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Some Theorems on Recurrent Markov Processes

Kumiko Kitamura

Department of Mathematics, Faculty of Science,
Ochanomizu University
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§0. Introduction.

The important and beautiful connection between Markov processes and analytic potential theory was investigated by many mathematicians, for instance, Kakutani, Doob and Hunt. There is now a large literature on potential theory for transient Markov processes. Recurrent potential theory was also developed for mainly random walks and Markov chains. However, potential theory for recurrent Markov processes with continuous state spaces was not studied to our satisfaction. In 1960 T. Ueno put forward a new approach to study the recurrent Markov process and derived the important results on the Green potential and capacity. He introduced the invariant measure m for the process using a nice measure and the Green measure. Further he obtained the density function of the Green measure relative to the measure m.

Following Ueno's assumption, in this paper we shall show some results on recurrent Markov processes. We derive that in §2 there exists the equilibrium measure for the kernel of the density by assuming symmetry of the density, and that in §3 the Green capacity defined by Ueno is the Choquet capacity. In the final section we resolved the condenser problem on processes with Brownian hitting measures completely.

§1. Preliminaries.

Let R be a separable Hausdorff locally compact space containing at least two points and satisfying

(R.1) For each point $x \in R$, we can take a countable base of neighborhoods of x consisting of arcwise connected open sets,

(R.2) R is connected.

We denote by $\boldsymbol{B}$ the topological Borel field of subsets of R.

A measurable function $w(t)$ from $[0, \infty)$ to R is called the path function if it is right continuous and has left limits. We denote by W the set of all such functions. $X(t, w)$ (or simply $X(t)$) is a function on W defined by $X(t, w) = w(t)$, $t \geqq 0$. The hitting time σ_A for a set $A \in \boldsymbol{B}$ is defined by

$$\sigma_A(w)=\inf\{t\geqq 0 \mid X(t, w)\in A\}, \qquad \text{if such } t \text{ exists,}$$

$$=\infty, \qquad \text{otherwise.}$$

We denote by $\mathscr{B}$, the smallest Borel field of subsets of W containing $\{w \mid X(t, w)\in A\}$ for all $A\in \boldsymbol{B}$ and $t\geqq 0$. For a $\mathscr{B}$-measurable function σ taking values in $[0, \infty)$, we define the shift transformation θ_σ by

$$(\theta_\sigma w)(s)=w(s+\sigma(w)).$$

From this definition we get

$$X(s, \theta_\sigma(w))=X(s+\theta_\sigma(w), w).$$

Let $\mathscr{B}_t$ denote the smallest Borel field on W for which the functions $\{X(s); 0\leqq s\leqq t\}$ are measurable. A random time σ is called a Markov time, if $\{w \mid \sigma(w)<t\}\in \mathscr{B}_t$ for $0\leqq t<\infty$. It can be proved that a hitting time σ_A for a closed or open set A is a Markov time, and that $X(\sigma)$ is $\mathscr{B}_{\sigma+}$-measurable for a Markov time σ, where $\mathscr{B}_{\sigma+}=\bigcap_{n=1}^{\infty} \mathscr{B}_{\sigma+1/n}$.

Let $\{P_x(\cdot), x\in R\}$ be a system of probability measures on satisfying

(P.1) $P_x(E)$ is a $\boldsymbol{B}$-measurable function of x for each $E\in \mathscr{B}$,

(P.2) $P_x(\{w \mid X(0, w)=x\})=1$ for each $x\in R$,

(P.3) quasi-left continuity: if $\{\sigma_n\}$ is a sequence of Markov times increasing monotonely with P_x-probability 1, then we have

$$P_x(\{w \mid \lim_{n\to\infty} X(\sigma_n(w), w)=X(\sigma_\infty(w), w), \sigma_\infty(w)<\infty\})$$
$$=P_x(\{w \mid \sigma_\infty(w)<\infty\}),$$

where $\sigma_\infty(w)=\lim_{n\to\infty} \sigma_n(w)$,

(P.4) Markov property: for any bounded $\mathscr{B}$-measurable function $F(w)$ and $x\in R$, we have

$$E_x(F\circ\theta_t \mid \mathscr{B}_t)=E_{x(t,w)}(F) \qquad \text{a.s. } (P_x),$$

where $E_x(\cdot)$ is the expectation of $\cdot$ with respect to P_x, and $E_x(\cdot \mid \mathscr{B}_t)$ is the conditional expectation with respect to the Borel field $\mathscr{B}_t$.

We call the system $\{W, \mathscr{B}, P_x(\cdot)\}$ a Markov process on R. In order to study a broad class of recurrent Markov processes Ueno [4] introduced the following assumptions (X.1)$\sim$(X.5). In this paper we follow his assumptions.

(X.1) Recurrence: The process hits any set $A\in \boldsymbol{B}$ containing an inner point with probability 1, i.e.

$$P_x(X(t, w)\in A \quad \text{for some } 0\leqq t<\infty)=1, \qquad \text{for any } x\in R.$$

We define the hitting measure $h_A(x, \cdot)$ for the set $A\in \boldsymbol{B}$ by

$$h_A(x, S)=P_x(X(\sigma_A(w), w)\in S, \quad \sigma_A(w)<\infty), \qquad x\in R, S\in \boldsymbol{B}.$$

It is to be called the harmonic measure in classical potential theory.

(X.2) For any continuous function f on A,

$$h_A f(x)=\int h_A(x, dy)f(y)$$

is continuous in $A^c=R-A$, where A is a closed set in R containing an inner point.

(X.3) Maximum Principle: For any non-negative continuous function f in A, $h_A f(x)$ is either strictly positive or 0 for all points x of any one component of A^c, where A is a closed set in R containing an inner point.

(X.4) For any continuous function f on R, the resolvent operator

$$G_\alpha f(x)=E_x\left(\int_0^\infty e^{-\alpha t}f(X(t, w))dt\right)$$

is continuous on R.

(X.5) There is no point of positive holding time, that is, there is no such $x\in R$ that

$$P_x(\sigma>0)>0, \qquad \sigma=\inf\{t\geqq 0 \mid X(t)\in R-\{x\}\}.$$

REMARK. By our assumption (X.4) the strong Markov property, that is,

$$E_x(F\circ\theta_\sigma \mid \mathcal{B}_{\sigma+})=E_{x(\sigma)}(F) \qquad \text{a.s. } (P_x), \tag{1.1}$$

holds for any $\mathcal{B}$-measurable function F and a Markov time σ. Now, we introduce the Green measure

$$G_L(x, A)=E_x\left(\int_0^{\sigma_L}\chi_A(X(t))dt\right), \qquad x\in R,\ A\in\boldsymbol{B}, \tag{1.2}$$

for any closed set L containing an inner point, where χ_A takes 1 on A, 0 on A^c respectively.

LEMMA. 1.1. (X.1) *and* (X.4) *imply*

$$G(x, A)\leqq M(A, L)<\infty, \qquad x\in R,$$

for any closed set L containing an inner point and any $A\in\boldsymbol{B}$ with compact closure, where $M(A, L)$ is a constant which is depending on A and L.

This is Lemma 1.1 in Ueno [4].

Next we define the collection $\mathcal{F}$ which is used in the subsequent sections. Let $\mathcal{F}$ be the family of all $\{K, L\}$ satisfying the following conditions:

(F.1) K and L are closed subsets of R and contain inner points,

(F.2) K is compact,

(F.3) K is contained in one component of L^c.

Moreover we say that $\{K, L\}\in\mathcal{F}$ belongs to $\mathcal{F}_0$ if L is contained in one component of K^c.

LEMMA 1.2. *For each $\{K, L\}\in\mathcal{F}$ there is a unique pair of measures μ_L^K and μ_K^L with total mass 1 on K and L respectively, satisfying*

$$\mu_L^K(\cdot)=\mu_K^L h_K(\cdot)=\int_L \mu_K^L(dx)h_K(x, \cdot),$$

$$\mu_K^L(\cdot)=\mu_L^K h_L(\cdot)=\int_K \mu_L^K(dx)h_L(x, \cdot).$$

This is Lemma 2.1 in Ueno's paper.

Applying these measures μ_L^K, μ_K^L, Ueno [4] introduces his own Green capacity. For $\{K, L\}$ and $\{K', L'\}$ in $\mathfrak{F}$ we make the following definitions.

$$C_{(K,L)}(K', L)=\mu_L^K h_{K',L}(K'), \tag{1.3}$$

$$C_{(K',L)}(K, L)=C_{(K,L)}(K', L)^{-1}, \qquad \text{when } K'\subset K, \tag{1.4}$$

where

$$h_{K,L}(x, E)=P_x(\sigma_K<\sigma_L, X(\sigma_K)\in E), \qquad E\in \boldsymbol{B}. \tag{1.5}$$

The measure $h_{K,L}$ is the conditional hitting probability for which the path attains the set K before the set L.

$$C_{(K,L)}(K', L)=C_{(K,L)}(K\cup K', L)\cdot C_{(K\cup K',L)}(K', L), \tag{1.6}$$

when $\{K, L\}\leftrightarrow\{K', L\}$, where the notation $\{K, L\}\leftrightarrow\{K', L\}$ denotes $\{K\cup K', L\}\in\mathfrak{F}$.

For a sequence $\alpha=(\{K_1, L_1\}, \{K_2, L_2\}, \cdots, \{K_n, L_n\})$ of satisfying $\{K, L\}\leftrightarrow\{K_1, L_1\}\leftrightarrow\cdots\leftrightarrow\{K_n, L_n\}\leftrightarrow\{K', L'\}$

$$C^{\alpha}_{(K,L)}(K', L')=C_{(K,L)}(K_1, L_1)\cdot C_{(K_1,L_1)}(K_2, L_2)\cdot\cdots\cdot C_{(K_n,L_n)}(K', L').$$

Lemma 3.2 in Ueno [4] shows that such $C^{\alpha}_{(K,L)}(K', L')$ does not depend on the choice of α. Now, fixing any $\{K_0, L_0\}\in\mathfrak{F}$, we call $C(K, L)=C_{(K_0,L_0)}(K, L)$ the Green capacity of K with respect to L. Then we introduce the measure

$$m(\cdot)=C(K, L)\int_K \mu_L^K(dx)G(x, \cdot)+C(L, K)\int_L \mu_K^L(dx)G(x, \cdot).$$

This measure is independent of the choice of $\{K, L\}\in\mathfrak{F}$, and takes positive value for each Borel set with inner points.

Since every Green measure $G(x, \cdot)$ is absolutely continuous relative to m by Theorem 4.1 of Ueno, it has a density function $g_L(x, y)$ satisfying

$$G_L(x, A)=\int_A g_L(x, y)m(dy). \tag{1.7}$$

For the kernel $N(x, y)$, $x, y\in R$, the potential $N\mu$ of the measure μ on R is defined by

$$N\mu(x)=\int N(x, y)\mu(dy).$$

A $\boldsymbol{B}$-measurable function f defined on R is said to be superharmonic in an open set D, if

(i) f is lower semi-continuous on D,

(ii) for every $x\in D$ and every open ball $V\subset D$ with the center x

$$f(x)\geqq\int_{V^c}h_{V^c}(x, dy)f(y).$$

In particular when f satisfies condition (ii) only, we say that it is superharmonic in the wide sense.

LEMMA 1.3. *Let f be a non-negative Borel measurable function on R. Then*

$$G_L f(x)=\int_R G_L(x, dy)f(y)$$

is superharmonic on L in the wide sense. (Ueno. Lemma 4.1.)

§ 2. Equilibrium measures on recurrent Markov processes.

In this section we add following two assumptions regarding the density function of the Green measure.

(A.1) $g_L(x, y)$ is lower semi-continuous with respect to x.

(A.2) symmetry: $g_L(x, y)=g_L(y, x)$

holds almost everywhere relative to m. Then we obtain

THEOREM 2.1. *There exists the sequence $\{\mu_n\}$ of measures such that $g_L\mu_n$ tends to one monotonely on L^c as $n\to\infty$.*

PROOF. Let C_n be compact subsets of L^c such that C_n approach to L^c. For every $x\in L^c$ we choose a closed ball $B_r(x)$ with center x and radius r contained in L^c. To simplify the notation let τ be the hitting time for the set $B_r(x)^c$. Then we have $P_x(\tau<\sigma_L)=1$ because of the right continuity of paths. Applying (1.2) we get

$$\begin{aligned}G_L(x, B_r(x))&=E_x\Big(\int_0^{\sigma_L}\chi_{B_r(x)}(X(t))dt\Big)\\&\geqq E_x\Big(\int_0^{\tau}\chi_{B_r(x)}(X(t))dt\Big)\\&=E_x(\tau).\end{aligned}$$

Since $P_x(\tau>0)=1$ from the right continuity of paths, we see $E_x(\tau)>0$. Therefore

$$G_L(x, B_r(x))>0. \tag{2.1}$$

Now we can take $x\in\mathring{C}_n$ and $B_r(x)\subset\mathring{C}_n$ for a sufficiently large number n and so

$$nG_L\chi_{C_n}(x)\geqq nG_L(x, B_r(x)),$$

where $\mathring{C}_n$ denotes the interior of the set C_n. Thus by (2.1) $nG_L\chi_{C_n}(x)$ tends to infinity. Setting $f_n=1\wedge nG_L\chi_{C_n}(x)$, where $a\wedge b=\min\{a, b\}$, we obtain $f_n\uparrow 1$ as $n\to\infty$.

Furthermore such f_n is non-negative and superharmonic on L^c because

$G_L \chi_{C_n}(x)$ is so by Lemma 1.3, Fatou's lemma and (A.1). Hence f_n is excessive as well known (Dynkin [2]. p. 16). Here a Borel measurable function g is called excessive on L^c if $Q^t g(x) \leqq g(x)$ and $\lim_{t \to 0} Q^t g(x) = g(x)$ for every $x \in L^c$, where

$$Q^t(x, E) = P_x(X(t) \in E,\ \sigma_L > t), \qquad E \in \boldsymbol{B}.$$

Then

$$G_L(x, E) = \int_0^\infty Q^t(x, E)\,dt \tag{2.2}$$

and the semigroup property

$$Q^{t+s} = Q^s Q^t, \qquad s, t > 0 \tag{2.3}$$

hold by the Markov property. In view of (2.2) and (2.3), we have

$$\begin{aligned} Q^t f_n(x) &\leqq n Q^t G_L(x, C_n) \\ &= n \int Q^t(x, dy) \int_0^\infty Q^s(y, C_n)\,ds \\ &= n \int_0^\infty Q^{t+s}(x, C_n)\,ds \\ &= n\left(\int_0^\infty Q^s(x, C_n)\,ds - \int_0^t Q^s(x, C_n)\,ds\right) \\ &\leqq n G_L(x, C_n). \end{aligned}$$

Since $n G_L(x, C_n) \leqq M(C_n, L) < \infty$ by Lemma 1.1, we get

$$\lim_{t \to \infty} n\left(\int_0^\infty Q^s(x, C_n)\,ds - \int_0^t Q^s(x, C_n)\,ds\right) = 0.$$

Therefore

$$\lim_{t \to \infty} Q^t f_n(x) = 0, \qquad x \in L^c \tag{2.4}$$

holds as desired.

Choose a positive number r arbitrarily. Then (2.3) implies that

$$\int_0^r Q^t(f_n - Q^s f_n)(x)\,dt = \int_0^s Q^t f_n(x)\,dt - \int_r^{r+s} Q^t f_n(x)\,dt.$$

By letting $r \to \infty$ and noting (2.4), we have

$$\int_0^\infty Q^t(f_n - Q^s f_n)(x)\,dt = \int_0^s Q^t f_n(x)\,dt.$$

That is,

$$G_L\left(\frac{f_n - Q^s f_n}{s}\right)(x) = \frac{1}{s}\int_0^s Q^t f_n(x)\,dt. \tag{2.5}$$

Set $\varphi_n = n(f_n - Q^{1/n} f_n)$. Then using (2.5), we get

$$G_L \varphi_n(x) = n \int_0^{1/n} Q^t f_n(x)\,dt.$$

On the other hand since

$$n\int_0^{1/n} Q^t f_n(x)dt \geqq n\int_0^{1/n} Q^t f_m(x)dt \geqq m\int_0^{1/m} Q^t f_m(x)dt$$

for $n \geqq m$, $G_L\varphi_n(x)$ is also nondecreasing in n. And we have

$$G_L\varphi_n(x) = n\int_0^{1/n} Q^t f_n(x)dt \geqq n\int_0^{1/n} Q^t f_m(x)dt . \tag{2.6}$$

Now $Q^t f_m(x)$ approaches increasing to $f_m(x)$ as $t \downarrow 0$ because f_m is excessive, so by letting $n \uparrow \infty$ in the identity (2.6) we have $\lim_{n\to\infty} G_L\varphi_n(x) \geqq f_m(x)$ and therefore $\lim_{n\to\infty} G_L\varphi_n(x) \geqq 1$ as $m \uparrow \infty$. That is, $G_L\varphi_n(x) \uparrow 1$.

Let μ_n be the measure on L^c given by $\mu_n(dx) = \varphi_n(x)m(dx)$. Then by (1.4)

$$G_L\varphi_n(x) = \int G_L(x, dy)\varphi_n(y) = \int g_L(x, y)m(d)\varphi_n(y) = \int g_L(x, y)\mu_n(dy) .$$

This completes the proof of the theorem.

Applying (1.5) and the strong Markov property we have

$$G_L(x, E) = G_{L\cup K}(x, E) + \int h_{K,L}(x, dy)G_L(y, E)$$

for $x \in R$ and $E \in \boldsymbol{B}$. By (1.7) the fundamental identity

$$g_L(x, y) = g_{L\cup K}(x, y) + \int h_{K,L}(x, dz)g_L(z, y) \tag{2.7}$$

is obtained almost everywhere in y relative to the measure m.

THEOREM 2.2. *There exists a Radon measure μ such that*

$$g_L\mu = h_{K,L}1 = P.(\sigma_K < \sigma_L), \qquad \text{a.e. } (m) \text{ on } L^c .$$

PROOF. By Theorem 2.1 there is a sequence of measures $\{\nu_n\}$ on L^c such that $g_L\nu_n \uparrow 1$ on L^c. Set

$$\mu_n = \nu_n h_{K,L} = \int \nu_n(dy)h_{K,L}(y, \cdot) .$$

Observe that such μ_n is concentrated on K. Then it follows from the fundamental identity (2.7) and the assumption (A.2) that

$$g_L\mu_n(x) = h_{K,L}g_L\nu_n(x) \qquad \text{a.e. } (m) \text{ on } L^c . \tag{2.8}$$

Therefore by letting $n \to \infty$ in (2.8) we have

$$g_L\mu_n \uparrow h_{K,L}1 \qquad \text{a.e. } (m) \text{ on } L^c . \tag{2.9}$$

Next it will be shown that $\sup_n \mu_n(C) < \infty$ for every compact subset C of L^c. Let G be a compact neighborhood of C. Combining (A.2) with (1.7) we get

$$\int_G g_L\mu_n(x)m(dx)=\int G_L(y, G)\mu_n(dy) \tag{2.10}$$

$$\geqq\int_C G_L(y, G)\mu_n(dy).$$

Since $G_L(y, G)$ is lower semi-continuous on L^c by the assumption (A.1) and Fatou's lemma, we can set $\min_{y\in C} G_L(y, G)=\delta$. Let y_0 denote the point giving the minimum value to $G_L(y, G)$ on C. Then there is an open ball such that $B_r(y_0)\subset G$ because G is a compact neighborhood. $G_L(y_0, B_r(y_0))$ is positive by (2.1) and hence

$$\delta=G_L(y_0, G)\geqq G_L(y_0, B_r(y_0))>0. \tag{2.11}$$

Now

$$\int_G g_L\mu_n(x)m(dx)\geqq\delta\mu_n(C)$$

by (2.10) and

$$\int g_L\mu_n(x)m(dx)\leqq\int_G h_{K,L}1(x)m(dx)\leqq m(G)<\infty$$

follows from (2.9). Consequently $\sup_n \mu_n(C)<\infty$ as desired.

By the result shown above, there is a Radon measure μ on L^c and strictly increasing sequence $\{n_j\}$ of positive integers such that μ_{n_j} converges vaguely to μ. Again we take a compact subset C of L^c arbitrarily. Then we have

$$\int_C g_L\mu_{n_j}(x)m(dx)=\int_K G_L(y, C)\mu_{n_j}(x). \tag{2.12}$$

By applying the identity (2.9) to the left part of (2.12), we get

$$\lim_{j\to\infty}\int_C g_L\mu_{n_j}(x)m(dx)=\int_C h_{K,L}1(x)m(dx). \tag{2.13}$$

On the other hand we can consider $G_L(x, C)$ as the limit of the increasing sequence of simple functions on K since $G_L(x, C)$ is a $\boldsymbol{B}$-measurable function on L^c. Namely setting $f_n=\sum_{i=1}^n a_i\chi_{E_i}$, where $\sum_{i=1}^n E_i=K$, $a_i\geqq 0$, we obtain $f_n(x)\uparrow G_L(x, C)$ on K. Then

$$\lim_{n\to\infty}\sum_{i=1}^n a_i\mu_{n_j}(E_i)=\int_K G_L(x, C)\mu_{n_j}(dx).$$

By Lemma 1.1 we have $\lim_{j\to\infty}\int_K G_L(x, C)\mu_{n_j}(dy)<\infty$ and hence

$$\lim_{j\to\infty}\int_K G_L(x, C)\mu_{n_j}(dx)=\lim_{j\to\infty}\lim_{n\to\infty}\sum_{i=1}^n a_i\mu_{n_j}(E_i) \tag{2.14}$$

$$=\lim_{n\to\infty}\lim_{j\to\infty}\sum_{i=1}^n a_i\mu_{n_j}(E_i)$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n} a_i\mu(E_i)$$

$$=\int_K G_L(x, C)\mu(dx).$$

Consequently it follows from (2.13), (2.14) and (A.2) that

$$\int_C h_{K,L}1(x)m(dx)=\int_K G_L(x, C)\mu(dx)$$

$$=\int_C\int_K g_L(x, y)\mu(dx)m(dy)$$

$$=\int_C g_L\mu(x)m(dx),$$

which completes the proof of the theorem.

We say the measure μ on a domain D the equilibrium measure on $K\subset D$ when for the kernel $U(x, y)$ the potential $U\mu$ equals to a constant value α on K^r, where K^r is the set of points which are regular for K. By Theorem 2.2 $g_L\mu=1$ a.e. (m) on K^r. That is, μ is the equilibrium measure for the kernel $g_L(x, y)$ in the wide sense.

§ 3. Some properties on the Green capacity of Ueno.

The objective of this section is to show that the Green capacity $C(K, L)$ defined by Ueno is the Choquet capacity.

THEOREM 3.1. *For $\{K, L\}$ and $\{K', L\}$ in $\mathfrak{F}$ if $\{K\cap K', L\}$ is contained in $\mathfrak{F}$, then*

$$C(K\cup K', L)+C(K\cap K', L)\leqq C(K, L)+C(K', L).$$

This property is called strong subadditivity.

PROOF. By (1.5) and (1.6)

$$C(K', L)=C(K, L)\cdot C_{(K,L)}(K\cup K', L)\cdot C_{(K\cup K',L)}(K', L)$$

$$=\frac{C(K, L)C_{(K\cup K',L)}(K', L)}{C_{(K\cup K',L)}(K, L)}.$$

Therefore applying (1.4) to $C_{(K\cup K',L)}(K, L)$ and $C_{(K\cup K',L)}(K', L)$, we have

$$C(K, L)+C(K', L)=C(K, L)+\frac{C(K, L)\mu_L^{K\cup K'}h_{K',L}(K')}{\mu_L^{K\cup K'}h_{K,L}(K)} \tag{3.1}$$

$$=C(K\cup K', L)\Big(\int\mu_L^{K\cup K'}(dx)P_x(\sigma_K<\sigma_L)+\int\mu_L^{K\cup K'}(dx)P_x(\sigma_{K'}<\sigma_L)\Big).$$

Now $\sigma_{K\cup K'}=\sigma_K\wedge\sigma_{K'}$ and $\sigma_{K\cap K'}\geqq\sigma_K\vee\sigma_{K'}$ and so

$$P.(\sigma_{K\cap K'}<\sigma_L)+P.(\sigma_{K\cup K'}<\sigma_L)\leqq P.(\sigma_K<\sigma_L)+P.(\sigma_{K'}<\sigma_L). \tag{3.2}$$

Consequently (3.1) and (3.2) imply

$$C(K,L)+C(K',L)\geqq C(K\cup K',L)\int\mu_L^{K\cup K'}(dx)(P_x(\sigma_{K\cap K'}<\sigma_L)+P_x(\sigma_{K\cup K'}<\sigma_L))$$

$$=C(K\cup K',L)(C_{(K\cup K',L)}(K\cap K',L)+C_{(K\cup K',L)}(K\cup K',L))$$

$$=C(K\cap K',L)+C(K\cup K',L)$$

as desired.

THEOREM 3.2. (i) *Suppose* $\{K_n, L\}\in\mathcal{F}$ *for each* n *and* $K_n\uparrow K$ *where* $\{K, L\}\in\mathcal{F}$. *Then*

$$\lim_{n\to\infty}C(K_n,L)=C(K,L)\,.$$

(ii) *Suppose* $\{K_n, L\}\in\mathcal{F}$ *for each* n *and* $K_n\downarrow K$ *where* $\{K, L\}\in\mathcal{F}$. *Then*

$$\lim_{n\to\infty}C(K_n,L)=C(K,L)\,.$$

PROOF. (i) By definition (1.4) the capacity can be expressed as

$$C(K_n,L)=C(K,L)\int\mu_L^K(dx)P_x(\sigma_{K_n}<\sigma_L)\,. \tag{3.3}$$

By the hypothesis $\sigma_{K_n}\downarrow\sigma_K$, so $\{\sigma_{K_n}<\sigma_L\}\uparrow\{\sigma_K<\sigma_L\}$ and hence $P.(\sigma_{K_n}<\sigma_L)\uparrow P.(\sigma_K<\sigma_L)$. Consequently it follows from (3.3) that $C(K_n, L)\uparrow C(K, L)$ as desired.

(ii) As before we have

$$C(K_n,L)=C(K_1,L)\int\mu_L^{K_1}(dx)P_x(\sigma_{K_n}<\sigma_L)\,. \tag{3.4}$$

For any $x\in K_1$ by the hypothesis $\sigma=\lim_{n\to\infty}\sigma_{K_n}$ exists, and so from the quasi-left continuity (P.3) we get

$$P_x(\lim_{n\to\infty}X(\sigma_{K_n})=X(\sigma),\ \sigma<\infty)=P_x(\sigma<\infty)\,.$$

Then since $P_x(\sigma_K<\infty)=1$ by the recurrence (X.1), we have $P_x(\sigma<\infty)=1$ and hence

$$P_x(\lim_{n\to\infty}X(\sigma_{K_n})=X(\sigma))=1\,. \tag{3.5}$$

Next we will prove that

$$P.(X(\sigma)\in K)=1 \qquad \text{on } K_1\,. \tag{3.6}$$

Suppose now $P_x(X(\sigma)\notin K=\bigcap_{n=1}^{\infty}K_n)>0$. Then $P_x(X(\sigma)\notin K_{n_0}$ for some $n_0\geqq1)>0$ and so (3.5) implies $P_x(X(\sigma_{K_n})\notin K_{n_0}$ for some $n\geqq n_0\geqq1)>0$. Therefore $P_x(X(\sigma_{K_n})\notin K_n$ for some $n\geqq1)>0$, but this contradicts $X(\sigma_{K_n})\in K_n$ and so (3.6) holds.

By (3.6) we have $P.(\sigma=\sigma_K)=1$ and hence $P.(\sigma_{K_n}\uparrow\sigma_K)=1$ on K_1. Thus we get

$$P.(\sigma_{K_n}<\sigma_L \text{ for all } n \text{ and } \sigma_K>\sigma_L)=0 \qquad \text{on } K_1\,. \tag{3.7}$$

Since there exists $\delta=\lim_{n\to\infty} P.(\sigma_{K_n}<\sigma_L)$, it follows from (3.7) that

$$\begin{aligned}\delta=\lim_{n\to\infty} P.(\sigma_{K_n}<\sigma_L)&=P.(\bigcap_{n=1}^{\infty}\{\sigma_{K_n}<\sigma_L\})\\ &=P.(\sigma_{K_n}<\sigma_L \text{ for all } n)\\ &=P.(\sigma_{K_n}<\sigma_L \text{ for all } n \text{ and } \sigma_K<\sigma_L)\\ &\leqq P.(\sigma_K<\sigma_L).\end{aligned}$$

Trivially $\delta\geqq P.(\sigma_K<\sigma_L)$ holds and hence we have

$$\lim_{n\to\infty} P.(\sigma_{K_n}<\sigma_L)=P.(\sigma_K<\sigma_L) \quad \text{on } K_1.$$

Consequently from (3.4) $C(K_n, L)\downarrow C(K, L)$ as desired.

Let $\mathcal{K}$ be the class of all compact subsets of R. Then a function C is called a Choquet capacity provided:

(i) if $A, B\in\mathcal{K}$ and $A\subset B$, then $C(A)\leqq C(B)$;

(ii) given $A\in\mathcal{K}$ and $\varepsilon>0$ there exists an open set $G\supset A$ such that for every $B\in\mathcal{K}$ with $A\subset B\subset G$ one has $C(B)-C(A)<\varepsilon$.

(iii) $C(A\cup B)+C(A\cap B)\leqq C(A)+C(B)$ for all $A, B\in\mathcal{K}$.

Combining Theorem 3.1 with Theorem 3.2, we can conclude that our Green capacity satisfies the conditions of the Choquet capacity.

§ 4. Condenser measures on Markov processes with Brownian hitting measures.

We say that a Markov process has Brownian hitting measures, if it satisfies (X.4) and

(X.B) For any closed set $L\neq R$ with an inner point and $x\in L^c$, $h_L(x, \cdot)$ coincides with the classical harmonic measure of L viewed from x with respect to the connected component of L^c containing x.

In this section we consider the case in which the state space is a two dimensional Euclidian space R^2. It can be proved that (X.B) combined with (X.4) implies (X.1), (X.2), (X.3), (X.5) and the continuity of path functions.

Let D be a domain with compact boundary ∂D of positive logarithmic capacity. The classical Green function $g^D(x, y)$ of D is given by

$$g^D(x, y)=\log\frac{1}{|x-y|}-\int_{\partial D} h_{\partial D}(x, dz)\log\frac{1}{|z-y|}+\gamma_D(x), \quad x, y\in D, \tag{4.1}$$

where $\gamma_D(x)$ is a non-negative continuous function of x and converges to 0 if $x\in D$ tends to a regular point of the boundary ∂D. By making use of (4.1), for any closed set $L\neq R$ with an inner point we define $g^L(x, y)$ by

$$g^L(x, y)=\log\frac{1}{|x-y|}-\int_L h_L(x, dz)\log\frac{1}{|z-y|}+\gamma_L(x), \tag{4.2}$$

where

$$\gamma_L(x)=\gamma_D(x), \quad \text{if } x \text{ belongs to a connected component } D \text{ of } L^C,$$
$$=0, \quad \text{otherwise}.$$

Then it is known that

$$g^L(x, y)=g^{D_\alpha}(x, y), \quad \text{if } x \text{ and } y \text{ belong to the same component } D_\alpha \text{ of } L^C,$$
$$=0, \quad \text{otherwise, excepting on the set of irregular points of } \partial L.$$

Our principal results depend on the following two lemmas due to Ueno [4].

LEMMA 4.1. *For each* $\{K, L\}$ *in* $\mathfrak{F}$, μ_L^K *is the equilibrium measure of* K *with respect to the classical Green kernel* g^L *of the component.*

LEMMA 4.2. *For any closed set* $L \neq R$ *with an inner point, we can take the classical Green kernel* $g^L(x, y)$ *for* $g_L(x, y)$ *in* (1.7), *or*

$$G_L(x, E)=\int_E g^L(x, y)m(dy), \qquad E\in \boldsymbol{B},\ x\in R.$$

These are Theorem 5.1 and Theorem 5.2 in Ueno [4]. Then we obtain

THEOREM 4.1. *For* $\{K, L\}$ *in* $\mathfrak{F}$, μ_L^K *is the equilibrium measure of* K *with respect to* g_L *in the wide sense.*

PROOF. By Lemma 4.2, we have

$$\int_E g_L\mu_L^K(x)m(dx)=\int G_L(y, E)\mu_L^K(dy)$$
$$=\int_E g^L\mu_L^K(x)m(dx), \qquad E\in \boldsymbol{B}.$$

Therefore $g_L\mu_L^K=g^L\mu_L^K$, a.e. (m) on L^C. Consequently it follows from Lemma 4.1 that $g_L\mu_L^K=\alpha$, a.e. (m) on K^r. This completes the proof of the theorem.

The preceding theorem shows that μ_L^K is essentially the equilibrium measure on Markov processes with Brownian hitting measures.

Lastly we discuss the condenser problem on a Markov process with Brownian hitting measures on R^2. Let $\{K, L\}$ be a pair of sets in $\mathfrak{F}$ and let $U(x, y)$ be a potential kernel. A signed measure ν is called the condenser measure corresponding to K, L if ν is concentrated on $K\cup L$, $U\nu=\alpha$ on L^r and $U\nu=\alpha+\beta$ on K^r, where α, β are constants.

For transient Markov processes whose state spaces are over three dimensions, Chung and Getoor [1] studied the condenser theorem within the framework of probabilistic theory. Moreover for the two dimensional Brownian motion, the condenser measure was obtained by Port and Stone [3]. Here we concretely give the condenser measure on a Markov process with two dimensional Brownian hitting measures.

THEOREM 4.2. *For the logarithmic potential kernel* $k(x, y)=\log\frac{1}{|x-y|}$, $\mu_L^K-\mu_K^L$ *is the condenser measure corresponding to* K, L. *Then we have* $\alpha\leqq k(\mu_L^K-\mu_K^L)\leqq\alpha+\beta$ *on* R^2 *for some* $\alpha\leqq 0$, $\beta>0$.

PROOF. Integrating the identity (4.2) by μ_L^K, we can write

$$\int\mu_L^K(dx)k(x, y)=\int\mu_L^K(dx)g^L(x, y)+\iint\mu_L^K(dx)h_L(x, dz)k(z, y)-\int_{\partial K}\mu_L^K(dx)\gamma_L(x).$$

Since

$$\iint\mu_L^K(dx)h_L(x, dz)k(z, y)=\int\mu_L^K(dz)k(z, y)$$

by Lemma 1.2, it follows that

$$\int(\mu_L^K-\mu_K^L)(dx)k(x, y)=g^L\mu_L^K(y)-\int_{\partial K}\mu_L^K(dx)\gamma_L(x).$$

We may set $-\int_{\partial K}\mu_L^K(dx)\gamma_L(x)=\alpha$, for ∂K is compact and $\gamma_L(x)$ is continuous on L^C. Then we get

$$\int(\mu_L^K-\mu_K^L)(dx)k(x, y)=g^L\mu_L^K(y)+\alpha.$$

By applying Lemma 4.1 to this, we see

$$\int(\mu_L^K-\mu_K^L)(dx)k(x, y)=\alpha+\beta \qquad \text{on } K^r$$

for some constant β. Moreover from the definition of the classical Green function,

$$\int\mu_L^K(dx)g^L(x, y)=\int_{\partial K}\mu_L^K(dx)g^L(x, y)=0 \qquad \text{on } L^r \tag{4.3}$$

holds. Therefore

$$\int(\mu_L^K-\mu_K^L)(dx)k(x, y)=\alpha \qquad \text{on } L^r.$$

That is, $\mu_L^K-\mu_K^L$ is the condenser measure. Then it is trivial that $\alpha\leqq k(\mu_L^K-\mu_K^L)\leqq\alpha+\beta$, $\alpha\leqq 0$, $\beta>0$ follows on R^2.

In the Jordan decomposition of the condenser measure ν, let ν^+ and ν^- denote, respectively, the positive part and the negative part of ν. Then Chung and Getoor indicated that ν^+ is the equilibrium measure of K and ν^- is its balayage onto L. We say that the measure ν on L is the balayage onto L of the measure μ on K, if for the potential kernel U, $U\mu=U\nu$ holds on L^r. Now we show the similar property of the condenser measure on the Markov process with Brownian hitting measures.

Lemma 4.1 tells us that μ_L^K is the equilibrium measure with respect to g^L. Therefore it is enough to show $g^L\mu_L^K=g^L\mu_K^L$ on L^r. By (4.2) and Lemma 1.2, we have

$$g^L\mu_K^L(y)=\int \mu_K^L(dx)\Big\{\log\frac{1}{|x-y|}-\int h_L(xd,z)\log\frac{1}{|z-y|}+\gamma_L(x)\Big\}$$

$$=\int \mu_K^L(dx)\log\frac{1}{|x-y|}-\int \mu_L^K h_L(dx)h_L(x,dz)\log\frac{1}{|z-y|}+\int_{\partial L}\mu_K^L(dx)\gamma_L(x)$$

$$=\int \mu_K^L(dx)\log\frac{1}{|x-y|}-\int \mu_L^K h_L(dz)\log\frac{1}{|z-y|}+\int_{\partial L}\mu_K^L(dx)\gamma_L(x)$$

$$=\int_{\partial L}\mu_K^L(dx)\gamma_L(x)\,.$$

Since $\int_{\partial L}\mu_K^L(dx)\gamma_L(x)=0$ from the definition of γ_L, we see $g^L\mu_K^L=0$ on R^2. Therefore by (4.3) the desired result $g^L\mu_K^L=g^L\mu_L^K=0$ on L^r is obtained.

References

[1] Chung, K.L. and Getoor, R.K.: The condenser problem, Ann. probability **5** (1977), 82-86.
[2] Dynkin, E.B.: Markov Processes, Volume II, Springer-Verlag, Berlin, 1965.
[3] Port, S.C. and Stone, C.J.: Brownian Motion and Classical Potential Theory, Academic Press, New York, 1978.
[4] Ueno, T.: On recurrent Markov processes, Kodai Math. Seminar Report 12 (1960), 109-142.