G}\lambda+\check{G}\tau_{np})-\check{G}\tau_{np}$. Let us choose $\sigma_{np}\in\check{\mathscr{L}}(F\cap(H_n\cup O_{np}))$ satisfying

$$\check{G}\sigma_{np}=\check{G}\sigma'_n\wedge(\check{G}\lambda+\check{G}\tau_{np}) \text{ n.e. on } F\cap(H_n\cup O_{np}) .$$

Then

$$\check{G}\sigma_{np}=\check{G}\tau_{np}=\check{G}\lambda \text{ n.e. on } F\cap O_{np} , \quad \check{G}\sigma_{np}-\check{G}\tau_{np}=0 \text{ n.e. on } F\cap H_n .$$

Hence $0\leqq\check{G}\sigma_{np}-\check{G}\tau_{np}\leqq\check{G}\lambda$ n.e. on X. Further

$$\lim_{p\to\infty}(\check{G}\sigma_{np}-\check{G}\tau_{np})=\check{G}\lambda \text{ n.e. on } \bigcup_{p=1}^{\infty} O_{np}=F\backslash A(K_n) ,$$
$$\lim_{p\to\infty}(\check{G}\sigma_{np}-\check{G}\tau_{np})=0 \text{ n.e. on } F\cap H_n$$

and

$$0\leqq\lim(\check{G}\sigma_{np}-\check{G}\tau_{np})\leqq\check{G}\lambda \text{ n.e. on } F .$$

Accordingly $\lim_{n\to\infty}\lim_{p\to\infty}(\check{G}\sigma_{np}-\check{G}\tau_{np})=0$ n.e. on $F\cap(\bigcup_{n=1}^{\infty}H_n)$ and hence

$$\lim_{n\to\infty}\lim_{p\to\infty}(\check{G}\sigma_{np}-\check{G}\tau_{np})=0 \text{ n.e. on } F . \tag{3.9}$$

Let ν be a balayaged measure of μ on F. Then ν is a measure satisfying $\nu(N)=0$ for every negligible set N. Remarking that $\sigma_{np}\in\check{\mathscr{L}}(F)$ and $\tau_{np}\in\check{\mathscr{L}}(F)$, we obtain, by (3.9) and the Lebesgue's convergence theorem,

$$\begin{aligned}0&=\int\lim_{n\to\infty}\lim_{p\to\infty}(\check{G}\sigma_{np}-\check{G}\tau_{np})d\nu=\lim_{n\to\infty}\int\lim_{p\to\infty}(\check{G}\sigma_{np}-\check{G}\tau_{np})d\nu\\&=\lim_{n\to\infty}\lim_{p\to\infty}\int(\check{G}\sigma_{np}-\check{G}\tau_{np})d\nu=\lim_{n\to\infty}\lim_{p\to\infty}\int(\check{G}\sigma_{np}-\check{G}\tau_{np})d\mu .\end{aligned}$$

Consequently, for any $\varepsilon>0$ there is $n_0\in N$ such that

$$\lim_{p\to\infty}\int(\check{G}\sigma_{n_0,p}-\check{G}\tau_{n_0,p})d\mu<\varepsilon .$$

Put $\sigma_{n_0p}:=\sigma_p$ and $\tau_{n_0p}:=\tau_p$. Then the sequences $\{\sigma_p\}$ and $\{\tau_p\}$ satisfy (3.8).

PROPOSITION 5. *Let F be a closed set and x be a point in CF. Suppose that for each $\lambda\in\check{\mathscr{F}}(F)$ and each real number $\varepsilon>0$ there exist $n_0\in N$ and sequences $\{\sigma_p\}$ and $\{\tau_p\}$ of measures satisfying*

(i) $\{\sigma_p\}\subset\check{\mathscr{L}}$ *and* $\{\tau_p\}\subset\check{\mathscr{L}}(H_{n_0}\cap F)$,
(ii) $0\leqq(\check{G}\sigma_p-\check{G}\tau_p)\leqq\check{G}\lambda$ *n.e. on* X,

(iii) $\lim_{n\to\infty}(\check{G}\sigma_p-\check{G}\tau_p)$ *n.e. on* $F\backslash A(K_{n_0})$,
(iv) $\lim_{n\to\infty}(\check{G}\sigma_p(x)-\check{G}\tau_p(x))<\varepsilon$.
Then ε_x *is balayable on* F *with respect to* G.

PROOF. Let ν_n be a balayaged measure on $H_n\cap F$ with respect to G. Then $\nu_n\in M^+(H_n\cap F)$ satisfies

$$G\nu_n\leqq G\varepsilon_x \text{ on } X \quad\text{and}\quad G\nu_n=G\varepsilon_x \text{ n.e. on } H_n\cap F\,.$$

Since $\{\nu_n\}$ is vaguely bounded, we can choose a subsequence $\{\nu_{n_j}\}$ of which converges vaguely to $\nu\in M^+(F)$. It holds that

$$G\nu\leqq\liminf_{j\to\infty}G\nu_{n_j}\leqq G\varepsilon_x\,. \tag{3.10}$$

Let $\lambda\in\check{\mathscr{L}}(F)$ be arbitrary and ε be an arbitrary positive real number. Then we can choose $n_0\in N$, $\{\sigma_p\}$ and $\{\tau_p\}$ satisfying (i)$\sim$(iv). Remarking that for any $j\geqq n_0$ $\nu_{n_j}\in\check{\mathscr{L}}(H_{n_j}\cap F)$, $\tau_p\in\check{\mathscr{L}}(F\cap H_{n_0})\subset\check{\mathscr{L}}(F\cap H_{n_j})$ and $CK_{n_0}\cap(H_{n_j}\cap F)\subset F\backslash A(H_{n_0})$, we obtain

$$\begin{aligned}\int_{CK_{n_0}}\check{G}\lambda d\nu_{n_j}&\leqq\int\liminf_{p\to\infty}(\check{G}\sigma_p-\check{G}\tau_p)d\nu_{n_j}\\&\leqq\liminf_{p\to\infty}\int(\check{G}\sigma_p-\check{G}\tau_p)d\nu_{n_j}\\&=\liminf_{p\to\infty}\left(\int G\nu_{n_j}d(\sigma_p-\tau_p)\right)\\&\leqq\liminf_{p\to\infty}\left(\int G\varepsilon_x d(\sigma_p-\tau_p)\right)\\&=\lim_{p\to\infty}(\check{G}\sigma_p(x)-\check{G}\tau_p(x))<\varepsilon\,.\end{aligned}$$

Further, as $j\to\infty$,

$$\begin{aligned}\int\check{G}\lambda d\nu&\leqq\liminf_{j\to\infty}\int\check{G}\lambda d\nu_{n_j}\leqq\limsup_{j\to\infty}\int\check{G}\lambda d\nu_{n_j}\\&\leqq\limsup_{j\to\infty}\int_{Kn}\check{G}\lambda d\nu_{n_j}+\limsup_{j\to\infty}\int_{CK_n}\check{G}\lambda d\nu_{n_j}\\&<\int_{Kn}\check{G}\lambda d\nu+\varepsilon\leqq\int\check{G}\lambda d\nu+\varepsilon\,.\end{aligned}$$

Consequently

$$\lim_{j\to\infty}\int\check{G}\lambda d\nu_{n_j}=\int\check{G}\lambda d\nu\,.$$

On the other hand, let us λ_{n_j} be a balayaged measure of λ on H_{n_j} with respect to $\check{G}$. Then, putting $\bigcup_{j=1}^{\infty}H_{n_j}=H$,

$$\lim_{j\to\infty}\check{G}\lambda_{n_j}=\check{G}\lambda \text{ n.e. on } H\cap F \quad\text{and hence n.e. on } F\,.$$

From $\lambda\in\check{\mathscr{F}}(F)$ it follows that $\check{G}\lambda\leqq\lim_{j\to\infty}\check{G}\lambda_{n_j}$ n.e. on X. Especially,

$\check{G}\lambda \leqq \lim_{j\to\infty} \check{G}\lambda_{n_j}$ on CF and hence $\check{G}\lambda = \lim_{j\to\infty} \check{G}\lambda_{n_j}$ on CF. Thus we have

$$\int G\nu d\lambda = \int \check{G}\lambda d\nu = \lim_{j\to\infty} \int \check{G}\lambda d\nu_{n_j} = \lim_{j\to\infty} \int \check{G}\lambda_{n_j} d\nu_{n_j}$$
$$= \lim_{j\to\infty} \int G\varepsilon_x d\lambda_{n_j} = \lim_{j\to\infty} \check{G}\lambda_{n_j}(x) = \check{G}\lambda(x) = \int G\varepsilon_x d\lambda \,.$$

Since $\lambda \in \check{\mathscr{F}}(F)$ is arbitrary, we obtain $G\nu = G\varepsilon_x$ n.e. on F. Therefore, by (3.10), ν is a balayaged measure of ε_x on F with respect to G.

Using Theorem 1 and Propositions 4, 5, it is easy to prove the following theorem.

THEOREM 2. *Let F be a non-negligible closed set. Any $\mu \in M_K^+$ is balayable on F if and only if every $x \in CF$, for every $\lambda \in \check{\mathscr{F}}(F)$ and for every real number $\varepsilon > 0$ there exist $n_0 \in N$ and sequences $\{\sigma_p\}$ and $\{\tau_p\}$ of measures satisfying* (i)$\sim$(iv) *in Proposition* 5.

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